# Topological Algebras 

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Summer Semester 2018

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The primary sources for these notes are [11] and [3]. However, we also referred to [2] and [13]. The references to results from the theory of topological vector spaces appear in the following according to the enumeration used in [9] and [10].

## Introduction

The theory of topological algebras has its first roots in the famous works by Gelfand on "normed rings" of 1939 (see [4, 5, 6, 7]) followed by about fifteen years of successful activity on this subject which culminated in the publication of the book dealing with the commutative theory and its applications. From there the theory of normed and Banach algebras gained more and more importance (see [?] for a thorough account) until, with the development of the theories of topological rings and topological vector spaces, the investigation of general topological algebras became unavoidable. On the one hand, there was a great interest in better understanding which are the advantages of having in the same structure both the properties of topological rings and topological vector spaces. On the other hand it was desirable to understand how far one can go beyond normed and Banach algebras still retaining their distinguished features. The need for such an extension has been apparent since the early days of the theory of general topological algebras, more precisely with the introduction of locally multiplicative convex algebras by Arens in [1] and Michael in [12] (they introduced the notion independently). Moreover, it is worth noticing that the previous demand was due not only to a theoretical interest but also to concrete applications of this general theory to a variety of other disciplines (such as quantum filed theory and more in general theoretical physics). This double impact of the theory of topological algebras is probably the reason for which, after almost 80 years from its foundation, this is still an extremely active subject which is indeed recently enjoying very fast research developments.

## Chapter 1

## General Concepts

In this chapter we are going to consider vector spaces over the field $\mathbb{K}$ of real or complex numbers which is given the usual euclidean topology defined by means of the modulus.

### 1.1 Brief reminder about algebras over a field

Let us first recall the basic vocabulary needed to discuss about algebras.
Definition 1.1.1. $A \mathbb{K}$-algebra $A$ is a vector space over $\mathbb{K}$ equipped with an additional binary operation which is bilinear:

$$
\begin{array}{ccc}
A \times A & \rightarrow & A \\
(a, b) & \mapsto & a \cdot b
\end{array}
$$

called vector multiplication.
In other words, $(A,+, \cdot)$ is a ring such that the vector operations are both compatible with the multiplication by scalars in $\mathbb{K}$.

If a $\mathbb{K}$-algebra has an associative (resp. commutative) vector multiplication then it is said to be an associative (resp. commutative ) $\mathbb{K}$-algebra. Furthermore, if a $\mathbb{K}$-algebra $A$ has an identity element for the vector multiplication (called the unity of $A$ ), then $A$ is referred to as unital.

## Examples 1.1.2.

1. The real numbers form a unital associative commutative $\mathbb{R}$-algebra.
2. The complex numbers form a unital associative commutative $\mathbb{R}$-algebra.
3. Given $n \in \mathbb{N}$, the polynomial ring $\mathbb{R}\left[x_{1}, \ldots, x_{n}\right]$ (real coefficients and $n$ variables) equipped with pointwise addition and multiplication is a unital associative commutative $\mathbb{R}$-algebra.
4. The space $\mathcal{C}(X)$ of $\mathbb{K}$-valued continuous function on a topological space $X$ equipped with pointwise addition and multiplication is a unital associative commutative $\mathbb{K}$-algebra.
5. Given $n \in \mathbb{N}$, the ring $\mathbb{R}^{n \times n}$ of real square matrices of order $n$ equipped with the standard matrix addition and matrix multiplication is a unital associative $\mathbb{R}$-algebra but not commutative.
6. The set of quaternions $\mathbb{H}:=\{a+b i+c j+d k: a, b, c, d \in \mathbb{R}\}$ equipped with the componentwise addition and scalar multiplication is a real vector space with basis $\{1, i, j, k\}$. Let us equip $\mathbb{H}$ with the Hamilton product which is defined first on the basis elements by setting

$$
\begin{gathered}
i \cdot 1=1=1 \cdot i, \quad j \cdot 1=1=1 \cdot j, \quad k \cdot 1=1=1 \cdot k, \quad i^{2}=j^{2}=k^{2}=-1 \\
i j=k, \quad j i=-k, \quad j k=i, \quad k j=-i, \quad k i=j, \quad i k=-j,
\end{gathered}
$$

