metrizable lmc algebra by Theorem 3.1.3. Indeed, if $\underline{a}, \underline{b} \in \mathbb{K}^{\mathbb{N}}$, *then*

$$p_n(\underline{ab}) = \max_{k\leq n} |a_k b_k| \leq \max_{k\leq n} |a_k| \max_{k\leq n} |b_k| = p_n(\underline{a})p_n(\underline{b})$$

for all $n \in \mathbb{N}$. *Further, if* $p_n(\underline{a}) = 0$ *for all* $n \in \mathbb{N}$, *then*

$$\max_{k\leq n} |a_k| = 0, \forall n \in \mathbb{N} \Rightarrow |a_k| = 0, \forall k \in \mathbb{N} \Rightarrow a \equiv 0.$$

Moreover, $(\mathbb{K}^{\mathbb{N}}, \tau_{\mathcal{P}})$ *is sequentially complete and so complete (prove it yourself). Hence, it is a Fréchet lmc algebra.*

3) *The Arens–algebra* $L^{\omega}([0,1]) := \bigcap_{p\geq 1} L^p([0,1])$ *endowed with the topology* $\tau_{\mathcal{P}}$ *generated by* $\mathcal{P} := \{\|\cdot\|_p : p \in \mathbb{N}\}$ *is a Fréchet lc algebra which is not lmc. We have already showed that it is an lc algebra but not lmc. Metrizability comes from the fact that the family of seminorms is countable and increasing (Hölder–inequality). Proving completeness is more complicated which we will maybe see it later on.*

3.2 Locally bounded algebras

The TAs we are going to study in this section were first introduced by W. Zelazko in the 1960's and provide non-trivial examples of TAs whose underlying space is not necessarily locally convex (so they are neither necessarily lc algebras nor lmc algebras) but they still share several nice properties of Banach and/or lmc algebras.

Definition 3.2.1. *A TA is* locally bounded (lb) *if there exists a neighbourhood of the origin which is bounded. Equivalently, a locally bounded algebra is a TA which is in particular a locally bounded TVS (i.e. the space has a bounded neighbourhood of the origin).*

Recall that:

Definition 3.2.2. *A subset B of a TVS X is* bounded *if for any neighbourhood U of the origin in X there exists $\lambda > 0$ s.t. $B \subseteq \lambda U$ (i.e. B can be swallowed by any neighbourhood of the origin).*

This generalizes the concept of boundedness we are used to in the theory of normed and metric spaces, where a subset is bounded whenever we can find a ball large enough to contain it.

Example 3.2.3. *The subset* $Q := [0,1]^2$ *is bounded in* $(\mathbb{R}^2, \|\cdot\|)$ *as for any* $\varepsilon > 0$ *there exists* $\lambda > 0$ *s.t.* $Q \subseteq \lambda B_\varepsilon(o)$ *namely, if* $\varepsilon \geq \sqrt{2}$ *take* $\lambda = 1$*, otherwise take* $\lambda = \frac{\sqrt{2}}{\varepsilon}$*.*

Proposition 3.2.4. *Every Hausdorff locally bounded algebra is metrizable.*

Proof.
Let (A, τ) be a Hausdorff locally bounded algebra and $\mathcal{F}(o)$ its filter of neighbourhoods of the origin. Then there exists $U \in \mathcal{F}(o)$ bounded. W.l.o.g. we can assume that U is balanced. Indeed, if this is not the case, then we can replace it by some $V \in \mathcal{F}(o)$ balanced s.t. $V \subseteq U$. Then the boundedness of U provides that $\forall N \in \mathcal{F}(o)\ \exists \lambda > 0$ s.t. $U \subseteq \lambda N$ and so $V \subseteq \lambda N$, i.e. V is bounded and balanced.

The collection $\{\frac{1}{n}U : n \in \mathbb{N}\}$ is a countable basis of neighbourhoods of the origin o. In fact, for any $N \in \mathcal{F}(o)$ there exists $\lambda > 0$ s.t. $U \subseteq \lambda N$, i.e. $\frac{1}{\lambda}U \subseteq N$, and so $\frac{1}{n}U \subseteq \frac{1}{\lambda}U$ for all $n \geq \lambda$ as U is balanced. Hence, we obtain that for any $N \in \mathcal{F}(o)$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n}U \subseteq N$. Then we can apply Theorem 3.1.2 which gives that (A, τ) is a metrizable algebra. $\square$

The converse is not true in general as for example the countable product of 1–dimensional metrizable TVS is metrizable but not locally bounded.

