Proposition 3.3.3. *Let E be a vector space over $\mathbb{K}$ endowed with the projective topology τ_{proj} w.r.t. the family $\{(E_\alpha, \tau_\alpha), f_\alpha, I\}$, where each (E_α, τ_α) is a TVS over $\mathbb{K}$ and each f_α a linear mapping from E to E_α. Then τ_{proj} is Hausdorff if and only if for each $0 \neq x \in E$, there exists an $\alpha \in I$ and a neighbourhood U_α of the origin in (E_α, τ_α) such that $f_\alpha(x) \notin U_\alpha$.*

Proof.
Suppose that (E, τ_{proj}) is Hausdorff and let $0 \neq x \in E$. By Proposition 1.3.2, there exists a neighbourhood U of the origin in E not containing x. Then, by (3.7), there exists a finite subset $F \subseteq I$ and, for any $\alpha \in F$, there exists U_α neighbourhood of the origin in (E_α, τ_α) s.t. $\bigcap_{\alpha \in F} f_\alpha^{-1}(U_\alpha) \subseteq U$. Hence, as $x \notin U$, there exists $\alpha \in F$ s.t. $x \notin f_\alpha^{-1}(U_\alpha)$, i.e. $f_\alpha(x) \notin U_\alpha$. Conversely, suppose that there exists $\alpha \in I$ and a neighbourhood V_α of the origin in (E_α, τ_α) such that $f_\alpha(x) \notin V_\alpha$. Let $\mathcal{B}_\alpha$ be a basis of neighbourhoods of the origin in (E_α, τ_α). Then there exists $U_\alpha \in \mathcal{B}_\alpha$ such that $U_\alpha \subseteq V_\alpha$. Hence, $x \notin f_\alpha^{-1}(U_\alpha)$ and $f_\alpha^{-1}(U_\alpha) \in \mathcal{B}_{proj}$ (see (3.7)), that is, we have found a neighbourhood of the origin in (E, τ_{proj}) not containing x. This implies, by Proposition 1.3.2, that τ_{proj} is a Hausdorff topology. □

Coming back to the context of TAs, we have the following result.

Theorem 3.3.4. *Let E be a $\mathbb{K}-$algebra endowed with the projective topology τ_{proj} w.r.t. the family $\{(E_\alpha, \tau_\alpha), f_\alpha, I\}$, where each (E_α, τ_α) is a TA over $\mathbb{K}$ (resp. a TA with continuous multiplication) and each f_α a homomorphism from E to E_α. Then (E, τ_{proj}) is a TA (resp. a TA with continuous multiplication).*

Proof.
As each (E_α, τ_α) is a TVS, it is easy to verify that (E, τ_{proj}) is a TVS. Therefore, it remains to show that left and right multiplication are both continuous. For any $x \in E$, consider the left multiplication $\ell_x : E \to E$. For each $\alpha \in I$ we get that:

$$\forall\, y \in E,\ (f_\alpha \circ \ell_x)(y) = f_\alpha(xy) = f_\alpha(x) f_\alpha(y) = \ell_{f_\alpha(x)}(f_\alpha(y)) = (\ell_{f_\alpha(x)} \circ f_\alpha)(y). \tag{3.8}$$

Since $f_\alpha(x) \in E_\alpha$ and (E_α, τ_α) is a TA, we have that $\ell_{f_\alpha(x)} : E_\alpha \to E_\alpha$ is continuous and so $\ell_{f_\alpha(x)} \circ f_\alpha$ is continuous. Hence, by (3.8), we have that $f_\alpha \circ \ell_x$ is continuous for all $\alpha \in I$ and so by the previous lemma we have that ℓ_x is continuous. Similarly, we get the continuity of the right multiplication in E. Hence, (E, τ_{proj}) is a TA.

