3.3.3 Arens-Michael decomposition

This section will be devoted to the Arens-Michael decomposition theorem, which was independently conceived by Arens and Michael in the early days of the theory of TAs (1952). This result is so important because it provides a device to reduce basic questions about lmc algebra to analogous ones for the corresponding factor Banach algebras. Since the theory of Banach algebras has been heavily studied, being able to reduce to Banach algebras is very advantageous and so much desirable.

Before stating the Arens-Michael decomposition theorem, let us recall the completion theorem for TVS and two useful lemmas about projective limit algebras.

Theorem 3.3.17.
Let X be a Haudorff TVS. Then there exists a complete Hausdorff TVS $\hat{X}$ and a mapping $i : X \to \hat{X}$ with the following properties:
a) The mapping i is a topological monomorphism.
b) The image of X under i is dense in $\hat{X}$.
c) For every complete Hausdorff TVS Y and for every continuous linear map $f : X \to Y$, there is a continuous linear map $\hat{f} : \hat{X} \to Y$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle i} & \nearrow{\scriptstyle \hat{f}} & \\ \hat{X} & & \end{array}$$

is commutative. Furthermore:
I) Any other pair $(\hat{X}_1, i_1)$, consisting of a complete Hausdorff TVS $\hat{X}_1$ and of a mapping $i_1 : X \to \hat{X}_1$ such that properties (a) and (b) hold substituting $\hat{X}$ with $\hat{X}_1$ and i with i_1, is topologically isomorphic to $(\hat{X}, i)$. This means that there is a topological isomorphism j of $\hat{X}$ onto $\hat{X}_1$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{i_1} & \hat{X}_1 \\ \downarrow{\scriptstyle i} & \nearrow{\scriptstyle j} & \\ \hat{X} & & \end{array}$$

is commutative.
II) Given Y and f as in property (c), the continuous linear map $\hat{f}$ is unique.

Lemma 3.3.18. *Let $\{(A_{\mathcal{S}}, \tau_{proj}), g_\alpha, J\}$ be the projective limit of the projective system $\mathcal{S} := \{(A_\alpha, \tau_\alpha), g_{\alpha\beta}, J\}$ of TAs. Then a basis of neighbourhoods of the* *origin in* $(A_{\mathcal{S}}, \tau_{proj})$ *is given by*

$$\widetilde{\mathcal{B}}_{proj} := \{g_\alpha^{-1}(V_\alpha) : V_\alpha \in \mathcal{B}_\alpha, \alpha \in J\},$$

where each $\mathcal{B}_\alpha$ *is a basis of neighbourhoods of the origin in* (A_α, τ_α)*.*

Proof.
By the continuity of the g_α's, we know that $\widetilde{\mathcal{B}}_{proj}$ is a collection of neighbourhoods of the origin in $(A_{\mathcal{S}}, \tau_{proj})$. We want to show that it is a basis.

By (3.7), we have that a basis of neighbourhoods of the origin in $(A_{\mathcal{S}}, \tau_{proj})$ is given by

$$\mathcal{B}_{proj} := \left\{ \bigcap_{\beta \in F} g_\beta^{-1}(V_\beta) : \; F \subseteq I \text{ finite}, V_\beta \in \mathcal{B}_\beta, \forall\, \beta \in F \right\}.$$

As J is directed, for any finite subset F of I there exists $\alpha \in J$ such that $\beta \leq \alpha$ for all $\beta \in F$. Then we have that $g_\beta = g_{\beta\alpha} \circ g_\alpha$ for all $\beta \in F$ and so

$$\bigcap_{\beta \in F} g_\beta^{-1}(V_\beta) = \bigcap_{\beta \in F} (g_{\beta\alpha} \circ g_\alpha)^{-1}(V_\beta) = \bigcap_{\beta \in F} g_\alpha^{-1}(g_{\beta\alpha}^{-1}(V_\beta)) = g_\alpha^{-1}\left(\bigcap_{\beta \in F} g_{\beta\alpha}^{-1}(V_\beta) \right).$$