Corollary 3.2.5. *Every complete Hausdorff lb algebra has continuous multiplication.*

Proof. Since local boundedness and Hausdorffness imply metrizability, Proposition 3.1.16 ensures that the multiplication is continuous. $\square$

The concept of lb TVS and so of lb TA can be characterized through extensions of the notion of norm, which will allow us to see how some results can be extended from Banach algebras to complete lb algebras.

Definition 3.2.6. *Let* X *be a* $\mathbb{K}$*–vector space. A map* $\|\cdot\| : X \to \mathbb{R}^+$ *is said to be a* quasi-norm *if*
1. $\forall x \in X : \|x\| = 0 \iff x = 0$,
2. $\forall x \in X \forall \lambda \in \mathbb{K} : \|\lambda x\| = |\lambda| \, \|x\|$,
3. $\exists k \geq 1 : \|x + y\| \leq k(\|x\| + \|y\|), \forall x, y \in X$.
If $k = 1$ *this coincides with the notion of norm.*

Example 3.2.7.
Let $0 < p < 1$ *and consider the space* $L^p([0,1])$ *with* $\|\cdot\|_p : L^p([0,1]) \to \mathbb{R}^+$ *defined by* $\|f\|_p := (\int_0^1 |f(x)|^p \, dx)^{\frac{1}{p}}$ *for all* $f \in L^p([0,1])$*. Then the Minkowski*

inequality does not hold but we still have that $\|f+g\|_p \leq 2^{\frac{1-p}{p}}(\|f\|+\|g\|)$ *for all* $f,g \in L^p([0,1])$ *and so that* $\|\cdot\|_p$ *is a quasi-norm.*

Proposition 3.2.8. *Let* (X,τ) *be a Hausdorff TVS. Then* (X,τ) *is lb if and only if* τ *is induced by a quasi-norm on* X.

Proof.
Assume that (X,τ) is lb and $\mathcal{F}(o)$ is its filter of neighbourhoods of the origin. Then there exists balanced and bounded $U \in \mathcal{F}(o)$ and $\mathcal{B} := \{\alpha U : \alpha > 0\}$ is a basis of neighbourhoods of the origin in (X,τ) because for any $N \in \mathcal{F}(o)$ there exists $\lambda > 0$ s.t. $U \subseteq \lambda N \Rightarrow \mathcal{B} \ni \frac{1}{\lambda}U \subseteq N$. Consider the Minkowski functional $p_U(x) := \inf\{\alpha > 0 : x \in \alpha U\}$. In the proof of Lemma 2.2.7 we have already seen that if U is absorbing and balanced, then $0 \leq p_U(x) < \infty$ and $p_U(\lambda x) = |\lambda|\, p_U(x)$ for all $x \in X$ and all $\lambda \in \mathbb{K}$. If $p_U(x) = 0$, then $x \in \alpha U$ for all $\alpha > 0$ and so $x \in \bigcap_{\alpha>0} \alpha U = \{o\}$, i.e. $x = o$. Since X is a TVS, $\exists\ V \in \mathcal{F}(o)$ s.t. $V + V \subseteq U$ and also $\exists\ \alpha > 0$ s.t. $\alpha U \subseteq V$ as $\mathcal{B}$ is a basis of neighbourhoods. Therefore, $\alpha U + \alpha U \subseteq V + V \subseteq U$ and taking $k \geq \max\{1, \frac{1}{\alpha}\}$, we obtain $U + U \subseteq \frac{1}{\alpha}U \subseteq kU$ as U is balanced.

Let $x, y \in X$ and $\rho > p_U(x), \delta > p_U(y)$, then $x \in \rho U, y \in \delta U$ since U is balanced, and so $\frac{x}{\rho}, \frac{y}{\delta} \in U$. Thus,

$$\frac{x+y}{\rho+\delta} = \frac{\rho}{\rho+\delta}\frac{x}{\rho} + \frac{\delta}{\rho+\delta}\frac{y}{\delta} \in U + U \subseteq kU.$$

and we obtain $x + y \in k(\rho+\delta)U$ which implies $p_U(x+y) \leq k(\rho+\delta)$. As $\rho > p_U(x)$ and $\delta > p_U(y)$ were chosen arbitrarily, we conclude $p_U(x+y) \leq k(p_U(x) + p_U(y))$. Hence, p_U is a quasi-norm.