and then it is extended to all quaternions by using the distributive property and commutativity with real quaternions. Note that the multiplication formulas are equivalent to $i^{2}=j^{2}=k^{2}=i j k=-1$.
Then $\mathbb{H}$ is a unital, associative but non-commutative $\mathbb{R}$-algebra since e.g. $i j=k$ but $j i=-k$.
7. The three-dimensional Euclidean space $\mathbb{R}^{3}$ equipped with componentwise addition and scalar multiplication and with the vector cross product $\wedge$ as multiplication is a non-unital, non-associative, non-commutative $\mathbb{R}$-algebra. Non-associative since e.g. $(i \wedge j) \wedge j=k \wedge j=-i$ but $i \wedge(j \wedge j)=i \wedge 0=0$, non-commutative since e.g. $i \wedge j=k$ but $j \wedge i=-k$ and non-unital because if there was a unit element $u$ then for any $x \in \mathbb{R}^{3}$ we would have $u \wedge x=x=x \wedge u$, which is equivalent to say that $x$ is perpendicular to itself and so that $x=0$. (Here $i=(1,0,0)$, $j=(0,1,0)$ and $k=(0,0,1))$.
If we replace the vector cross product by the componentwise multiplication then $\mathbb{R}^{3}$ becomes a unital associative commutative $\mathbb{R}$-algebra with unity $(1,1,1)$.

Recall that:
Definition 1.1.3. Let $A$ be a $\mathbb{K}$-algebra. Then

1. $A$ subalgebra $B$ of $A$ is a linear subspace of $A$ closed under vector multiplication, i.e. $\forall b, b^{\prime} \in B, \quad b b^{\prime} \in B$.
2. A left ideal (resp. right ideal) $I$ of $A$ is a linear subspace of $A$ such that $\forall a \in A, \forall b \in I, a b \in I$ (resp. $\forall a \in A, \forall b \in I, b a \in I$. An ideal of $A$ is a linear subspace of $A$ which is simultaneously left and right ideal of $A$.
3. $A$ homomorphism between two $\mathbb{K}$-algebras $(A, \cdot)$ and $(B, *)$ is a linear map $\varphi: A \rightarrow B$ such that $\varphi(a \cdot b)=\varphi(a) * \varphi(b)$ for all $a, b \in A$. Its kernel $\operatorname{Ker}(\varphi)$ is an ideal of $A$ and its image $\varphi(A)$ is a subalgebra of $B$. A homomorphism between two unital $\mathbb{K}$-algebras has the additional property that $\varphi\left(1_{A}\right)=1_{B}$ where $1_{A}$ and $1_{B}$ are respectively the unit element in $A$ and the unit element in $B$.
4. The vector space $A_{1}=\mathbb{K} \times A$ equipped with the following operations:

$$
\begin{gathered}
(\lambda, a)+(\mu, b):=(\lambda+\mu, a+b), \forall \lambda, \mu \in \mathbb{K}, a, b \in A \\
\mu(\lambda, a):=(\mu \lambda, \mu a), \forall \lambda, \mu \in \mathbb{K}, a \in A \\
(\lambda, a) \cdot(\mu, b):=(\lambda \mu, \lambda b+\mu a+a b), \forall \lambda, \mu \in \mathbb{K}, a, b \in A
\end{gathered}
$$

is called the unitization of $A$.
Proposition 1.1.4. $A \mathbb{K}$-algebra $A$ can be always embedded in its unitization $A_{1}$ which is a unital algebra.

Proof. It is easy to check that $A_{1}$ fulfils the assumptions of $\mathbb{K}$-algebra and that the map

$$
e: A \rightarrow A_{1}, a \mapsto(0, a)
$$

is an injective homomorphism, i.e. a monomorphism. The unit element of $A_{1}$ is given by $(1, o)$ as $(\lambda, a) \cdot(1, o)=(\lambda, a)=(1, o) \cdot(\lambda, a), \forall \lambda \in \mathbb{K}, a \in A$. Identifying $a$ and $e(a)$ for any $a \in A$, we can see $A$ as a subalgebra of $A_{1}$.

