Chapter 4

Symmetric tensor algebras

As usual, we consider only vector spaces over the field $\mathbb{K}$ of real numbers or of complex numbers. The aim of this section is to present a way to explicitly construct an lmc algebra starting from the symmetric tensor algebra of a lc TVS. For this purpose, we will preliminarily introduce the concept of tensor product of vector spaces and then endow it with one of the many topologies which can be defined when the starting space carries an lc structure.

4.1 Tensor product of vector spaces

Let us start with a notion which is central in the definition of tensor product.

Definition 4.1.1.
*Let E, F, M be three vector spaces over $\mathbb{K}$ and $\phi : E \times F \to M$ be a bilinear map. E and F are said to be $\phi-$*linearly disjoint *if:*

(LD) *For any $r, s \in \mathbb{N}$, $x_1, \dots, x_r$ linearly independent in E and $y_1, \dots, y_s$ linearly independent in F, the set $\{\phi(x_i, y_j) : i = 1, \dots, r,\ j = 1, \dots, s\}$ consists of linearly independent vectors in M.*

or equivalently if:

(LD)' *For any $r \in \mathbb{N}$, any $\{x_1, \dots, x_r\}$ finite subset of E and any $\{y_1, \dots, y_r\}$ finite subset of F s.t. $\sum_{i=1}^{r} \phi(x_i, y_j) = 0$, we have that both the following conditions hold:*

- *if $x_1, \dots, x_r$ are linearly independent in E, then $y_1 = \dots = y_r = 0$*
- *if $y_1, \dots, y_r$ are linearly independent in F, then $x_1 = \dots = x_r = 0$.*

Definition 4.1.2. *A* tensor product *of two vector spaces E and F over $\mathbb{K}$ is a pair (M, ϕ) consisting of a vector space M over $\mathbb{K}$ and of a bilinear map $\phi : E \times F \to M$ (canonical map) s.t. the following conditions are satisfied:*

(TP1) *The image of $E \times F$ spans the whole space M.*

(TP2) *E and F are $\phi-$linearly disjoint.*

The following theorem guarantees that the tensor product of any two vector spaces always exists, satisfies the "universal property" and it is unique up to isomorphisms. For this reason, the tensor product of E and F is usually denoted by $E \otimes F$ and the canonical map by $(x, y) \mapsto x \otimes y$.

Theorem 4.1.3. *Let E, F be two vector spaces over $\mathbb{K}$.*
(a) There exists a tensor product of E and F.
(b) Let (M, ϕ) be a tensor product of E and F. Let G be any vector space over $\mathbb{K}$, and b any bilinear mapping of $E \times F$ into G. There exists a unique linear map $\tilde{b} : M \to G$ such that the diagram

$$\begin{tikzcd} E \times F \arrow[r, "b"] \arrow[d, "\phi"] & G \\ M \arrow[ur, "\tilde{b}"'] & \end{tikzcd}$$

is commutative.
(c) If (M_1, ϕ_1) and (M_2, ϕ_2) are two tensor products of E and F, then there is a bijective linear map u such that the diagram

$$\begin{tikzcd} E \times F \arrow[r, "\phi_2"] \arrow[d, "\phi_1"] & M_2 \\ M_1 \arrow[ur, "u"'] & \end{tikzcd}$$

is commutative.

Proof. (see [16, Theorem 4.1.4]) □

Examples 4.1.4.

1. *Let $n, m \in \mathbb{N}$, $E = \mathbb{K}^n$ and $F = \mathbb{K}^m$. Then $E \otimes F = \mathbb{K}^{nm}$ is a tensor product of E and F whose canonical bilinear map ϕ is given by:*

$$\begin{array}{rlcl} \phi : & E \times F & \to & \mathbb{K}^{nm} \\ & \Big((x_i)_{i=1}^n, (y_j)_{j=1}^m\Big) & \mapsto & (x_i y_j)_{1 \le i \le n, 1 \le j \le m}. \end{array}$$

2. *Let X and Y be two sets. For any functions $f : X \to \mathbb{K}$ and $g : Y \to \mathbb{K}$, we define:*

$$\begin{array}{rlcl} f \otimes g : & X \times Y & \to & \mathbb{K} \\ & (x, y) & \mapsto & f(x)g(y). \end{array}$$

Let E (resp. F) be the linear space of all functions from X (resp. Y) to $\mathbb{K}$ endowed with the pointwise addition and multiplication by scalars. We *denote by M the linear subspace of the space of all functions from $X \times Y$ to $\mathbb{K}$ spanned by the elements of the form $f \otimes g$ for all $f \in E$ and $g \in F$. Then M is actually a tensor product of E and F, i.e. $M = E \otimes F$.*

Similarly to how we defined the tensor product of two vector spaces we can define the tensor product of an arbitrary number of vector spaces.