Let $B_1^{p_U} := \{x \in X : p_U(x) \leq 1\}$. Then we have $U \subseteq B_1^{p_U} \subseteq (1+\varepsilon)U$ for all $\varepsilon > 0$. Indeed, if $x \in U$, then $p_U(x) \leq 1$ and so $x \in B_1^{p_U}$. If $x \in B_1^{p_U}$, then $p_U(x) \leq 1$ and so $\forall \varepsilon > 0\ \exists\ \alpha$ with $\alpha \leq 1 + \varepsilon$ s.t. $x \in \alpha U$. This gives that $x \in (1+\varepsilon)U$ as U is balanced and so $\alpha U \subseteq (1+\varepsilon)U$. Since $\{\varepsilon B_1^{p_U} : \varepsilon > 0\}$ is a basis of τ_{p_U}, this implies $\tau = \tau_{p_U}$.
Conversely, assume that $\tau = \tau_q$ for a quasi-norm q on X and $\mathcal{F}^q(o)$ its filter of neighbourhoods of the origin. The collection $\mathcal{B} := \{\varepsilon B_1^q : \varepsilon > 0\}$ is a basis of neighbourhoods of the origin in (X,τ) (by Theorem 1.2.6). Let us just show that $\forall N \in \mathcal{F}^q(o)\ \exists V \in \mathcal{F}^q(o)$ s.t. $V + V \subseteq N$. Indeed, $\frac{1}{2k}B_1^q + \frac{1}{2k}B_1^q \subseteq B_1^q$ because if $x, y \in B_1^q$, then

$$q\left(\frac{x+y}{2k}\right) = \frac{1}{2k}q(x+y) \leq \frac{k(q(x)+q(y))}{2k} \leq \frac{2k}{2k} = 1$$

and so $\frac{x+y}{2k} \in B_1^q$. Then for all $N \in \mathcal{F}^q(o)$ there is some $\varepsilon > 0$ s.t. $\varepsilon B_1^q \subseteq N$ and so $\frac{\varepsilon}{2k} B_1^q + \frac{\varepsilon}{2k} B_1^q \subseteq \varepsilon B_1^q \subseteq N$. Since $\mathcal{B}$ is a basis for τ_q, for any $N \in \mathcal{F}^q(o)$ there exists $\varepsilon > 0$ s.t. $\varepsilon B_1^q \subseteq N$, which implies $B_1^q \subseteq \frac{1}{\varepsilon} N$. Therefore, B_1^q is bounded and so τ_q is a lb TVS. □

Using the previous proposition and equipping the space in Example 3.2.7 with pointwise multiplication, we get an example of lb but not lc algebra (see Sheet 5). An example of lc but not lb algebra is given by the following.

Example 3.2.9. *Let K be any compact subset of $(\mathbb{R}, \|\cdot\|)$ and let us consider the algebra $\mathcal{C}^\infty(K)$ of all real valued infinitely differentiable functions on K equipped with pointwise operations. Using the same technique as in Example 3.1.17, we can show that $\mathcal{C}^\infty(K)$ endowed with the topology τ_K, generated by the family $\{r_n : n \in \mathbb{N}_0\}$ where $r_n(f) := \sup\limits_{j=0,\dots,n} \sup_{x\in K} |(D^{(j)} f)(x)|$ for any $f \in \mathcal{C}^\infty(K)$, is a Fréchet lmc algebra, i.e. an lc metrizable and complete algebra.*

Denote now by $\mathcal{C}^\infty(\mathbb{R})$ the space of all real valued infinitely differentiable functions on $\mathbb{R}$ and by $\mathcal{C}_c^\infty(K)$ its subset consisting of all the functions $f \in \mathcal{C}^\infty(\mathbb{R})$ whose support lies in K, i.e.