Set $W_\alpha := \bigcap_{\beta \in F} g_{\beta\alpha}^{-1}(V_\beta)$. Since for all $\beta \in F$ the map $g_{\beta\alpha} : A_\alpha \to A_\beta$ is continuous, we get that for all $\beta \in F$ the set $g_{\beta\alpha}^{-1}(V_\beta)$ is a neighbourhood of the origin in (A_α, τ_α) and so is W_α. Then there exists $V_\alpha \in \mathcal{B}_\alpha$ such that $V_\alpha \subseteq W_\alpha$. Hence, we obtain

$$\bigcap_{\beta \in F} g_\beta^{-1}(V_\beta) = g_\alpha^{-1}(W_\alpha) \supseteq g_\alpha^{-1}(V_\alpha)$$

and so we have showed that for any $M \in \mathcal{B}_{proj}$ there exists $\widetilde{M} \in \widetilde{\mathcal{B}}_{proj}$ such that $\widetilde{M} \subseteq M$, i.e. $\widetilde{\mathcal{B}}_{proj}$ is a basis of neighbourhoods of the origin in $(A_{\mathcal{S}}, \tau_{proj})$. □

Lemma 3.3.19. *Let* $\{(A_{\mathcal{S}}, \tau_{proj}), g_\alpha, J\}$ *be the projective limit of the projective system* $\mathcal{S} := \{(A_\alpha, \tau_\alpha), g_{\alpha\beta}, J\}$ *of TAs and* W *a linear subspace of* $A_{\mathcal{S}}$*. Then*

$$\overline{W}^{\tau_{proj}} = \bigcap_{\alpha \in J} g_\alpha^{-1}\left(\overline{g_\alpha(W)}^{\tau_\alpha} \right) = \text{projlim}(\mathcal{S}_1),$$

where $\mathcal{S}_1$ *denotes the projective system* $\left\{ \overline{g_\alpha(W)}^{\tau_\alpha}, g_{\alpha\beta} \restriction_{\overline{g_\beta(W)}}, J \right\}$ *of TAs (here* $\overline{g_\alpha(W)}^{\tau_\alpha}$ *is intended as endowed with the relative topology induced by* τ_α*).*

In particular, if W is closed in $(A_{\mathcal{S}}, \tau_{proj})$ then

$$W = \text{projlim}(\mathcal{S}_2) = \text{projlim}(\mathcal{S}_1),$$

where $\mathcal{S}_2$ denotes the projective system $\left\{g_\alpha(W), g_{\alpha\beta}\restriction_{g_\beta(W)}, J\right\}$ of TAs (here $g_\alpha(W)$ is intended as endowed with the relative topology induced by τ_α).

Proof.
Since $\mathcal{S}$ is a projective system of TAs, by Definition 3.3.12, we have that for any $\alpha \leq \beta$ the map $g_{\alpha\beta} : A_\beta \to A_\alpha$ is a continuous homomorphism fulfilling

$$g_{\alpha\alpha} = \text{id}, \ \forall \alpha \in J \tag{3.9}$$

and

$$g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma}, \ \forall \alpha \leq \beta \leq \gamma \text{ in } J. \tag{3.10}$$

Also, by Definition 3.3.14 we have that for any $\alpha \in J$ the map $g_\alpha : A_{\mathcal{S}} \to A_\alpha$ is a continuous homomorphism such that

$$g_\alpha = g_{\alpha\beta} \circ g_\beta, \ \forall \alpha \leq \beta \ \text{ in } J. \tag{3.11}$$

For any $\alpha \in J$, we have that $g_\alpha(W) \subseteq A_\alpha$ and so (3.9) provides that $g_{\alpha\alpha}\restriction_{g_\alpha(W)} = \text{id}\restriction_{g_\alpha(W)}$. Moreover, for any $\alpha \leq \beta \leq \gamma$ in J, the relation (3.11) implies that $g_{\beta\gamma}(g_\gamma(W)) \subseteq g_\beta(W)$, which in turn gives that for any $x \in g_\gamma(W)$:

$$g_{\alpha\beta}\restriction_{g_\beta(W)} \left(g_{\beta\gamma}\restriction_{g_\gamma(W)}(x)\right) = g_{\alpha\beta}(g_{\beta\gamma}(x)) \overset{(3.10)}{=} g_{\alpha\gamma}(x) = g_{\alpha\gamma}\restriction_{g_\gamma(W)}(x).$$