If each (E_α, τ_α) is a TA with continuous multiplication, then by combining Remark 3.3.1 and Proposition 1.4.1 we can conclude that (E, τ_{proj}) is a TA. □

Proposition 3.3.5. *Let E be a $\mathbb{K}-$algebra endowed with the projective topology τ_{proj} w.r.t. the family $\{(E_\alpha, \tau_\alpha), f_\alpha, I\}$, where each (E_α, τ_α) is an lc (resp. lmc) algebra over $\mathbb{K}$ and each f_α a homomorphism from E to E_α. Then (E, τ_{proj}) is an lc (resp. lmc) algebra.*

Proof.
By assumption, we know that each (E_α, τ_α) is a TA and so Theorem 3.3.4 ensures that (E, τ_{proj}) is a TA, too. Moreover, as each (E_α, τ_α) is an lc (resp. lmc) algebra, there exists a basis $\mathcal{B}_\alpha$ of convex (resp. m-convex) neighbourhoods of the origin in (E_α, τ_α). Then the corresponding $\mathcal{B}_{proj}$ (see (3.7)) also consists of convex (resp. m-convex) neighbourhoods of the origin in (E, τ_{proj}). In fact, any $B \in \mathcal{B}_{proj}$ is of the form $B = \bigcap_{\alpha \in F} f_\alpha^{-1}(U_\alpha)$ for some $F \subseteq I$ finite and $U_\alpha \in \mathcal{B}_\alpha, \forall\, \alpha \in F$. Since all the U_α's are convex (resp. m-convex) and the preimage of a convex (resp. m-convex) set under a homomorphism is convex (resp. m-convex by Proposition 2.1.3-d)), we get that B is a finite intersection of convex (resp. m-convex) sets and so it is convex (resp. m-convex). □

Corollary 3.3.6. *Let (A, τ) be an lc (resp. lmc) algebra and M a subalgebra of A. If we endow M with the relative topology τ_M induced by A, then (M, τ_M) is an lc (resp. lmc) algebra.*

Proof.
Recalling that τ_M coincides with the projective topology on M induced by $\mathrm{id} : M \to A$ (see Corollary 1.4.2), the conclusion easily follows from the previous proposition (applied for $I = \{1\}$, $E_1 = A$ and $\tau_1 = \tau$, $E = M$ and $f_1 = \mathrm{id}$). □

Corollary 3.3.7. *Any subalgebra of a Hausdorff TA is a Hausdorff TA.*

Proof. This is a direct application of Proposition 3.3.3 and Corollary 1.4.2. □

Example 3.3.8. *Let $\{(E_\alpha, \tau_\alpha) : \alpha \in I\}$ be a family of TAs over $\mathbb{K}$. Then the Cartesian product $F = \prod_{\alpha \in I} E_\alpha$ equipped with coordinatewise operation is a $\mathbb{K}-$algebra. Consider the canonical projections $\pi_\alpha : F \to E_\alpha$ defined by $\pi_\alpha(x) := x_\alpha$ for any $x = (x_\beta)_{\beta \in I} \in F$, which are all homomorphisms. Then the product topology τ_{prod} on F is the coarsest topology for which all the canonical projections are continuous and so coincides with the projective* *topology on F w.r.t. $\{(E_\alpha, \tau_\alpha), \pi_\alpha : \alpha \in I\}$* [2]. *Hence, by Theorem 3.3.4 we have that (F, τ_{prod}) is a TA.*

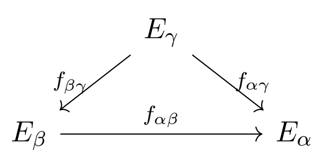
Recalling that a cartesian product of complete Hausdorff TAs endowed with the product topology is a complete Hausdorff TA and applying Proposition 3.3.5, Corollary 3.3.6 and Proposition 3.3.3 to the previous example, we can easily prove the following properties

- any Cartesian product of lc (resp. lmc) algebras endowed with the product topology is an lc (resp. lmc) algebra
- any subalgebra of a Cartesian product of lc (resp. lmc) endowed with the relative topology is a TA of the same type
- a cartesian product of Hausdorff TAs endowed with the product topology is a Hausdorff TA.

3.3.2 Projective systems of TAs and their projective limit

In this section we are going to discuss the concept of projective system (resp. projective limit) first for just $\mathbb{K}$-algebras and then for TAs.

Definition 3.3.9. *Let $(I, <)$ be a directed partially ordered set (i.e. for all $\alpha, \beta \in I$ there exists $\gamma \in I$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$). A* projective system of algebras *$\{E_\alpha, f_{\alpha\beta}, I\}$ is a family of $\mathbb{K}-$algebras $\{E_\alpha, \alpha \in I\}$ together with a family of homomorphisms $f_{\alpha\beta} : E_\beta \to E_\alpha$ defined for all $\alpha \leq \beta$ in I such that $f_{\alpha\alpha}$ is the identity on E_α and $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$ for all $\alpha \leq \beta \leq \gamma$, i.e. the diagram*



commutes.