$$\mathcal{C}_c^\infty(K) := \{f \in \mathcal{C}^\infty(\mathbb{R}) : supp(f) \subseteq K\},$$

where $supp(f)$ denotes the support of the function f, that is the closure in $(\mathbb{R}, \|\cdot\|)$ of the subset $\{x \in \mathbb{R} : f(x) \neq 0\}$. Then it is easy to see that $\mathcal{C}_c^\infty(K) = \mathcal{C}^\infty(K)$ and this is a linear subspace of $\mathcal{C}^\infty(\mathbb{R})$. Indeed, for any $f, g \in \mathcal{C}_c^\infty(K)$ and any $\lambda \in \mathbb{R}$, we clearly have $f + g \in \mathcal{C}^\infty(\mathbb{R})$ and $\lambda f \in \mathcal{C}^\infty(\mathbb{R})$ but also $supp(f+g) \subseteq supp(f) \cup supp(g) \subseteq K$ and $supp(\lambda f) = supp(f) \subseteq K$, which gives $f + g, \lambda f \in \mathcal{C}_c^\infty(K)$.

Let $\mathcal{C}_c^\infty(\mathbb{R})$ be the union of the subspaces $\mathcal{C}_c^\infty(K)$ as K varies in all possible ways over the family of compact subsets of $\mathbb{R}$, i.e. $\mathcal{C}_c^\infty(\mathbb{R})$ consists of all the functions belonging to $\mathcal{C}^\infty(\mathbb{R})$ having compact support (this is what is actually encoded in the subscript "c"). In particular, the space $\mathcal{C}_c^\infty(\mathbb{R})$ is usually called space of test functions *and plays an essential role in the theory of distributions.*

Consider a sequence $(K_j)_{j\in\mathbb{N}}$ of compact subsets of $\mathbb{R}$ s.t. $K_j \subseteq K_{j+1}, \forall j \in \mathbb{N}$ and $\bigcup_{j=1}^\infty K_j = \mathbb{R}$. Then $\mathcal{C}_c^\infty(\mathbb{R}) = \bigcup_{j=1}^\infty \mathcal{C}_c^\infty(K_j)$, as an arbitrary compact subset K of $\mathbb{R}$ is contained in K_j for some sufficiently large j, and we have that $\mathcal{C}_c^\infty(K_j) \subseteq \mathcal{C}_c^\infty(K_{j+1})$. For any $j \in \mathbb{N}$, we endow $\mathcal{C}_c^\infty(K_j)$ with the topology $\tau_j := \tau_{K_j}$ defined as above. Then $(\mathcal{C}_c^\infty(K_j), \tau_{K_j})$ is a Fréchet lmc algebra and $\tau_{j+1} \restriction_{\mathcal{C}_c^\infty(K_j)} = \tau_j$. Denote by τ_{ind} the finest lc topology on $\mathcal{C}_c^\infty(\mathbb{R})$ such that all the inclusions $\mathcal{C}_c^\infty(K_j) \subseteq \mathcal{C}_c^\infty(\mathbb{R})$ are continuous (τ_{ind} does not depend on *the choice of the sequence of compact sets K_j's provided they fill $\mathbb{R}$). Then it is possible to show that $(\mathcal{C}_c^\infty(\mathbb{R}), \tau_{ind})$ is a complete lc algebra but not Baire. Hence, Proposition 3.1.13 provides that $(\mathcal{C}_c^\infty(\mathbb{R}), \tau_{ind})$ is not metrizable and so not lb by Proposition 3.2.4.*

Definition 3.2.10. *Let X be a $\mathbb{K}$–vector space and $0 < \alpha \leq 1$. A map $q : X \to \mathbb{R}^+$ is an* α–norm *if*
1. $\forall x \in X : q(x) = 0 \iff x = 0$,
2. $\forall x \in X \forall \lambda \in \mathbb{K} : q(\lambda x) = |\lambda|^\alpha q(x)$,
3. $\forall x, y \in X : q(x + y) \leq q(x) + q(y)$.
If $\alpha = 1$, this coincides with the notion of norm.

Definition 3.2.11. *A TVS (X, τ) is* α–normable *if τ can be induced by an α–norm for some $0 < \alpha \leq 1$.*

In order to understand how $\alpha-$norms relates to lb spaces we need to introduce a generalization of the concept of convexity.