Endowing each $g_\beta(W)$ with the subspace topology induced by τ_β, we have that $g_{\alpha\beta}\restriction_{g_\beta(W)}$ is continuous for any $\alpha \leq \beta$ in J. Hence, we have showed that $\mathcal{S}_2$ is a projective system of TAs.

By the continuity of the $g_{\alpha\beta}$'s for all $\alpha \leq \beta$ in J, we also get that

$$g_{\beta\gamma}(\overline{g_\gamma(W)}) \subseteq \overline{g_{\beta\gamma}(g_\gamma(W))} \overset{(3.11)}{=} \overline{g_\beta(W)}, \forall \beta \leq \gamma \text{ in } J \tag{3.12}$$

Therefore, for any $\alpha \leq \beta \leq \gamma$ in J and for any $x \in \overline{g_\gamma(W)}$ we obtain that

$$g_{\alpha\beta}\restriction_{\overline{g_\beta(W)}} \left(g_{\beta\gamma}\restriction_{\overline{g_\gamma(W)}}(x)\right) = g_{\alpha\beta}(g_{\beta\gamma}(x)) \overset{(3.10)}{=} g_{\alpha\gamma}(x) = g_{\alpha\gamma}\restriction_{\overline{g_\gamma(W)}}(x).$$

Hence, we have showed that $\mathcal{S}_1$ is a projective system of TAs, too.

Then

$$
\begin{aligned}
\operatorname{projlim}(\mathcal{S}_1) &= \Big\{x := (x_\alpha)_{\alpha\in J} : x_\alpha \in \overline{g_\alpha(W)},\ \forall \alpha \in J \text{ and} \\
&\qquad x_\alpha = g_{\alpha\beta}(x_\beta),\ \forall\, \alpha \leq \beta \text{ in } J\} \\
&\overset{(3.12)}{=} \Big\{x := (x_\alpha)_{\alpha\in J} : x_\alpha \in \overline{g_\alpha(W)}, \alpha \in J\Big\} \\
&= \{x \in A_{\mathcal{S}} : g_\alpha(x) \in \overline{g_\alpha(W)}, \forall \alpha \in J\} = \bigcap_{\alpha\in J} g_\alpha^{-1}(\overline{g_\alpha(W)})
\end{aligned}
$$

and similarly $\operatorname{projlim}(\mathcal{S}_2) = \bigcap_{\alpha\in J} g_\alpha^{-1}(g_\alpha(W))$.

For any $\alpha \in J$, the continuity of g_α provides that $g_\alpha(\overline{W}) \subseteq \overline{g_\alpha(W)}$ and so $\overline{W} \subseteq g_\alpha^{-1}(\overline{g_\alpha(W)}))$. Hence,

$$
\overline{W} \subseteq \bigcap_{\alpha\in J} g_\alpha^{-1}(\overline{g_\alpha(W)})) = \operatorname{projlim}(\mathcal{S}_1).
$$

Conversely, suppose that $x \in \operatorname{projlim}(\mathcal{S}_1)$. Then $x \in g_\alpha^{-1}(\overline{g_\alpha(W)}))$ for all $\alpha \in J$, that means $g_\alpha(x) \in \overline{g_\alpha(W)}$ for all $\alpha \in J$. Hence, for each $\alpha \in J$, we have that for any neighbourhood V_α of the origin in (A_α, τ_α), the following holds $(g_\alpha(x) + V_\alpha) \cap g_\alpha(W) \neq \emptyset$ and so $(x + g_\alpha^{-1}(V_\alpha)) \cap W \neq \emptyset$. This gives by Lemma 3.3.18 that for any U neighbourhood of the origin in $(A_{\mathcal{S}}, \tau_{proj})$ the sets $x+U$ and W have non-empty intersection, i.e. $x \in \overline{W}$. We have therefore showed that $\overline{W} = \operatorname{projlim}(\mathcal{S}_1)$.