[2] We could have also directly showed that the equivalence of the two topologies using their basis of neighbourhoods of the origin. Indeed

$$\mathcal{B}_{proj} \overset{(3.7)}{=} \left\{ \bigcap_{\alpha \in F} \pi_\alpha^{-1}(U_\alpha) : F \subseteq I \text{ finite}, U_\alpha \in \mathcal{B}_\alpha, \forall \alpha \in F \right\}$$
$$= \left\{ \prod_{\alpha \in F} U_\alpha \times \prod_{\alpha \in I \setminus F} E_\alpha : F \subseteq I \text{ finite}, U_\alpha \in \mathcal{B}_\alpha, \forall \alpha \in F \right\} = \mathcal{B}_{prod}.$$

Definition 3.3.10. *Given a projective system of algebras* $\mathcal{S} := \{E_\alpha, f_{\alpha\beta}, I\}$, *we define the* projective limit of $\mathcal{S}$ *(or the* projective limit algebra associated with $\mathcal{S}$*) to be the triple* $\{E_{\mathcal{S}}, f_\alpha, I\}$, *where*

$$E_{\mathcal{S}} := \left\{ x := (x_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} E_\alpha : x_\alpha = f_{\alpha\beta}(x_\beta),\ \forall\, \alpha \leq \beta \text{ in } I \right\}$$

and, for any $\alpha \in I$, $f_\alpha : E_{\mathcal{S}} \to E_\alpha$ *is defined by* $f_\alpha := \pi_\alpha \restriction_{E_{\mathcal{S}}}$ *(where* $\pi_\alpha : \prod_{\beta\in I} E_\beta \to E_\alpha$ *is the canonical projection, see Example 3.3.8).*

It is easy to see from the previous definitions that $E_{\mathcal{S}}$ is a subalgebra of $\prod_{\alpha\in I} E_\alpha$. Indeed, for any $x, y \in E_{\mathcal{S}}$ and for any $\lambda \in \mathbb{K}$ we have that for all $\alpha \leq \beta$ in I the following hold

$$\lambda x_\alpha + y_\alpha = \lambda f_{\alpha\beta}(x_\beta) + f_{\alpha\beta}(y_\beta) = f_{\alpha\beta}(\lambda x_\beta + y_\beta)$$

and

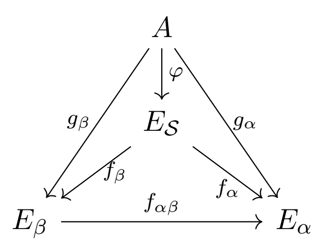
$$x_\alpha y_\alpha = f_{\alpha\beta}(x_\beta) f_{\alpha\beta}(y_\beta) = f_{\alpha\beta}(x_\beta y_\beta),$$

i.e. $\lambda x + y, xy \in E_{\mathcal{S}}$. Note that the f_α's are not necessarily surjective and also that

$$f_\alpha = f_{\alpha\beta} \circ f_\beta,\ \forall \alpha \leq \beta \text{ in } I,$$

since for all $x := (x_\alpha)_{\alpha\in I} \in E_{\mathcal{S}}$ we have $f_\alpha(x) = x_\alpha = f_{\alpha\beta}(x_\beta) = f_{\alpha\beta}(f_\beta(x))$.

Also, we can show that $\{E_{\mathcal{S}}, f_\alpha, I\}$ satisfies the following universal property: given a $\mathbb{K}-$algebra A and a family of homomorphism $\{g_\alpha : A \to E_\alpha, \alpha \in I\}$ such that $g_\alpha = f_{\alpha\beta} \circ g_\beta$ for all $\alpha \leq \beta$ in I, there exists a unique homomorphism $\varphi : A \to E_{\mathcal{S}}$ such that $g_\alpha = f_\alpha \circ \varphi$ for all $\alpha \in I$, i.e. the diagram



commutes. In fact, the map $\varphi : A \to E_{\mathcal{S}}$ defined by $\varphi(a) := (g_\alpha(a))_{\alpha\in I}$ for all $a \in A$ is a homomorphism such that $(f_\alpha \circ \varphi)(a) = (\varphi(a))_\alpha = g_\alpha(a)$, for all $a \in A$. Moreover, if there exists $\varphi' : A \to E_{\mathcal{S}}$ such that $g_\alpha = f_\alpha \circ \varphi'$ for all $\alpha \in I$, then for all $a \in A$ we get

$$\varphi(a) = (g_\alpha(a))_{\alpha\in I} = \big((f_\alpha \circ \varphi')(a)\big)_{\alpha\in I} = \big((\varphi'(a))_\alpha\big)_{\alpha\in I} = \varphi'(a),$$

i.e. $\varphi' \equiv \varphi$ on A.