### 1.2 Definition and main properties of a topological algebra

Definition 1.2.1. $A \mathbb{K}$-algebra $A$ is called a topological algebra (TA) if $A$ is endowed with a topology $\tau$ which makes the vector addition and the scalar multiplication both continuous and the vector multiplication separately continuous. (Here $\mathbb{K}$ is considered with the euclidean topology and, $A \times A$ and $\mathbb{K} \times A$ with the corresponding product topologies.)

If the vector multiplication in a TA is jointly continuous then we just speak of a TA with a continuous multiplication. Recall that jointly continuous implies separately continuous but the converse is false in general. In several books, the definition of TA is given by requiring a jointly continuous vector multiplication but we prefer here the more general definition according to [11].

An alternative definition of TA can be given in connection to TVS. Let us recall the definition:

Definition 1.2.2. A vector space $X$ over $\mathbb{K}$ is called a topological vector space (TVS) if $X$ is provided with a topology $\tau$ which is compatible with the vector space structure of $X$, i.e. $\tau$ makes the vector addition and the scalar multiplication both continuous. (Here $\mathbb{K}$ is considered with the euclidean topology and, $X \times X$ and $\mathbb{K} \times X$ with the corresponding product topologies.)

Then it is clear that
Definition 1.2.3. A topological algebra over $\mathbb{K}$ is a TVS over $\mathbb{K}$ equipped with a separately continuous vector multiplication.

Therefore, TAs inherit all the advantageous properties of TVS. In the following we will try to characterize topologies which make a $\mathbb{K}$-algebra into a TA. To do that we will make use of the results already available from the theory of TVS and see the further properties brought in by the additional structure of being a TA.

In this spirit, let us first recall that the topology of a TVS is always translation invariant that means, roughly speaking, that any TVS topologically looks about any point as it does about any other point. More precisely:

## Proposition 1.2.4.

The filter ${ }^{1} \mathcal{F}(x)$ of neighbourhoods of $x$ in a TVS $X$ coincides with the family of the sets $O+x$ for all $O \in \mathcal{F}(o)$, where $\mathcal{F}(o)$ is the filter of neighbourhoods of the origin o (i.e. neutral element of the vector addition).
(see [9, Corollary 2.1.9]]). This result easily implies that:
Proposition 1.2.5. Let $X, Y$ be two t.v.s. and $f: X \rightarrow Y$ linear. The map $f$ is continuous if and only if $f$ is continuous at the origin o.

Proof. (see [9, Corollary 2.1.15-3]]).
Thus, the topology of a TVS (and in particular the one of a TA) is completely determined by the filter of neighbourhoods of any of its points, in particular by the filter of neighbourhoods of the origin o or, more frequently, by a base of neighbourhoods of the origin $o$. We would like to derive a criterion on a collection of subsets of a $\mathbb{K}$-algebra $A$ which ensures that it is a basis of neighbourhoods of the origin $o$ for some topology $\tau$ making $(A, \tau)$ a TA. To this aim let us recall the following result from TVS theory:

[^0]Theorem 1.2.6. A filter $\mathcal{F}$ of a vector space $X$ over $\mathbb{K}$ is the filter of neighbourhoods of the origin for some topology $\tau$ making $X$ into a TVS iff

1. The origin belongs to every set $U \in \mathcal{F}$
2. $\forall U \in \mathcal{F}, \exists V \in \mathcal{F}$ s.t. $V+V \subset U$
3. $\forall U \in \mathcal{F}, \forall \lambda \in \mathbb{K}$ with $\lambda \neq 0$ we have $\lambda U \in \mathcal{F}$
4. $\forall U \in \mathcal{F}, U$ is absorbing.
5. $\forall U \in \mathcal{F}, \exists V \in \mathcal{F}$ balanced s.t. $V \subset U$.

Proof. (see [9, Theorem 2.1.10]).
Recall that:
Definition 1.2.7. Let $U$ be a subset of a vector space $X$.