If W is closed, then $W = \overline{W}$. However, we have

$$
W \subseteq \bigcap_{\alpha\in J} g_\alpha^{-1}(g_\alpha(W)) \subseteq \bigcap_{\alpha\in J} g_\alpha^{-1}(\overline{g_\alpha(W)}) = \operatorname{projlim}(\mathcal{S}_1) = \overline{W} = W
$$

i.e. $W = \operatorname{projlim}(\mathcal{S}_2) = \operatorname{projlim}(\mathcal{S}_1)$. □

Suppose now that (E, τ) is a Hausdorff lmc algebra. Then, by Theorem 2.1.11 there exists a basis $\mathcal{M} := \{U_\alpha\}_{\alpha\in I}$ of neighbourhoods of the origin in (E, τ) consisting of m-barrels. For each $\alpha \in I$, let p_α be the Minkowski functional of U_α. Then we have showed in Section 2.2 that $\{p_\alpha\}_{\alpha\in I}$ is a family of submultiplicative seminorms on E generating τ. For each $\alpha \in I$, we define $N_\alpha := \ker(p_\alpha)$ which is a closed ideal in (E, τ). Then we can take the quotient $E_\alpha := E/N_\alpha$ and endow it with the quotient norm $q_\alpha(\rho_\alpha(x)) := \inf_{y\in N_\alpha} p_\alpha(x - y)$ where $\rho_\alpha : E \to E_\alpha$ denotes the corresponding quotient map. With a similar proof to the one of Proposition 1.4.9 we can prove that is (E_α, q_α) is a normed algebra. Taking the completion $(\hat{E}_\alpha, \hat{q}_\alpha)$ of each (E_α, q_α), we get a family of Banach algebras. If we denote by $i_\alpha : E_\alpha \to \hat{E}_\alpha$ the canonical injection (which is an injective continuous and open homomorphism), then $\overline{\rho_\alpha} := i_\alpha \circ \rho_\alpha$ is a continuous open homomorphism. For convenience, from now on we will just denote (E_α, q_α) by E_α and $(\hat{E}_\alpha, \hat{q}_\alpha)$ by $\hat{E}_\alpha$.

We define a partial order on I by setting:

$$\alpha \leq \beta \Leftrightarrow U_\beta \subseteq U_\alpha \Leftrightarrow p_\alpha(x) \leq p_\beta(x), \forall x \in X.$$

Then $(I, \leq)$ is directed because $\mathcal{M}$ is a basis and so for any $\alpha, \beta \in I$ we have $U_\alpha \cap U_\beta \in \mathcal{M}$, i.e. there exists $\gamma \in I$ such that $U_\gamma \subseteq U_\alpha \cap U_\beta$ and so $U_\gamma \subseteq U_\alpha$ and $U_\gamma \subseteq U_\beta$, i.e. $\alpha \leq \gamma$ and $\beta \leq \gamma$. Also, for any $\alpha \leq \beta$ in I we have $N_\beta \subseteq N_\alpha$ and hence

$$\begin{array}{rlcl} f_{\alpha\beta}: & E_\beta & \to & E_\alpha \\ & x + N_\beta & \mapsto & x + N_\alpha \end{array}$$

is a well-defined surjective homomorphism and the following holds

$$\rho_\alpha = f_{\alpha\beta} \circ \rho_\beta, \ \forall\, \alpha \leq \beta \,\text{in}\, I. \tag{3.13}$$

Then all $f_{\alpha\beta}$'s are continuous homomorphisms and for any $\alpha \leq \beta \leq \gamma$ in I and any $x \in E_\beta$, we have

$$f_{\alpha\beta}(f_{\beta\gamma}(x + N_\gamma)) = f_{\alpha\beta}(x + N_\beta) = x + N_\alpha = f_{\alpha\gamma}(x + N_\gamma),$$

i.e. $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$.