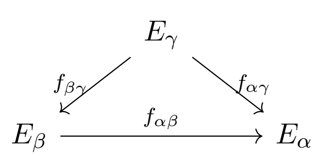
These considerations allows to easily see that one can give the following more general definition of projective limit algebra.

Definition 3.3.11. *Given a projective system of algebras $\mathcal{S} := \{E_\alpha, f_{\alpha\beta}, I\}$, a* projective limit of $\mathcal{S}$ *(or a* projective limit algebra associated with $\mathcal{S}$*) is a triple $\{E, h_\alpha, I\}$, where E is a $\mathbb{K}-$algebra; for any $\alpha \in I$, $h_\alpha : E \to E_\alpha$ is a homomorphisms such that $h_\alpha = f_{\alpha\beta} \circ h_\beta$, $\forall \alpha \leq \beta$ in I; and the following universal property holds: for any $\mathbb{K}-$algebra A and any family of homomorphism $\{g_\alpha : A \to E_\alpha, \alpha \in I\}$ such that $g_\alpha = f_{\alpha\beta} \circ g_\beta$ for all $\alpha \leq \beta$ in I, there exists a unique homomorphism $\varphi : A \to E$ such that $g_\alpha = h_\alpha \circ \varphi$ for all $\alpha \in I$.*

It is easy to show that $\{E, h_\alpha, I\}$ is unique up to (algebraic) isomorphisms, i.e. if $\{\tilde{E}, \tilde{h}_\alpha, I\}$ fulfills Definition 3.3.11 then there exists a unique isomorphism between E and $\tilde{E}$. This justifies why in Definition 3.3.10 we have called $\{E_{\mathcal{S}}, f_\alpha, I\}$ *the* projective limit of $\mathcal{S}$. (Indeed, we have already showed that $\{E_{\mathcal{S}}, f_\alpha, I\}$ fulfills Definition 3.3.11.)

The definitions introduced above for algebras can be easily adapted to the category of TAs.

Definition 3.3.12. *Let $(I, <)$ be a directed partially ordered set. A* projective system of TAs $\{(E_\alpha, \tau_\alpha), f_{\alpha\beta}, I\}$ *is a family of $\mathbb{K}$-algebras $\{(E_\alpha, \tau_\alpha) : \alpha \in I\}$ together with a family of continuous homomorphisms $f_{\alpha\beta} : E_\beta \to E_\alpha$ defined for all $\alpha \leq \beta$ in I such that $f_{\alpha\alpha}$ is the identity on E_α and $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$ for all $\alpha \leq \beta \leq \gamma$, i.e. the diagram*



commutes. Equivalently, a projective system of TAs is a projective system of algebras $\{E_\alpha, f_{\alpha\beta}, I\}$ in which each E_α is endowed with a topology τ_α making (E_α, τ_α) into a TA and all the homomorphisms $f_{\alpha\beta}$ continuous.

Definition 3.3.13. *Given a projective system $\mathcal{S} := \{(E_\alpha, \tau_\alpha), f_{\alpha\beta}, I\}$ of TAs, we define the* projective limit of $\mathcal{S}$ *(or the* projective limit TA associated with $\mathcal{S}$*) to be the triple $\{(E_{\mathcal{S}}, \tau_{proj}), f_\alpha, I\}$ where $\{E_{\mathcal{S}}, f_\alpha, I\}$ is the projective limit algebra associated with $\{E_\alpha, f_{\alpha\beta}, I\}$ and τ_{proj} is the projective topology on $E_{\mathcal{S}}$ w.r.t. the family $\{(E_\alpha, \tau_\alpha), f_\alpha : \alpha \in I\}$.*

Similarly, to the algebraic case, one could give the following more general definition of projective limit TA.

