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REAL ALGEBRAIC GEOMETRY–WS 2014/15

Solution to Exercise 1 – Sheet 5

Prove that there are $2^{\aleph_0}$ pairwise distinct Archimedean orderings on $\mathbb{Q}(x)$. (Recall that $2^{\aleph_0}$ is the cardinality of the continuum.)

Proof.
Let $\mathcal{Q}(\mathbb{Q}(x))$ be the collection of all orderings on $\mathbb{Q}(x)$. We aim to prove that

$$|\mathcal{Q}(\mathbb{Q}(x))| = 2^{\aleph_0}.$$

Let us denote by $(\mathbb{R}, <_{\mathbb{R}})$ the set $\mathbb{R}$ with its unique ordering $\sum \mathbb{R}^2$ and by $\tilde{\mathbb{Q}}^r$ the relative algebraic closure of $\mathbb{Q}$ in $\mathbb{R}$. Consider $e, \pi \in \mathbb{R}\backslash\tilde{\mathbb{Q}}^r$ and the corresponding extensions of $\mathbb{Q}$, i.e. $\mathbb{Q}(e)$ and $\mathbb{Q}(\pi)$.

Since $\mathbb{Q}(e) \subset \mathbb{R}$, we have that $P_e := \mathbb{Q}(e) \cap \sum \mathbb{R}^2$ is an ordering on $\mathbb{Q}(e)$ and we denote by $(\mathbb{Q}(e), <_{P_e})$ the correspondent ordered field. In the same way, $P_\pi := \mathbb{Q}(\pi) \cap \sum \mathbb{R}^2$ is an ordering on $\mathbb{Q}(\pi)$ and we denote by $(\mathbb{Q}(\pi), <_{P_\pi})$ the correspondent ordered field. Note that both the orderings P_e and P_π are Archimedean since $(\mathbb{R}, <_{\mathbb{R}})$ is an Archimedean field.

Consider now the following field homomorphism $\varphi : \mathbb{Q}(e) \to \mathbb{Q}(\pi)$ defined by $\varphi(q) = q$ for any $q \in \mathbb{Q}$ and $\varphi(e) = \pi$. Note that φ is actually a field isomorphism, since e and π are both transcendental over $\mathbb{Q}$ and so $\mathbb{Q}(e)$ and $\mathbb{Q}(\pi)$ are both isomorphic to $\mathbb{Q}(x)$.

Let Q_π be the cone given by the pull-back of P_π through φ, i.e.

$$Q_\pi := \varphi^{-1}(P_\pi) = \{x \in \mathbb{Q}(e) : \varphi(x) \in P_\pi\}.$$

Then the cone Q_π is an ordering on $\mathbb{Q}(e)$ (see Sheet 2–Ex 3.e) and we denote by $(\mathbb{Q}(e), <_{Q_\pi})$ the corresponding ordered field. In other words, for any $x, y \in \mathbb{Q}(e)$ we have $x <_{Q_\pi} y$ if and only if $\varphi(x) <_{P_\pi} \varphi(y)$. Note that Q_π is also Archimedean since P_π is Archimedean. Indeed, for any $r \in \mathbb{Q}(e)$ there exists $n \in \mathbb{N}$ such that $\varphi(r) <_{P_\pi} n = \varphi(n)$, which is equivalent to $r <_{Q_\pi} n$ by definition of Q_π.

Let us show now that $Q_\pi \neq P_e$.
Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $q \in \mathbb{Q}$ such that $e <_{\mathbb{R}} q <_{\mathbb{R}} \pi$. In particular, this gives that $e <_{P_e} q$ and $q <_{P_\pi} \pi$. Note that by the definition of φ and of Q_π we get that:

$$q <_{P_\pi} \pi \iff \varphi(q) <_{P_\pi} \varphi(e) \iff q <_{Q_\pi} e.$$

Therefore, we proved that $e <_{P_e} q$ but $q <_{Q_\pi} e$. Hence, we have constructed two distinct Archimedean orderings Q_π and P_e on $\mathbb{Q}(e)$.

We can repeat the same procedure replacing π by any other $t \neq e$ transcendental over $\mathbb{Q}$ to obtain a new Archimedean ordering Q_t on $\mathbb{Q}(e)$ which is different from P_e. In this way, we have explicitly constructed as many pairwise distinct Archimedean orderings on $\mathbb{Q}(e)$ as the number of transcendentals over $\mathbb{Q}$. In Sheet 4–Ex 3.c, we proved that the field of real algebraic numbers is countable, i.e. $|\tilde{\mathbb{Q}}^r| = \aleph_0$. Therefore, recalling that $|\mathbb{R}| = 2^{\aleph_0}$, we have that $|\mathbb{R} \setminus \tilde{\mathbb{Q}}^r| = 2^{\aleph_0}$, i.e. the set of transcendental numbers has cardinality $2^{\aleph_0}$.
So far, we showed that there exist at least $2^{\aleph_0}$ pairwise distinct Archimedean orderings on $\mathbb{Q}(e)$ and so on $\mathbb{Q}(x)$, since $\mathbb{Q}(x) \cong \mathbb{Q}(e)$, i.e. $|\mathcal{Q}(\mathbb{Q}(x))| \geq 2^{\aleph_0}$.

On the other hand, any ordering on $\mathbb{Q}(x)$ is a subset of $\mathbb{Q}(x)$ and $|\mathbb{Q}(x)| = \aleph_0$. Hence, $|\mathcal{Q}(\mathbb{Q}(x))| \leq |\mathcal{P}(\mathbb{Q}(x))| = 2^{\aleph_0}$, where $\mathcal{P}(\mathbb{Q}(x))$ denotes the power set of $\mathbb{Q}(x)$. $\square$