Definition 3.2.12. *Let $0 < \alpha \leq 1$ and X a $\mathbb{K}-$vector space.*

- *A subset V of X is* $\alpha-$convex *if for any $x, y \in V$ we have $tx + sy \in V$ for all $t, s > 0$ such that $t^\alpha + s^\alpha = 1$.*
- *A subset V of X is* absolutely $\alpha-$convex *if for any $x, y \in V$ we have $tx + sy \in V$ for all $t, s \in \mathbb{K}$ such that $|t|^\alpha + |s|^\alpha \leq 1$.*
- *For any $W \subseteq X$, $\Gamma_\alpha(W)$ denotes the smallest absolutely $\alpha-$convex subset of X containing W, i.e.*

$$\Gamma_\alpha(W) := \left\{ \sum_{i=1}^n \lambda_i w_i : n \in \mathbb{N}, w_i \in W, \lambda_i \in \mathbb{K} \ s.t. \sum_{i=1}^n |\lambda_i|^\alpha \leq 1 \right\}.$$

Proposition 3.2.13. *Let (X, τ) be a TVS and $0 < \alpha \leq 1$. Then (X, τ) is α–normable if and only if there exists an α–convex, bounded neighbourhood of the origin.*

Proof.
Suppose that τ is induced by an α–norm q, i.e. the collection of all $B_r^q := \{x \in X : q(x) \leq r\}$ for all $r > 0$ is a basis of neighbourhoods of the origin for τ. Then for any $x, y \in B_1^q$ and any $t, s \in \mathbb{K}$ such that $|t|^\alpha + |s|^\alpha \leq 1$ we have that

$$q(tx + sy) \leq |t|^\alpha q(x) + |s|^\alpha q(y) \leq |t|^\alpha + |s|^\alpha \leq 1,$$

i.e. B_1^q is absolutely $\alpha-$convex. Also, the definition of $\alpha-$norm easily implies that

$$\forall \rho > 0, \forall x \in B_1^q, \ q(\rho^{\frac{1}{\alpha}}) = \rho q(x) \leq \rho$$

and so that $B_1^q \subseteq \rho^{-\frac{1}{\alpha}} B_\rho^q$. Hence, B_1^q is a bounded absolutely $\alpha-$convex neighbourhood of the origin.

Conversely, suppose that V is an α–convex bounded neighbourhood of the origin in (X, τ).

Claim 1: W.l.o.g. we can always assume that V is absolutely α–convex. Then, as we showed in the proof of Proposition 3.2.8, the Minkowski functional p_V of V is a quasi-norm generating τ. Hence, defining $q(x) := p_V(x)^\alpha, \forall x \in X$ we can prove that

Claim 2: q is an $\alpha-$norm.

Now $V \subseteq B_1^q$ because for any $x \in V$ we have that $q(x) \leq 1$. Also, for any $x \in B_1^q$ we have that $p_V(x) \leq 1$ and so for any $\varepsilon > 0$ there exists $\rho > 0$ s.t. $x \in \rho V$ and $\rho < p_V(x) + \varepsilon \leq 1 + \varepsilon$. Then $x \in \rho V \subseteq (1+\varepsilon)V$ as V is balanced. Then we have just showed that

$$\forall \varepsilon > 0, V \subseteq B_1^q \subseteq (1+\varepsilon)V,$$

which in turn provides that τ is generated by q.

Let us now complete the proof by showing both claims.

Proof. of Claim 1

By assumption V is α–convex bounded neighbourhood of the origin in (X, τ). If V is also balanced, then there is nothing to prove as V is already absolutely α–convex. If V is not balanced, then we can replace it with $\Gamma_\alpha(W)$ for some W balanced neighbourhood of the origin in X such that $W \subseteq V$ (the existence of such a W is given by Theorem 1.2.6 as (X, τ) is a TVS). In fact, we can show that $\Gamma_\alpha(W) \subseteq V$, which provides in turn that $\Gamma_\alpha(W)$ is both bounded and absolutely α–convex.