Definition 4.1.5. *Let $n \in \mathbb{N}$ with $n \geq 2$ and $E_1, \dots, E_n$ vector spaces over $\mathbb{K}$. A tensor product of $E_1, \dots, E_n$ is a pair (M, ϕ) consisting of a vector space M over $\mathbb{K}$ and of a multilinear map $\phi : E_1 \times \cdots \times E_n \to M$ (canonical map) s.t. the following conditions are satisfied:*

(TP1) *The image of $E_1 \times \cdots \times E_n$ spans the whole space M.*

(TP2) *$E_1, \dots, E_n$ are $\phi-$linearly disjoint, i.e. for any $r_1, \dots, r_n \in \mathbb{N}$ and for any $x_1^{(i)}, \dots, x_{r_i}^{(i)}$ linearly independent in E_i ($i = 1, \dots, n$), the set*

$$\left\{\phi\left(x_{j_1}^{(1)}, \dots, x_{j_n}^{(n)}\right) : j_1 = 1, \dots, r_1, \dots, j_n = 1, \dots, r_n\right\}$$

consists of linearly independent vectors in M.

Recall that a map is multilinear if it is linear in each of its variables. As for the case $n = 2$ it is possible to show that:

(a) There always exists a tensor product of $E_1, \dots, E_n$.

(b) The universal property holds for $E_1 \otimes \cdots \otimes E_n$.

(c) $E_1 \otimes \cdots \otimes E_n$ is unique up to isomorphisms.

4.2 The $\pi-$topology on the tensor product of lc TVS

Given two locally convex TVS E and F, there are various ways to construct a topology on the tensor product $E \otimes F$ which makes the vector space $E \otimes F$ in a TVS. Indeed, starting from the topologies on E and F, one can define a topology on $E \otimes F$ either relying directly on the seminorms on E and F, or using an embedding of $E \otimes F$ in some space related to E and F over which a natural topology already exists. The first method leads to the so-called $\pi-$topology. The second method may lead instead to a variety of topologies, which we are not going to investigate in this course.

Definition 4.2.1 ($\pi-$topology)**.**
Given two locally convex TVS E and F, we define the $\pi-$topology *(or* projective topology*) on $E \otimes F$ to be the finest locally convex topology on this vector space for which the canonical mapping $E \times F \to E \otimes F$ is continuous. The space $E \otimes F$ equipped with the $\pi-$topology will be denoted by $E \otimes_\pi F$.*

A basis of neighbourhoods of the origin in $E \otimes_\pi F$ is given by the family:

$$\mathcal{B} := \{\mathrm{conv}_b(U_\alpha \otimes V_\beta) : U_\alpha \in \mathcal{B}_E,\, V_\beta \in \mathcal{B}_F\},$$

where $\mathcal{B}_E$ (resp. $\mathcal{B}_F$) is a basis of neighbourhoods of the origin in E (resp. in F), $U_\alpha \otimes V_\beta := \{x \otimes y \in E \otimes F : x \in U_\alpha, y \in V_\beta\}$. In fact, on the one hand, the $\pi-$topology is by definition locally convex and so it has a basis $\mathcal{B}$ of convex balanced neighbourhoods of the origin in $E \otimes F$. Then, as the canonical mapping ϕ is continuous w.r.t. the $\pi-$topology, we have that for any $C \in \mathcal{B}$ there exist $U_\alpha \in \mathcal{B}_E$ and $V_\beta \in \mathcal{B}_F$ s.t. $U_\alpha \times V_\beta \subseteq \phi^{-1}(C)$. Hence, $U_\alpha \otimes V_\beta = \phi(U_\alpha \times V_\beta) \subseteq C$ and so $\mathrm{conv}_b(U_\alpha \otimes V_\beta) \subseteq \mathrm{conv}_b(C) = C$ which yields that the topology generated by $\mathcal{B}_\pi$ is finer than the $\pi-$topology. On the other hand, the canonical map ϕ is continuous w.r.t. the topology generated by $\mathcal{B}_\pi$, because for any $U_\alpha \in \mathcal{B}_E$ and $V_\beta \in \mathcal{B}_F$ we have that $\phi^{-1}(\mathrm{conv}_b(U_\alpha \otimes V_\beta)) \supseteq \phi^{-1}(U_\alpha \otimes V_\beta) = U_\alpha \times V_\beta$ which is a neighbourhood of the origin in $E \times F$. Hence, the topology generated by $\mathcal{B}_\pi$ is coarser than the $\pi-$topology.