Hence, $\{(E_\alpha, q_\alpha), f_{\alpha\beta}, I\}$ is a projective system of normed algebras. Moreover, for any $\alpha \leq \beta$ in I there exists $\overline{f_{\alpha\beta}} : \hat{E}_\beta \to \hat{E}_\alpha$ continuous and linear such that $\overline{f_{\alpha\beta}} \circ i_\beta = i_\alpha \circ f_{\alpha\beta}$ where i_α (resp. i_β) denotes the embedding of E_α (resp. E_β) in $\hat{E}_\alpha$ (resp. $\hat{E}_\beta$). Then it is easy to check that $\{(\hat{E}_\alpha, \hat{q}_\alpha), \overline{f_{\alpha\beta}}, I\}$ is a projective system of Banach algebras.

We are ready now for the Arens-Michael decomposition theorem.

Theorem 3.3.20. *Let (E, τ) be a Hausdorff lmc algebra and $\mathcal{M} := \{U_\alpha\}_{\alpha \in I}$ a basis of neighbourhoods of the origin in (E, τ) consisting of m-barrels. Consider the projective system $\{E_\alpha, f_{\alpha\beta}, I\}$ of normed algebras and the projective system $\{\hat{E}_\alpha, \overline{f_{\alpha\beta}}, I\}$ of Banach algebras introduced above. Then there exist the following topological monomorphisms*

$$E \hookrightarrow \text{projlim}\{(E_\alpha, q_\alpha), f_{\alpha\beta}, I\} \hookrightarrow \text{projlim}\{(\hat{E}_\alpha, q_\alpha), \overline{f_{\alpha\beta}}, I\} \cong \hat{E}. \tag{3.14}$$

If in addition (E, τ) is complete, then the maps in (3.14) are all topological isomorphisms. In this case, the expression $E = \text{projlim}\{(\hat{E}_\alpha, q_\alpha), \overline{f_{\alpha\beta}}, I\}$ is called the Arens-Michael decomposition *of E w.r.t. $\mathcal{M}$.*

Proof.
For convenience, let us denote by $\mathcal{P}$ and $\hat{\mathcal{P}}$ the projective systems $\{E_\alpha, f_{\alpha\beta}, I\}$ and $\{\hat{E}_\alpha, \overline{f_{\alpha\beta}}, I\}$, respectively.

For any $x \in E$, let us define $\phi(x) := (\rho_\alpha(x))_{\alpha\in I}$. Then $\phi(E) \subseteq \text{projlim}(\mathcal{P})$. Indeed, for any $x \in E$ and $\alpha \leq \beta$ in I we have

$$\Big(\phi(x)\Big)_\alpha = \rho_\alpha(x) \overset{(3.13)}{=} f_{\alpha\beta}(\rho_\beta(x)) = f_{\alpha\beta}\left(\Big(\phi(x)\Big)_\beta\right).$$

Then the following hold:
• ϕ is a homomorphism, as each ρ_α is a homomorphism and projlim($\mathcal{P}$) is equipped with coordinatewise operations. Let us just show that ϕ is multiplicative: for all $x, y \in E$,

$$\phi(xy) = (\rho_\alpha(xy))_{\alpha\in I} = (\rho_\alpha(x)\rho_\alpha(y))_{\alpha\in I} = (\rho_\alpha(x))_{\alpha\in I}\,(\rho_\alpha(y))_{\alpha\in I} = \phi(x)\phi(y).$$