Let $z \in \Gamma_\alpha(W)$. Then $z = \sum_{i=1}^n \lambda_i w_i$ for some $n \in \mathbb{N}, w_i \in W$, and some $\lambda_i \in \mathbb{K}$ s.t. $\sum_{i=1}^n |\lambda_i|^\alpha \leq 1$. Take $\rho > 0$ such that $\rho^\alpha = \sum_{i=1}^n |\lambda_i|^\alpha$ and for each $\in \{1, \ldots, n\}$ set $\varepsilon_i := \frac{\lambda_i}{|\lambda_i|}\rho$. Then

$$z = \sum_{i=1}^n \lambda_i w_i = \sum_{i=1}^n \frac{|\lambda_i|}{\rho} \varepsilon_i w_i. \tag{3.5}$$

As $\rho^\alpha \leq 1$, we have $\rho \leq 1$ and so $|\varepsilon_i| \leq 1$. Then by the balancedness of W, for each $i \in \{1, \ldots, n\}$, we get that $\varepsilon_i w_i \in W \subset V$. Since $\sum_{i=1}^n \left(\frac{|\lambda_i|}{\rho}\right)^\alpha = 1$ and V is $\alpha-$convex, (3.5) provides that $z \in V$.

Proof. of Claim 2

Since p_V is a quasi-norm on X, we have that $\forall x \in X,\ p_V(x) \geq 0$, which clearly implies that $\forall x \in X,\ q(x) = p_V(x)^\alpha \geq 0$. Moreover, we have that $x = 0$ if and only if $p_V(x) = 0$, which is equivalent to $q(x) = 0$. The positive homogeneity of p_V gives in turn that

$$\forall x \in X,\ \forall \lambda \in \mathbb{K},\ q(\lambda x) = p_V(\lambda x)^\alpha = |\lambda|^\alpha p_V(x)^\alpha = |\lambda|^\alpha q(x). \tag{3.6}$$

To show the triangular inequality for q, let us fix $x, y \in X$ and choose $\rho, \sigma \in \mathbb{R}^+$ such that $\rho > p_V(x)$ and $\sigma > p_V(y)$. Then there exist $\lambda, \mu > 0$ such that $x \in \lambda V$, $\lambda < \rho$ and $y \in \mu V$, $\mu < \sigma$. These together with the balancedness of V imply that $x \in \rho V$ and $y \in \sigma V$. Hence, we have $\frac{x}{\rho}, \frac{y}{\sigma} \in V$ and so, by the $\alpha-$convexity of V we can conclude that

$$\frac{x+y}{(\rho^\alpha + \sigma^\alpha)^{\frac{1}{\alpha}}} = \frac{\rho}{(\rho^\alpha + \sigma^\alpha)^{\frac{1}{\alpha}}} \cdot \frac{x}{\rho} + \frac{\sigma}{(\rho^\alpha + \sigma^\alpha)^{\frac{1}{\alpha}}} \cdot \frac{y}{\sigma} \in V.$$

Then $p_V\left(\frac{x+y}{(\rho^\alpha+\sigma^\alpha)^{\frac{1}{\alpha}}}\right) \leq 1$ and so $q\left(\frac{x+y}{(\rho^\alpha+\sigma^\alpha)^{\frac{1}{\alpha}}}\right) \leq 1$. Then, by using (3.6), we get that $\left(\frac{1}{(\rho^\alpha+\sigma^\alpha)^{\frac{1}{\alpha}}}\right)^\alpha \cdot q(x+y) \leq 1$, that is $q(x+y) \leq \rho^\alpha + \sigma^\alpha$. Since this holds for all $\rho, \sigma \in \mathbb{R}^+$ such that $\rho > p_V(x)$ and $\sigma > p_V(y)$, we obtain that $q(x+y) \leq p_V(x)^\alpha + p_V(y)^\alpha$.

□

Corollary 3.2.14. *Every α–normable TVS is lb.*

The converse also holds and in proving it the following notion turns out to be very useful.

Definition 3.2.15. *If (X, τ) is an lb TVS, then for any balanced, bounded, neighbourhood U of the origin in X we define*

$$C(U) := \inf\{\lambda : U + U \subseteq \lambda U\}.$$

The concavity module $C(X)$ of X *is defined as follows*

$$C(X) := \inf\{C(U) : U \textit{ balanced, bounded, neighbourhood of } o \textit{ in } X\}.$$

Theorem 3.2.16. *Let (X, τ) be a TVS. Then (X, τ) is lb if and only if τ is induced by some α–norm for some $0 < \alpha \leq 1$.*

Proof. The sufficiency is given by the previous corollary. As for the necessity, it is possible to show that if (X, τ) is lb then there exists a bounded $\alpha-$convex neighbourhood of the origin for all $0 < \alpha < \alpha_0$, where $\alpha_0 := \frac{\log 2}{\log C(X)}$ (see Sheet 5). Hence, the conclusion follows by Proposition 3.2.13. □

In the context of lb algebras, it might happen that the $\alpha-$norm defining the topology is actually submultiplicative. This is actually the case if the considered algebra is complete.