The $\pi-$topology on $E \otimes F$ can be described by means of the seminorms defining the locally convex topologies on E and F.

Theorem 4.2.2. *Let E and F be two locally convex TVS and let $\mathcal{P}$ (resp. $\mathcal{Q}$) be a family of seminorms generating the topology on E (resp. on F). The $\pi-$topology on $E \otimes F$ is generated by the family of seminorms*

$$\{p \otimes q : p \in \mathcal{P}, q \in \mathcal{Q}\},$$

where for any $p \in \mathcal{P}, q \in \mathcal{Q}, \theta \in E \otimes F$ we define:

$$(p \otimes q)(\theta) := \inf\left\{\sum_{k=1}^{r} p(x_k)q(y_k) : \theta = \sum_{k=1}^{r} x_k \otimes y_k, \, , x_k \in E, y_k \in F, r \in \mathbb{N}\right\}.$$

Proof. (see [16, Proposition 4.3.10 and Theorem 4.3.11]) □

The seminorm $p \otimes q$ on $E \otimes F$ defined in the previous proposition is called *tensor product of the seminorms p and q* (or *projective cross seminorm*)

Proposition 4.2.3. *Let E and F be two locally convex TVS. $E \otimes_\pi F$ is Hausdorff if and only if E and F are both Hausdorff.*

In analogy with the algebraic case (see Theorem 4.1.3-b), we also have a universal property for the space $E \otimes_\pi F$.

Proposition 4.2.4.
Let E, F be locally convex spaces. The $\pi-$topology on $E \otimes_\pi F$ is the unique locally convex topology on $E \otimes F$ such that the following property holds:

(UP) *For every locally convex space G, the algebraic isomorphism between the space of bilinear mappings from $E \times F$ into G and the space of all linear mappings from $E \otimes F$ into G (given by Theorem 4.1.3-b) induces an algebraic isomorphism between $B(E, F; G)$ and $L(E \otimes F; G)$, where $B(E, F; G)$ denotes the space of all continuous bilinear mappings from $E \times F$ into G and $L(E \otimes F; G)$ the space of all continuous linear mappings from $E \otimes F$ into G.*

Proof. Let τ be a locally convex topology on $E \otimes F$ such that the property (UP) holds. Then (UP) holds in particular for $G = (E \otimes F, \tau)$. Therefore, by Theorem 4.1.3-b) the identity $\mathrm{id} : E \otimes F \to E \otimes F$ is the unique linear map such that the diagram

$$\begin{tikzcd} E \times F \arrow[r, "\phi"] \arrow[d, "\phi"] & E \otimes_\tau F \\ E \otimes_\tau F \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

commutes. Hence, we get that $\phi : E \times F \to E \otimes_\tau F$ has to be continuous.

This implies that $\tau \subseteq \pi$ by definition of $\pi-$topology. On the other hand, (UP) also holds for $G = (E \otimes F, \pi)$.

$$\begin{tikzcd} E \times F \arrow[r, "\phi"] \arrow[d, "\phi"] & E \otimes_\pi F \\ E \otimes_\tau F \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

Hence, since by definition of $\pi-$topology $\phi : E \times F \to E \otimes_\pi F$ is continuous, the $id : E \otimes_\tau F \to E \otimes_\pi F$ has to be also continuous. This means that $\pi \subseteq \tau$, which completes the proof. □

4.3 Tensor algebra and symmetric tensor algebra of a vs

Let V be a vector space over $\mathbb{K}$. For any $k \in \mathbb{N}$, we define the $k-th$ *tensor power* of V as

$$V^{\otimes k} := \underbrace{V \otimes \cdots \otimes V}_{\text{k-times}}$$

and we take by convention $V^{\otimes 0} := \mathbb{K}$. Then it is possible to show that there exists the following algebraic isomorphism:

$$\forall\, n, m \in \mathbb{N}, \quad V^{\otimes n} \otimes V^{\otimes m} \cong V^{\otimes (n+m)} \tag{4.1}$$

We can pack together all tensor powers of V in a unique vector space:

$$T(V) := \bigoplus_{k=0}^{\infty} V^{\otimes k}.$$

We define a multiplication over $T(V)$ which makes it into a unital $\mathbb{K}-$algebra.