• ϕ is injective, because

$$\phi(x) = 0 \Rightarrow \rho_\alpha(x) = 0,\ \forall\alpha\in I \Rightarrow x \in N_\alpha,\ \forall\alpha \in I \Rightarrow p_\alpha(x) = 0,\ \forall\alpha\in I \Rightarrow x = 0,$$

where in the last implication we used that E is Hausdorff and so $\{p_\alpha\}_{\alpha\in I}$ is a separating family of seminorms.
• ϕ is continuous, because Lemma 3.3.18 guarantees that for any neighbourhood U of the origin in projlim($\mathcal{P}$), there exist $\alpha \in I$ and a neighbourhood V_α of the origin in E_α such that $f_\alpha^{-1}(V_\alpha) \subseteq U$. Then $\rho_\alpha^1(V_\alpha) = \phi^{-1}(f_\alpha^{-1}(V_\alpha)) \subseteq \phi^{-1}(U)$ and so, by the continuity of ρ_α we have that $\phi^{-1}(U)$ is a neighborhood of the origin in E.
• ϕ is an open map. Indeed, recalling that $\mathcal{M} := \{U_\alpha\}_{\alpha\in I}$ is a basis of neighbourhoods of the origin in (E,τ), we can show that for any $\alpha \in I$ the set $V := \left(\rho_\alpha\left(\frac{1}{2}U_\alpha\right) \times \prod_{\beta\in I\setminus\{\alpha\}} E_\beta\right) \cap \phi(E)$ is a neighbourhood of the origin in projlim($\mathcal{P}$) such that $V \subseteq \phi(U_\alpha)$. Fix $\alpha \in I$. Then the openness of ρ_α implies that $\rho_\alpha(\frac{1}{2}U_\alpha)$ is a neighbourhood of the origin in E_α and so $\rho_\alpha\left(\frac{1}{2}U_\alpha\right) \times \prod_{\beta\in I\setminus\{\alpha\}} E_\beta$ is a neighbourhood of the origin in $\prod_{\gamma\in I} E_\gamma$ endowed with the product topology. Hence, V is a neighbourhood of the origin in projlim($\mathcal{P}$).

Moreover, for any $x := (x_\gamma)_{\gamma\in I} \in V$ we have that:
a) $x \in \phi(E)$, i.e. there exists $y \in E$ such that $\phi(y) = x$
b) $x_\alpha \in \rho_\alpha\left(\frac{1}{2}U_\alpha\right)$
c) $x_\beta \in E_\beta$ for all $\beta \neq \alpha$ in I.
Then

$$\rho_\alpha(y) = f_\alpha(\phi(y)) \overset{(a)}{=} f_\alpha(x) = x_\alpha \overset{(b)}{\in} \rho_\alpha\left(\frac{1}{2}U_\alpha\right),$$

which implies that there exists $z \in \frac{1}{2}U_\alpha$ such that $\rho_\alpha(y) = \rho_\alpha(z)$. Therefore, $y = z + w$ for some $w \in N_\alpha$, which gives in turn

$$|p_\alpha(y) - p_\alpha(z)| \leq p_\alpha(y - z) \leq p_\alpha(w) = 0,$$

and so $p_\alpha(y) = p_\alpha(z) \leq \frac{1}{2} < 1$, i.e. $y \in U_\alpha$. Hence, $x \overset{(a)}{=} \phi(y) \in \phi(U_\alpha)$, that gives $V \subseteq \phi(U_\alpha)$.

We have then just showed that $\phi : E \hookrightarrow \text{projlim}(\mathcal{P})$ is a topological monomorphism.

Now, by using Theorem 3.3.17, we get that for any $\alpha \leq \beta$ in I the diagram

$$\begin{CD} E_\beta @>{i_\beta}>> \hat{E}_\beta \\ @V{f_{\alpha\beta}}VV @VV{\overline{f_{\alpha\beta}}}V \\ E_\alpha @>{i_\alpha}>> \hat{E}_\alpha \end{CD}$$

commutes, where i_α and i_β are topological monomorphisms such that $\overline{i_\alpha(E_\alpha)} = \hat{E}_\alpha$ and $\overline{i_\beta(E_\beta)} = \hat{E}_\beta$. Then [4, E.III.53, Corollary 1] ensures that there exists a unique topological monomorphism $j : \text{projlim}(\mathcal{P}) \hookrightarrow \text{projlim}(\hat{\mathcal{P}})$ such that the following diagram commutes

$$\begin{CD} \text{projlim}(\mathcal{P}) @>{j}>> \text{projlim}(\hat{\mathcal{P}}) \\ @V{f_\alpha}VV @VV{\overline{f_\alpha}}V \\ E_\alpha @>{i_\alpha}>> \hat{E}_\alpha \end{CD} \tag{3.15}$$