Definition 3.2.17. *An* $\alpha-$normed algebra *is a* $\mathbb{K}-$*algebra endowed with the topology induced by a submultiplicative* $\alpha-$*norm.*

Theorem 3.2.18. *Any lb Hausdorff complete algebra can be made into an* $\alpha-$*normed algebra for some* $0 < \alpha \leq 1$.

Proof. Sketch
Let (X, τ) be a Hausdorff complete lb algebra. For convenience let us assume that X is unital but the proof can be adapted also to the non-unital case.

As (X, τ) is lb, Theorem 3.2.16 ensures that the exists $0 < \alpha \leq 1$ such that τ is induced by an $\alpha-$norm q. Consider the space $L(X)$ of all linear continuous operators on X equipped with pointwise addition and scaler multiplication and with the composition as multiplication. Then the operator norm on $L(X)$ defined by $\|\ell\| := \sup_{x \in X \setminus \{o\}} \frac{q(\ell(x))}{q(x)}$ for all $\ell \in L(X)$ is a submultiplicative $\alpha-$norm. Since (X, q) is complete, it is possible to show that it is topologically isomorphic to $(L(X), \|\cdot\|)$. If we denote by φ such an isomorphism, we then get that (X, p) with $p(x) := \|\varphi(x)\|$ for all $x \in X$ is an $\alpha-$normed algebra. ☐

Proposition 3.2.19. *Let* (X, τ) *be an lb Hausdorff TA. Show that if* (X, τ) *has jointly continuous multiplication, then* (X, τ) *is* α*–normable.*

Proof. (see Sheet 6) ☐

3.3 Projective limit algebras

The class of topological algebras which we are going to introduce in this section consisits of algebras obtained as a projective limit of a family of TAs and then endowed with the so-called projective topology associated to the natural system of maps given by the projective limit construction. Therefore, we are first going to introduce in general the notion of projective topology w.r.t. a family of maps, then we will focus on the projective limit construction from both an algebraic and topological point of view.

3.3.1 Projective topology

Let $\{(E_\alpha, \tau_\alpha) : \alpha \in I\}$ be a family of TVSs over $\mathbb{K}$ (I is an arbitrary index set). Let E be a vector space over the same field $\mathbb{K}$ and, for each $\alpha \in I$, let $f_\alpha : E \to E_\alpha$ be a linear mapping. The *projective topology* τ_{proj} on E w.r.t.

the family $\{(E_\alpha, \tau_\alpha), f_\alpha, I\}$ is the coarsest topology on E for which all the mappings f_α $(\alpha \in I)$ are continuous.

It is easy to check that (E, τ_{proj}) is a TVS and that a basis of neighbourhoods of the origin is given by:

$$\mathcal{B}_{proj} := \left\{ \bigcap_{\alpha \in F} f_\alpha^{-1}(U_\alpha) : F \subseteq I \text{ finite}, U_\alpha \in \mathcal{B}_\alpha, \forall \alpha \in F \right\}, \tag{3.7}$$

where $\mathcal{B}_\alpha$ is a basis of neighbourhoods of the origin in (E_α, τ_α).

Remark 3.3.1. *Note that the projective topology τ_{proj} coincides with the initial topology given by the map*

$$\begin{array}{cccc} \varphi : & E & \to & \left(\prod_{\alpha \in I} E_\alpha\right) \\ & x & \mapsto & (f_\alpha(x))_{\alpha \in I}. \end{array}$$

Recall that the initial topology is defined as the coarsest topology on E such that φ is continuous or equivalently as the topology on E generated by the collection of all $\varphi^{-1}(U)$ when U is a neighbourhood of the origin in $\left(\prod_{\alpha \in I} E_\alpha, \tau_{prod}\right)$.

Let us first introduce some properties of the projective topology in the TVS setting.