First of all, let us observe that for any $k \in \mathbb{N}_0$ there is a natural embedding $i_k : V^{\otimes k} \to T(V)$. For sake of notational convenience, in the following we will identify each $g \in V^{\otimes k}$ with $i_k(g)$. Then every element $f \in T(V)$ can be expressed as $f = \sum_{k=0}^{N} f_k$ for some $N \in \mathbb{N}_0$ and $f_k \in V^{\otimes k}$ for $k = 0, \dots, N$. Using the isomorphism given by (4.1), for any $j, k \in \mathbb{N}$ we can define the following bilinear operation:

$$\begin{array}{cccc} \cdot : & V^{\otimes k} \times V^{\otimes j} & \to & V^{\otimes (k+j)} \\ & ((v_1 \otimes \cdots \otimes v_k), (w_1 \otimes \cdots \otimes w_j)) & \mapsto & v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_j. \end{array} \tag{4.2}$$

Hence, we get a multiplication $\cdot : T(V) \times T(V) \to T(V)$ just by defining for all $f, g \in T(V)$, say $f = \sum_{k=0}^{N} f_k$ and $g = \sum_{j=0}^{M} g_j$ for some $N, M \in \mathbb{N}_0$, $f_k \in V^{\otimes k}$, $g_j \in V^{\otimes j}$,

$$f \cdot g := \sum_{k=0}^{N} \sum_{j=0}^{M} f_k \cdot g_j,$$

where $f_k \cdot g_j$ is the one defined in (4.2). Then we easily see that:

a) $\cdot$ is bilinear on $T(V) \times T(V)$ as it is bilinear on each $V^{\otimes k} \times V^{\otimes j}$ for all $j, k \in \mathbb{N}_0$.

b) $\cdot$ is associative, i.e. $\forall f, g, h \in T(V)$, $(f \cdot g) \cdot h = f \cdot (g \cdot h)$. Indeed, if $f = \sum_{k=0}^{N} f_k$, $g = \sum_{j=0}^{M} g_j$, $h = \sum_{l=0}^{S} h_l$ with $N, M, S \in \mathbb{N}_0$, $f_k \in V^{\otimes k}$, $g_j \in V^{\otimes j}$, $h_l \in V^{\otimes l}$, then

$$(f \cdot g) \cdot h = \sum_{k=0}^{N} \sum_{j=0}^{M} \sum_{l=0}^{S} (f_k \cdot g_j) \cdot h_l = \sum_{k=0}^{N} \sum_{j=0}^{M} \sum_{l=0}^{S} f_k \cdot (g_j \cdot h_l) = f \cdot (g \cdot h),$$

where we have just used that $V^{\otimes (k+j)} \otimes V^{\otimes l} \cong V^{\otimes (k+j+l)} \cong V^{\otimes k} \otimes V^{\otimes (j+l)}$ by (4.1).

c) $1 \in \mathbb{K}$ is the identity for the multiplication $\cdot$, since $\mathbb{K} = V^{\otimes 0}$ and for all $f = \sum_{k=0}^{N} f_k \in T(V)$ we have $1 \cdot f = \sum_{k=0}^{n} (1 \cdot f_k) = \sum_{k=0}^{N} f_k = f$.

Hence, $(T(V), \cdot)$ is a unital $\mathbb{K}-$algebra, which is usually called the *tensor algebra* of V.

Remark 4.3.1. *If $\{x_i\}_{i\in\Omega}$ is a basis of the vector space V, then each element of $V^{\otimes k}$ can be identified with a polynomial of degree k in the non-commuting variables $\{x_i\}_{i\in\Omega}$ and with coefficients in $\mathbb{K}$. Hence, $T(V)$ is identified with the non-commutative polynomial ring $\mathbb{K}\langle x_i, i \in \Omega\rangle$.*

Proposition 4.3.2. *Let V be a vector space over $\mathbb{K}$. For any unital $\mathbb{K}-$algebra $(A, *)$ and any linear map $f : V \to A$, there exists a unique $\mathbb{K}-$algebra homomorphism $\bar{f} : T(V) \to A$ such that the following diagram commutes*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "i_1"] & A \\ T(V) \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

where i_1 is the natural embedding of $V = V^{\otimes 1}$ into $T(V)$.

Proof.
For any $k \in \mathbb{N}$, we define

$$\begin{array}{rccc} f_k : & \underbrace{V \times \cdots \times V}_{k\,\text{times}} & \to & A \\ & (v_1, \ldots, v_k) & \mapsto & f(v_1) * \cdots * f(v_k) \end{array}$$

which is multilinear by the linearity of f. For $k = 0$ we define

$$\begin{array}{rccc} f_0 : & \mathbb{K} & \to & A \\ & r & \mapsto & r1_A. \end{array}$$