Setting $\psi = j \circ \phi$ we get a topological monomorphism from E to $\text{projlim}(\hat{\mathcal{P}})$ and so $\psi(E)$ is a linear subspace of $\text{projlim}(\hat{\mathcal{P}})$. Therefore, Lemma 3.3.19 provides that $\overline{\psi(E)} = \text{projlim}(\mathcal{Q})$, where $\mathcal{Q} := \left\{ \overline{\overline{f_\alpha}(\psi(E))}, \overline{f_{\alpha\beta}} \restriction_{\overline{\overline{f_\beta}(\psi(E))}}, I \right\}$. By the commutativity of the diagram (3.15), we know that

$$\overline{f}_\alpha(\psi(E)) = \overline{f}_\alpha(j(\phi(E))) = i_\alpha(f_\alpha(\phi(E))) = i_\alpha(\rho_\alpha(E)) = i_\alpha(E_\alpha).$$

Hence, $\overline{\overline{f_\alpha}(\psi(E))} = \overline{i_\alpha(E_\alpha)} = \hat{E}_\alpha$ and so

$$\overline{\psi(E)} = \text{projlim}(\mathcal{Q}) = \text{projlim}(\hat{\mathcal{P}}).$$

This together with the fact that projlim($\hat{\mathcal{P}}$) is complete (see Remark 3.3.15-c)) implies that $\hat{E}$ is topologically isomorphic to projlim($\hat{\mathcal{P}}$) by Theorem 3.3.17-I). Therefore, we have proved that

$$E \overset{\phi}{\hookrightarrow} \text{projlim}(\mathcal{P}) \overset{j}{\hookrightarrow} \text{projlim}(\hat{\mathcal{P}}) \cong \hat{E}.$$

If in addition E is complete, then $E = \hat{E}$ and so ϕ and j must be also isomorphisms. □

Using Remark 3.3.15, we can easily derive from Theorem 3.3.20 the following

Corollary 3.3.21.

a) *Every Hausdorff lmc algebra can be topologically embedded in a cartesian product of Banach algebras.*

b) *Every Fréchet lmc algebra is topologically isomorphic to the projective limit of a sequence of Banach algebras.*

Theorem 3.3.22. *Let (E, τ) be a Hausdorff complete lmc algebra and $\mathcal{M} := \{U_\alpha\}_{\alpha\in I}$ a basis of neighbourhoods of the origin in (E, τ) consisting of m-barrels. Then:*

a) *E is unital if and only if each component of its Arens-Michael decomposition w.r.t. $\mathcal{M}$ is a unital Banach algebra.*

b) *$x \in E$ is invertible if and only if its image into each component of the its Arens-Michael decomposition of E w.r.t. $\mathcal{M}$ is invertible.*

Proof.
Let $E = \text{projlim}\{\hat{E}_\alpha, \overline{f_{\alpha\beta}}, I\}$ be the *Arens-Michael decomposition* of E w.r.t. $\mathcal{M}$ (see Theorem 3.3.20).

a) Suppose that there exists $u \in E$ s.t. for all $y \in E$ we have $u\cdot y = y = y\cdot u$. For any $\alpha \in I$, set $u_\alpha := \overline{\rho}_\alpha(u) \in \hat{E}_\alpha$. By the surjectivity of $\overline{\rho}_\alpha$, we know that for any $x_\alpha \in \hat{E}_\alpha$ there exists $x \in E$ such that $\overline{\rho}_\alpha(x) = x_\alpha$ and so we get that:

$$x_\alpha \cdot u_\alpha = \overline{\rho}_\alpha(x)\overline{\rho}_\alpha(u) = \overline{\rho}_\alpha(x \cdot u) = \overline{\rho}_\alpha(x) = x_\alpha$$

and similarly we obtain $u_\alpha x_\alpha = x_\alpha$, i.e. each $\hat{E}_\alpha$ is unital.