Lemma 3.3.2. *Let E be a vector space over $\mathbb{K}$ endowed with the projective topology τ_{proj} w.r.t. the family $\{(E_\alpha, \tau_\alpha), f_\alpha, I\}$, where each (E_α, τ_α) is a TVS over $\mathbb{K}$ and each f_α a linear mapping from E to E_α. Let (F, τ) be an arbitrary TVS and g a linear mapping from F into E. The mapping $g : F \to E$ is continuous if and only if, for each $\alpha \in I$, $f_\alpha \circ g : F \to E_\alpha$ is continuous.*

Proof.
Suppose that $g : F \to E$ is continuous. Since by definition of τ_{proj} all f_α's are continuous, we have that for each $\alpha \in I$, $f_\alpha \circ g : F \to E_\alpha$ is continuous.

Conversely, suppose that for each $\alpha \in I$ the map $f_\alpha \circ g : F \to E_\alpha$ is continuous and let U be a neighbourhood of the origin in (E, τ_{proj}). Then there exists a finite subset F of I and for each $\alpha \in F$ there exists $U_\alpha \in \mathcal{B}_\alpha$ such that $\bigcap_{\alpha \in F} f_\alpha^{-1}(U_\alpha) \subseteq U$. Therefore, we obtain

$$g^{-1}(U) \supseteq g^{-1}\left(\bigcap_{\alpha \in F} f_\alpha^{-1}(U_\alpha)\right) = \bigcap_{\alpha \in F} g^{-1}\left(f_\alpha^{-1}(U_\alpha)\right) = \bigcap_{\alpha \in F} (f_\alpha \circ g)^{-1}(U_\alpha),$$

which yields that $g^{-1}(U)$ is a neighbourhood of the origin in (F, τ) since the continuity of all $f_\alpha \circ g$'s ensures that $(f_\alpha \circ g)^{-1}(U_\alpha)$ is a neighbourhood of the origin in (F, τ). □

Proposition 3.3.3. *Let E be a vector space over $\mathbb{K}$ endowed with the projective topology τ_{proj} w.r.t. the family $\{(E_\alpha, \tau_\alpha), f_\alpha, I\}$, where each (E_α, τ_α) is a TVS over $\mathbb{K}$ and each f_α a linear mapping from E to E_α. Then τ_{proj} is Hausdorff if and only if for each $0 \neq x \in E$, there exists an $\alpha \in I$ and a neighbourhood U_α of the origin in (E_α, τ_α) such that $f_\alpha(x) \notin U_\alpha$.*

Proof.
(in the next lecture!) □

Coming back to the context of TAs, we have the following result.

Theorem 3.3.4. *Let E be a $\mathbb{K}-$algebra endowed with the projective topology τ_{proj} w.r.t. the family $\{(E_\alpha, \tau_\alpha), f_\alpha, I\}$, where each (E_α, τ_α) is a TA over $\mathbb{K}$ (resp. a TA with continuous multiplication) and each f_α a homomorphism from E to E_α. Then (E, τ_{proj}) is a TA (resp. a TA with continuous multiplication).*

Proof.
As each (E_α, τ_α) is a TVS, it is easy to verify that (E, τ_{proj}) is a TVS. Therefore, it remains to show that left and right multiplication are both continuous. For any $x \in E$, consider the left multiplication $\ell_x : E \to E$. For each $\alpha \in I$ we get that:

$$\forall\, y \in E,\ (f_\alpha \circ \ell_x)(y) = f_\alpha(xy) = f_\alpha(x) f_\alpha(y) = \ell_{f_\alpha(x)}(f_\alpha(y)) = (\ell_{f_\alpha(x)} \circ f_\alpha)(y). \tag{3.8}$$

Since $f_\alpha(x) \in E_\alpha$ and (E_α, τ_α) is a TA, we have that $\ell_{f_\alpha(x)} : E_\alpha \to E_\alpha$ is continuous and so $\ell_{f_\alpha(x)} \circ f_\alpha$ is continuous. Hence, by (3.8), we have that $f_\alpha \circ \ell_x$ is continuous for all $\alpha \in I$ and so by the previous lemma we have that ℓ_x is continuous. Similarly, we get the continuity of the right multiplication in E. Hence, (E, τ_{proj}) is a TA.

If each (E_α, τ_α) is a TA with continuous multiplication, then by combining Remark 3.3.1 and Proposition 1.4.1 we can conclude that (E, τ_{proj}) is a TA. □