By the universal property of $V^{\otimes k}$, we have that there exists a unique linear map $\sigma_k : V^{\otimes k} \to A$ s.t. $\sigma_k(v_1 \otimes \cdots \otimes v_k) = f_k(v_1, \ldots, v_k) = f(v_1) * \cdots * f(v_k)$ and for $k = 0$ we have $\sigma_0(r) = f_0(r) = r1_A$, $\forall r \in \mathbb{K}$. Then, by the universal property of the direct sum, we get that there exists a unique linear map $\bar{f} : T(V) \to A$ such that $\bar{f}(i_k(v_1 \otimes \cdots \otimes v_k)) = \sigma_k(v_1 \otimes \cdots \otimes v_k)$

$$\begin{tikzcd} V \times \cdots \times V \arrow[r, "f_k"] \arrow[d] & A \\ V^{\otimes k} \arrow[ur, "\sigma_k"] \arrow[d, "i_k"] & \\ T(V) \arrow[uur, bend right, "\bar{f}"'] & \end{tikzcd}$$

In particular, for $k = 1$ we obtain that $\bar{f}(i_1(v)) = \sigma_1(v) = f(v)$.

It remains to show that $\bar{f}$ is a $\mathbb{K}-$algebra homomorphism from $T(V)$ to A. By construction of $\bar{f}$, we clearly have that $\bar{f}$ is linear and

$$\bar{f}(1_{T(V)}) = \bar{f}(i_0(1)) = \sigma_0(1) = f_0(1) = 1_A.$$

Let us prove now that for any $x, y \in T(V)$ we get $\bar{f}(x \cdot y) = \bar{f}(x) * \bar{f}(y)$. As $\bar{f}$ is linear, it is enough to show that for any $n, m \in \mathbb{N}$, any $x_1, \dots, x_n \in V$ and any $y_1, \dots, y_m \in V$, we get

$$\bar{f}((x_1 \otimes \cdots \otimes x_n) \cdot (y_1 \otimes \cdots \otimes y_m)) = \bar{f}(x_1 \otimes \cdots \otimes x_n) * \bar{f}(y_1 \otimes \cdots \otimes y_m). \quad (4.3)$$

Indeed, by just applying the properties of $\bar{f}$, we obtain that:

$$\bar{f}(x_1 \otimes \cdots \otimes x_n) = \sigma_n(x_1 \otimes \cdots \otimes x_n) = f(x_1) * \cdots * f(x_n)$$

and

$$\bar{f}(y_1 \otimes \cdots \otimes y_m) = \sigma_m(y_1 \otimes \cdots \otimes y_m) = f(y_1) * \cdots * f(y_m).$$

These together with the definition of multiplication in $T(V)$ give that:

$$\begin{aligned} \bar{f}(x_1 \otimes \cdots \otimes x_n) * \bar{f}(y_1 \otimes \cdots \otimes y_m) &= f(x_1) * \cdots * f(x_n) * f(y_1) * \cdots * f(y_m) \\ &= \bar{f}((x_1 \otimes \cdots \otimes x_n) \cdot (y_1 \otimes \cdots \otimes y_m)). \end{aligned}$$

$\square$

Consider now the ideal I in $(T(V), \cdot)$ generated by the elements $v \otimes w - w \otimes v$, for all $v, w \in V$. The tensor algebra $T(V)$ factored by this ideal I is denoted by $S(V)$ and called the *symmetric (tensor) algebra* of V. If we denote by π the quotient map from $T(V)$ to $S(V)$, then for any $k \in \mathbb{N}_0$ and any element $f = \sum_{i=1}^{n} f_{i1} \otimes \cdots \otimes f_{ik} \in V^{\otimes k}$ (here $n \in \mathbb{N}$, $f_{ij} \in V$ for $i = 1, \dots, n$, $j = 1, \dots, k$ and $n \geq 1$) we have that

$$\pi\left(\sum_{i=1}^{n} f_{i1} \otimes \cdots \otimes f_{ik}\right) = \sum_{i=1}^{n} f_{i1} \cdots f_{ik}.$$

We define the *$k-$th homogeneous component of* $S(V)$ to be the image of $V^{\otimes k}$ under π and we denoted it by $S(V)_k$. Note that $S(V)_0 = \mathbb{K}$ and $S(V)_1 = V$. Hence, we have

$$S(V) = \bigoplus_{k=0}^{\infty} S(V)_k$$

and so every element $f \in S(V)$ can expressed as $f = \sum_{k=0}^{N} f_k$ for some $N \in \mathbb{N}$, $f_k \in S(V)_k$ for $k = 0, \dots, N$.

Remark 4.3.3. *If $\{x_i\}_{i\in\Omega}$ is a basis of the vector space V, then each element of $S(V)_k$ can be identified with a polynomial of degree k in the commuting variables $\{x_i\}_{i\in\Omega}$ and with coefficients in $\mathbb{K}$. Hence, $S(V)$ is identified with the commutative polynomial ring $\mathbb{K}[x_i : i \in \Omega]$.*

The universal property of $S(V)$ easily follows from the universal property of $T(V)$.