1. $U$ is absorbing (or radial) if $\forall x \in X \exists \rho>0$ s.t. $\forall \lambda \in \mathbb{K}$ with $|\lambda| \leq \rho$ we have $\lambda x \in U$.
2. $U$ is balanced (or circled) if $\forall x \in U, \forall \lambda \in \mathbb{K}$ with $|\lambda| \leq 1$ we have $\lambda x \in U$.
(see also [9, Examples 2.1.12, Proposition 2.1.13]).
A first interesting consequence of Theorem 1.2.6 for TA is that
Lemma 1.2.8. For a TVS to be a TA with continuous multiplication it is necessary and sufficient that the vector multiplication is jointly continuous at the point $(o, o)$.

## Proof.

If $A$ is a TA with continuous multiplication, then clearly the multiplication is jointly continuous everywhere and so in particular at $(o, o)$. Conversely, let $A$ be a TVS with multiplication $M$ jointly continuous at the point $(o, o)$ and denote by $\mathcal{F}(o)$ the filter of neighbourhoods of the origin in $A$. Let $(o, o) \neq$ $(a, b) \in A \times A$ and $U \in \mathcal{F}(o)$. Then Theorem 1.2.6 guarantees that there exists $V \in \mathcal{F}(o)$ balanced and such that $V+V+V \subset U$. Moreover, the joint continuity of the multiplication at $(o, o)$ gives that there exists $U_{1}, U_{2} \in \mathcal{F}(0)$ such that $U_{1} U_{2} \subset V$. Taking $W:=U_{1} \cap U_{2}$ we have $W W \subseteq V$. Also, since $W$ is absorbing, there exists $\rho>0$ s.t. for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq \rho$ we have $\lambda a \in W, \lambda b \in W$. For $\theta:=\left\{\begin{array}{ll}\rho & \text { if } \rho \leq 1 \\ \frac{1}{\rho} & \text { if } \rho>1\end{array}\right.$, we have both $|\theta| \leq 1$ and $|\theta| \leq \rho$. Hence,

$$
\begin{aligned}
(a+\theta W)(b+\theta W) & \subseteq a b+a \theta W+W \theta b+\theta^{2} W W \subseteq a b+W W+W W+\theta^{2} V \\
& \subseteq a b+V+V+V \subseteq a b+U
\end{aligned}
$$

We showed that $\exists N \in \mathcal{F}(o)$ such that $M^{-1}(a b+U) \supseteq(a+N) \times(b+N)$ which proves that joint continuity of $M$ at the point $(a, b)$.

We are now ready to give a characterization for a basis ${ }^{2}$ of neighbourhoods of the origin in a TA (resp. TA with continuous multiplication).

Theorem 1.2.9. $A$ non-empty collection $\mathcal{B}$ of subsets of $a \mathbb{K}$-algebra $A$ is a basis of neighbourhoods of the origin for some topology making A into a TA if and only if
a) $\mathcal{B}$ is a basis of neighbourhoods of o for a topology making A into a TVS. b) $\forall U \in \mathcal{B}, \forall a \in A, \exists V, W \in \mathcal{B}$ s.t. $a V \subseteq U$ and $W a \subseteq U$.

## Proof.

Let $(A, \tau)$ be a TA and $\mathcal{B}$ be a basis of neighbourhoods of the origin of $A$. Then $(A, \tau)$ is in particular a TVS and so (a) holds. Also by definition of TA, the multiplication is separately continuous which means for any $a \in A$ the maps $L_{a}(y)=a y$ and $R_{a}(y)=y a$ are both continuous everywhere in $A$. Then by Proposition 1.2.5 they are continuous at o, i.e. $\forall U \in \mathcal{B}, \forall a \in A, \exists V, W \in \mathcal{B}$ s.t. $V \subset L_{a}^{-1}(U)$ and $W \subset R_{a}^{-1}(U)$, i.e. $a V \subseteq U$ and $W a \subseteq U$, that is (b).