Definition 3.3.14. *Given a projective system of TAs $\mathcal{S} := \{(E_\alpha, \tau_\alpha), f_{\alpha\beta}, I\}$, a* projective limit *of $\mathcal{S}$ (or a* projective limit TA *associated with $\mathcal{S}$) is a triple $\{(E, \tau), h_\alpha, I\}$ where (E, τ) is a TA; for any $\alpha \in I$, $h_\alpha : E \to E_\alpha$ is a continuous homomorphism such that $h_\alpha = f_{\alpha\beta} \circ h_\beta$, for all $\alpha \leq \beta$ in I; and the following universal property holds: for any TA (A, ω) and any family of continuous homomorphism $\{g_\alpha : A \to E_\alpha, \alpha \in I\}$ such that $g_\alpha = f_{\alpha\beta} \circ g_\beta$ for all $\alpha \leq \beta$ in I, there exists a unique continuous homomorphism $\varphi : A \to E$ such that $g_\alpha = h_\alpha \circ \varphi$ for all $\alpha \in I$.*

It is easy to show that $\{(E, \tau), h_\alpha, I\}$ is unique up to topological isomorphisms. We have already showed that $E_{\mathcal{S}}$ is an algebra such that the family of all $f_\alpha := \pi_\alpha \restriction_{E_{\mathcal{S}}}$ $(\alpha \in I)$ fulfills $f_\alpha = f_{\alpha\beta} \circ f_\beta$, $\forall \alpha \leq \beta$ in I. Endowing $E_{\mathcal{S}}$ with the projective topology τ_{proj} w.r.t. $\{(E_\alpha, \tau_\alpha), f_\alpha, I\}$, we get by Theorem 3.3.4 that $(E_{\mathcal{S}}, \tau_{proj})$ is a TA and that all f_α's are continuous. Also, for any TA (A, ω) and any family of continuous homomorphism $\{g_\alpha : A \to E_\alpha, \alpha \in I\}$ such that $g_\alpha = f_{\alpha\beta} \circ g_\beta$ for all $\alpha \leq \beta$ in I, we have already showed that $\varphi : A \to E_{\mathcal{S}}$ defined by $\varphi(a) := (g_\alpha(a))_{\alpha \in I}$ for all $a \in A$ is the unique homomorphism such that $g_\alpha = f_\alpha \circ \varphi$ for all $\alpha \in I$. But φ is also continuous because for any $U \in \mathcal{B}_{proj}$ we have $U = \bigcap_{\alpha \in F} f_\alpha^{-1}(U_\alpha)$ for some $F \subset I$ finite and some $U_\alpha \in \mathcal{B}_\alpha$ for all $\alpha \in F$ and so $\varphi^{-1}(U) = \bigcap_{\alpha \in F} \varphi^{-1}(f_\alpha^{-1}(U_\alpha)) = \bigcap_{\alpha \in F} (f_\alpha \circ \varphi)^{-1}(U_\alpha) = \bigcap_{\alpha \in F} g_\alpha^{-1}(U_\alpha) \in \mathcal{B}_\omega$. Hence, $\{(E_{\mathcal{S}}, \tau_{proj}), f_\alpha, I\}$ satisfies Definiton 3.3.14.

Remark 3.3.15.
From the previous definitions one can easily deduce the following:

a) the projective limit of a projective system of Hausdorff TAs is a Hausdorff TA (easily follows by Proposition 3.3.3).

b) the projective limit of a projective system of Hausdorff TAs $\{(E_\alpha, \tau_\alpha), f_{\alpha,\beta}, I\}$ is a closed subalgebra of $(\prod_{\alpha \in I} E_\alpha, \tau_{\text{prod}})$ (see Sheet 6).

c) the projective limit of a projective system of complete Hausdorff TAs is a complete Hausdorff TA (see Sheet 6).

Corollary 3.3.16. *A projective limit of lmc algebras is an lmc algebra.*

Proof.
Let $\{(E_\alpha, \tau_\alpha), f_{\alpha,\beta}, I\}$ be a projective system of lmc algebras. Then its projective limit $\{(E_{\mathcal{S}}, \tau_{proj}), f_\alpha, I\}$ is an lmc algebra by Proposition 3.3.5. □

This easy corollary brings us to a very natural but fundamental question: can any lmc algebra be written as a projective limit of a projective system of lmc algebras or at least as a subalgebra of such a projective limit? The whole next section will be devoted to show a positive answer to this question.