Proposition 4.3.4. *Let V be a vector space over $\mathbb{K}$. For any unital commutative $\mathbb{K}-$algebra $(A, *)$ and any linear map $\psi : V \to A$, there exists a unique $\mathbb{K}-$algebra homomorphism $\bar{\psi} : S(V) \to A$ such that the following diagram commutes*

$$\begin{tikzcd} V \arrow[r, "\psi"] \arrow[d, hook] & A \\ S(V) \arrow[ur, "\bar{\psi}"'] & \end{tikzcd}$$

i.e. $\bar{\psi} \restriction_V = \psi$.

Corollary 4.3.5. *Let V be a vector space over $\mathbb{K}$. The algebraic dual V^* of V is algebraically isomorphic to $\mathrm{Hom}(S(V), \mathbb{K})$.*

Proof. For any $\alpha \in \mathrm{Hom}(S(V), \mathbb{K})$ we clearly have $\alpha \restriction_V \in V^*$. On the other hand, by Proposition 4.3.4, for any $\ell \in V^*$ there exists a unique $\bar{\ell} \in \mathrm{Hom}(S(V), \mathbb{K})$ such that $\bar{\ell} \restriction_V = \ell$. □

4.4 An lmc topology on the symmetric algebra of a lc TVS

Let V be a vector space over $\mathbb{K}$. In this section we are going to explain how a locally convex topology τ on V can be naturally extended to a locally convex topology $\overline{\tau}$ on the symmetric algebra $S(V)$ (see [14]). Let us start by considering the simplest possible case, i.e. when τ is generated by a single seminorm.

Suppose now that ρ is a seminorm on V. Starting from the seminorm ρ on V, we are going to construct a seminorm $\bar{\rho}$ on $S(V)$ in three steps:

1. For $k \in \mathbb{N}$, let us consider the projective tensor seminorm on $V^{\otimes k}$ see Theorem 4.2.2, i.e.

$$\begin{aligned} \rho^{\otimes k}(g) &:= (\underbrace{\rho \otimes \cdots \otimes \rho}_{k\text{times}})(g) \\ &= \inf\left\{ \sum_{i=1}^{N} \rho(g_{i1}) \cdots \rho_k(g_{ik}) : g = \sum_{i=1}^{N} g_{i1} \otimes \cdots \otimes g_{ik},\ g_{ij} \in V,\ N \in \mathbb{N} \right\}. \end{aligned}$$

2. Denote by $\pi_k : V^{\otimes k} \to S(V)_k$ the quotient map π restricted to $V^{\otimes k}$ and define $\overline{\rho}_k$ to be the quotient seminorm on $S(V)_k$ induced by $\rho^{\otimes k}$, i.e.

$$\begin{aligned}\overline{\rho}_k(f) &:= \inf\{\rho^{\otimes k}(g) : g \in V^{\otimes k},\ \pi_k(g) = f\} \\ &= \inf\left\{\sum_{i=1}^{N} \rho(f_{i1}) \cdots \rho(f_{ik}) : f = \sum_{i=1}^{N} f_{i1} \cdots f_{ik}, f_{ij} \in V, N \in \mathbb{N}\right\}.\end{aligned}$$

Define $\overline{\rho}_0$ to be the usual absolute value on $\mathbb{K}$.

3. For any $h \in S(V)$, say $h = h_0 + \cdots + h_\ell$, $f_k \in S(V)_k$, $k = 0, \ldots, \ell$, define

$$\overline{\rho}(f) := \sum_{k=0}^{\ell} \overline{\rho}_k(f_k).$$

We refer to $\overline{\rho}$ as the *projective extension of ρ to $S(V)$.*

Proposition 4.4.1. *$\overline{\rho}$ is a seminorm on $S(V)$ extending the seminorm ρ on V and $\overline{\rho}$ is also submultiplicative i.e. $\overline{\rho}(f \cdot g) \leq \overline{\rho}(f)\overline{\rho}(g),\ \forall f, g \in S(V)$*

To prove this result we need an essential lemma:

Lemma 4.4.2. *Let $i, j \in \mathbb{N}$, $f \in S(V)_i$ and $g \in S(V)_j$. If $k = i + j$ then $\overline{\rho}_k(fg) \leq \overline{\rho}_i(f)\overline{\rho}_j(g)$.*