Conversely, suppose that $\mathcal{B}$ is a collection of subsets of a $\mathbb{K}$-algebra $A$ fulfilling (a) and (b). Then (a) guarantees that there exists a topology $\tau$ having $\mathcal{B}$ as basis of neighbourhoods of $o$ and such that $(A, \tau)$ is a TVS. Hence, as we have already observed, (b) means that both $L_{a}$ and $R_{a}$ are continuous at $o$ and so by Proposition 1.2.5 continuous everywhere. This yields that the vector multiplication on $A$ is separately continuous and so that $(A, \tau)$ is a TA.

Theorem 1.2.10. A non-empty collection $\mathcal{B}$ of subsets of $a \mathbb{K}$-algebra $A$ is a basis of neighbourhoods of the origin for some topology making $A$ into a TA with continuous multiplication if and only if
a) $\mathcal{B}$ is a basis of neighbourhoods of o for a topology making A into a TVS. b') $\forall U \in \mathcal{B}, \exists V \in \mathcal{B}$ s.t. $V V \subseteq U$.

Proof. (Sheet 1).

## Examples 1.2.11.

1. Every $\mathbb{K}$-algebra $A$ endowed with the trivial topology $\tau$ (i.e. $\tau=\{\emptyset, A\}$ ) is a TA.

[^1]2. Let $S$ be a non-emptyset and $\mathbb{K}^{S}$ be the set of all functions from $S$ to $\mathbb{K}$ equipped with pointwise operations and the topology $\omega$ of pointwise convergence (or simple convergence), i.e. the topology generated by
$$
\mathcal{B}:=\left\{W_{\varepsilon}\left(x_{1}, \ldots, x_{n}\right): n \in \mathbb{N}, x_{1}, \ldots, x_{n} \in S, \varepsilon>0\right\}
$$
where $W_{\varepsilon}\left(x_{1}, \ldots, x_{n}\right):=\left\{f \in \mathbb{K}^{S}: f\left(x_{i}\right) \in B_{\varepsilon}(0), i=1, \ldots, n\right\}$ and $B_{\varepsilon}(0)=\{k \in \mathbb{K}:|k| \leq \varepsilon\}$. Then $\left(\mathbb{K}^{S}, \omega\right)$ is a TA with continuous multiplication. Indeed, for any $n \in \mathbb{N}, x_{1}, \ldots, x_{n} \in S, \varepsilon>0$ we have that
\[

$$
\begin{aligned}
W_{\sqrt{\varepsilon}}\left(x_{1}, \ldots, x_{n}\right) W_{\sqrt{\varepsilon}}\left(x_{1}, \ldots, x_{n}\right) & =\left\{f g: f\left(x_{i}\right), g\left(x_{i}\right) \in B_{\sqrt{\varepsilon}}(0), i=1, \ldots, n\right\} \\
& \subseteq\left\{h: h\left(x_{i}\right) \in B_{\varepsilon}(0), i=1, \ldots, n\right\} \\
& =W_{\varepsilon}\left(x_{1}, \ldots, x_{n}\right) .
\end{aligned}
$$
\]

As it is also easy to show that $\left(\mathbb{K}^{S}, \omega\right)$ is a TVS, the conclusion follows by Theorem 1.2.10.

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[^0]:    ${ }^{1}$ A filter on a set $X$ is a family $\mathcal{F}$ of subsets of $X$ which fulfils the following conditions: (F1) the empty set $\emptyset$ does not belong to $\mathcal{F}$
    (F2) $\mathcal{F}$ is closed under finite intersections
    (F3) any subset of $X$ containing a set in $\mathcal{F}$ belongs to $\mathcal{F}$ (c.f. [9, Section 1.1.1]]).

[^1]:    ${ }^{2}$ A family $\mathcal{B}$ of subsets of $X$ is called a basis of a filter $\mathcal{F}$ if

    1. $\mathcal{B} \subseteq \mathcal{F}$
    2. $\forall A \in \mathcal{F}, \exists B \in \mathcal{B}$ s.t. $B \subseteq A$
    or equivalently if $\forall A, B \in \mathcal{B}, \exists C \in \mathcal{B}$ s.t. $C \subseteq A \cap B$ (c.f. [9, Section 1.1.1])