Conversely, suppose that for any $\alpha \in I$ there exists $u_\alpha \in \hat{E}_\alpha$ s.t. $y \cdot u_\alpha = y = u_\alpha \cdot y$ for all $y \in \hat{E}_\alpha$. Then $u := (u_\alpha)_{\alpha\in I}$ belongs to $\text{projlim}\{\hat{E}_\alpha, \overline{f_{\alpha\beta}}, I\}$ since for all $\alpha \leq \beta$ in I and for all $x_\alpha \in \hat{E}_\alpha$ we get:

$$\begin{aligned} x_\alpha \cdot \overline{f_{\alpha\beta}}(u_\beta) &= \overline{\rho}_\alpha(x) \cdot \overline{f_{\alpha\beta}}(u_\beta) = \overline{f_{\alpha\beta}}(\overline{\rho}_\beta(x)) \cdot \overline{f_{\alpha\beta}}(u_\beta) \\ &= \overline{f_{\alpha\beta}}(\overline{\rho}_\beta(x) \cdot u_\beta) = \overline{f_{\alpha\beta}}(\overline{\rho}_\beta(x)) = \overline{\rho}_\alpha(x) = x_\alpha, \end{aligned}$$

i.e. $\overline{f_{\alpha\beta}}(u_\beta) = u_\alpha$. As the multiplication in projlim$\{\hat{E}_\alpha, \overline{f_{\alpha\beta}}, I\}$ is defined coordinatewise, it is then clear that $u := (u_\alpha)_{\alpha\in I}$ is the identity element of the multiplication in projlim$\{\hat{E}_\alpha, \overline{f_{\alpha\beta}}, I\}$, which is therefore a unital algebra.

b) Suppose that u is the identity element of the multiplication in E and that $x \in E$ is invertible, i.e. there exists $y \in E$ s.t. $x \cdot y = u = y \cdot x$. For each $\alpha \in I$, we have already showed that $u_\alpha := \overline{\rho}_\alpha(u)$ is the identity element of the multiplication in $\hat{E}_\alpha$. Hence, we have

$$\overline{\rho}_\alpha(x) \cdot \overline{\rho}_\alpha(y) = \overline{\rho}_\alpha(x \cdot y) = \overline{\rho}_\alpha(u) = u_\alpha,$$

i.e. $\overline{\rho}_\alpha(x)$ is invertible in $\hat{E}_\alpha$.

Conversely, suppose that $x \in$ projlim$\{\hat{E}_\alpha, \overline{f_{\alpha\beta}}, I\}$ is s.t. for each $\alpha \in I$ the element $\overline{\rho}_\alpha(x)$ is invertible. Then for each $\alpha \in I$ there exists $y_\alpha \in \hat{E}_\alpha$ s.t. $\overline{\rho}_\alpha(x) \cdot y_\alpha = u_\alpha = y_\alpha \cdot \overline{\rho}_\alpha(x)$, where u_α is the identity element of the multiplication in $\hat{E}_\alpha$. Now as we have already showed that $u := (u_\alpha)_{\alpha\in I}$ is the identity element of the (coordinatewise) multiplication in projlim$\{\hat{E}_\alpha, \overline{f_{\alpha\beta}}, I\}$, it is enough to prove that $(y_\alpha)_{\alpha\in I} \in$ projlim$\{\hat{E}_\alpha, \overline{f_{\alpha\beta}}, I\}$. This is indeed true since for all $\alpha \leq \beta$ in I the following holds

$$\overline{\rho}_\alpha(x) \cdot \overline{f_{\alpha\beta}}(y_\beta) = \overline{f_{\alpha\beta}}(\overline{\rho}_\beta(x)) \cdot \overline{f_{\alpha\beta}}(y_\beta) = \overline{f_{\alpha\beta}}(\overline{\rho}_\beta(x) \cdot y_\beta) = \overline{f_{\alpha\beta}}(u_\beta) = y_\alpha,$$

and, hence, $\overline{f_{\alpha\beta}}(y_\beta) = y_\alpha$. □