Proof.
Let us consider a generic representation of $f \in S(V)_i$ and $g \in S(V)_j$, i.e. $f = \sum_p f_{p1} \cdots f_{pi}$ with $f_{pk} \in V$ for $k = 1, \ldots, i$ and $g = \sum_q g_{q1} \cdots g_{qj}$ with $g_{ql} \in V$ for $l = 1, \ldots, j$. Then $f \cdot g = \sum_{p,q} f_{p1} \cdots f_{pi} g_{q1} \cdots g_{qj}$, and so

$$\begin{aligned}\overline{\rho}_k(f \cdot g) &\leq \sum_{p,q} \rho(f_{p1}) \cdots \rho(f_{pi}) \rho(g_{q1}) \cdots \rho(g_{qj}) \\ &= \left(\sum_p \rho(f_{p1}) \cdots \rho(f_{pi})\right) \left(\sum_q \rho(g_{q1}) \cdots \rho(g_{qj})\right).\end{aligned}$$

Since this holds for any representation of f and g, we get $\overline{\rho}_k(fg) \leq \overline{\rho}_i(f)\overline{\rho}_j(g)$. □

Proof. (of Proposition 4.4.1).
It is quite straightforward to show that $\overline{\rho}$ is a seminorm on $S(V)$. Indeed

- Let $k \in \mathbb{K}$ and $f \in S(V)$. Consider any representation of f, say we take $f = \sum_{j=0}^{n} f_j$ with $n \in \mathbb{N}$ and $f_j \in S(V)_j$ for $j = 0, \ldots, n$. Then using the definition of $\overline{\rho}$ and the fact that $\overline{\rho}_k$ is a seminorm on $S(V)_k$ we get:

$$\overline{\rho}(kf) = \overline{\rho}\left(\sum_{j=0}^{n} kf_j\right) = \sum_{j=0}^{n} \overline{\rho}_j(kf_j) = |k| \sum_{j=0}^{n} \overline{\rho}_j(f_j) = |k|\overline{\rho}(f).$$

- Let $f, g \in S(V)$. Consider any representation of f and g, say we take $f = \sum_{j=0}^{n} f_j$, $g = \sum_{i=0}^{m} g_i$ with $n, m \in \mathbb{N}$, $f_j \in S(V)_j$ for $j = 0, \ldots, n$ and $g_i \in S(V)_i$ for $i = 0, \ldots, m$. Take $N := \max\{n, m\}$. Then we can rewrite $f = \sum_{j=0}^{N} f_j$ and $g = \sum_{i=0}^{N} g_i$, where $f_j = 0$ for $j = n+1, \ldots, N$ and $g_i = 0$ for $i = m+1, \ldots, N$. Therefore, using the definition of $\overline{\rho}$ and the fact that $\overline{\rho}_k$ is a seminorm on $S(V)_k$, we have

$$\overline{\rho}(f+g) = \overline{\rho}\left(\sum_{j=0}^{N} (f_j + g_j)\right) \leq \sum_{j=0}^{N} \overline{\rho}_j(f_j) + \sum_{j=0}^{N} \overline{\rho}_j(g_j) = \overline{\rho}(f) + \overline{\rho}(g).$$

Also, $\overline{\rho}_1 = \rho$, so $\overline{\rho}$ restricted to V coincides with ρ. Let us finally show that $\overline{\rho}$ is submultiplicative. Let $f = \sum_{i=0}^{m} f_i$, $g = \sum_{j=0}^{n} g_j$, $f_i \in S(V)_i$, $g_j \in S(V)_j$ and set $T := \{0, \ldots, m\} \times \{0, \ldots, n\}$. Then by using the definition of $\overline{\rho}$, the fact that $\overline{\rho}_k$ is a seminorm on $S(V)_k$ and Lemma 4.4.2 we obtain

$$\begin{aligned}
\overline{\rho}(f \cdot g) =& \overline{\rho}\left(\sum_{i=0}^{m}\sum_{j=0}^{n} f_i g_j\right) = \overline{\rho}\left(\sum_{k=0}^{m+n} \sum_{\substack{(i,j)\in T\\ i+j=k}} f_i g_j\right) = \sum_{k=0}^{m+n} \overline{\rho}_k \left(\sum_{\substack{(i,j)\in T\\ i+j=k}} f_i g_j\right)\\
&\leq \sum_{k=0}^{m+n} \sum_{\substack{(i,j)\in T\\ i+j=k}} \overline{\rho}_k(f_i g_j) \leq \sum_{k=0}^{m+n} \sum_{\substack{(i,j)\in T\\ i+j=k}} \overline{\rho}_i(f_i)\overline{\rho}_j(g_j) = \sum_{i=0}^{m}\sum_{j=0}^{n} \overline{\rho}_i(f_i)\overline{\rho}_j(g_j)\\
&= \left(\sum_{i=0}^{m} \overline{\rho}_i(f_i)\right)\left(\sum_{j=0}^{n} \overline{\rho}_j(g_j)\right) = \overline{\rho}(f)\overline{\rho}(g).
\end{aligned}$$

□

Let us now consider $(S(V), \overline{\rho})$ and any other submultiplicative seminormed unital commutative $\mathbb{K}-$algebra (A, σ). If $\alpha : (S(V), \overline{\rho}) \to (A, \sigma)$ is linear and continuous, then clearly $\alpha \restriction_V : (V, \rho) \to (A, \sigma)$ is also continuous. However, if $\psi : (V, \rho) \to (A, \sigma)$ is linear and continuous, then the unique extension $\overline{\psi}$ given by Proposition 4.3.4 need not be continuous . All one can say in general is the following lemma.

Lemma 4.4.3. *If $\psi : (V, \rho) \to (A, \sigma)$ is linear and continuous, namely $\exists$ $C > 0$ such that $\sigma(\psi(v)) \leq C\rho(v)$ $\forall$ $v \in V$, then for any $k \in \mathbb{N}$ we have $\sigma(\overline{\psi}(g)) \leq C^k \overline{\rho}_k(g)$ $\forall$ $g \in S(V)_k$.*

Proof.
Let $k \in \mathbb{N}$ and $g \in S(V)_k$. Suppose $g = \sum_{i=1}^N g_{i1} \cdots g_{ik}$ with $g_{ij} \in V$ for $j = 1, \dots, N$. Then $\overline{\psi}(g) = \sum_{i=1}^N \psi(g_{i1}) \cdots \psi(g_{ik})$, and so

$$\begin{aligned}\sigma(\overline{\psi}(g)) \leq \sigma\left(\sum_{i=1}^N \psi(g_{i1}) \cdots \psi(g_{ik})\right) &\leq \sum_{i=1}^N \sigma(\psi(f_{i1})) \cdots \sigma(\psi(g_{ik})) \\ \leq \sum_{i=1}^N C\rho(g_{i1}) \cdots C\rho(g_{ik}) &= C^k \sum_{i=1}^N \rho(g_{i1}) \cdots \rho(g_{ik}).\end{aligned}$$

As this holds for any representation of g, we get $\sigma(\overline{\psi}(g)) \leq C^k \overline{\rho}_k(g)$. □

Proposition 4.4.4. *If $\psi : (V, \rho) \to (A, \sigma)$ has operator norm ≤ 1, then the induced algebra homomorphism $\overline{\psi} : (S(V), \overline{\rho}) \to (A, \sigma)$ has operator norm $\leq \sigma(1)$.*

Recall that given a linear operator L between two seminormed spaces (W_1, q_1) and (W_2, q_2) we define the operator norm of L as follows:

$$\|L\| := \sup_{\substack{w \in W_1 \\ q_1(w) \leq 1}} q_2(L(w)).$$

Proof.
Suppose $\sigma \not\equiv 0$ on A (if this is the case then there is nothing to prove). Then there exists $a \in A$ such that $\sigma(a) > 0$. This together with the fact that σ is a submultiplicative seminorm gives that

$$\sigma(1) \geq 1. \tag{4.4}$$

Since $\|\psi\| \leq 1$, we have that $\sigma(\psi(v)) \leq \rho(v)$, $\forall\, v \in V$. Then we can apply Lemma 4.4.3 and get that

$$\forall\, k \in \mathbb{N},\ g \in S(V)_k,\ \sigma(\overline{\psi}(g)) \leq \overline{\rho}_k(g) \tag{4.5}$$

Now let $f \in S(V)$, i.e. $f = \sum_{k=0}^{m} f_k$ with $f_k \in S(V)_k$ for $k = 0, \dots, m$. Then

$$\begin{aligned}\sigma(\overline{\psi}(f)) = \sigma\left(\sum_{k=0}^{m} \overline{\psi}(f_k)\right) &\leq \sum_{k=0}^{m} \sigma(\overline{\psi}(f_k)) \overset{(4.5)}{\leq} \sigma(\overline{\psi}(f_0)) + \sum_{k=1}^{m} \overline{\rho}_k(f_k) \\ &= \sigma(f_0) + \sum_{k=1}^{m} \overline{\rho}_k(f_k) \leq \sigma(1)\overline{\rho}(f_0) + \sum_{k=1}^{m} \overline{\rho}_k(f_k) \\ &\overset{(4.4)}{\leq} \sigma(1)\overline{\rho}(f_0) + \sum_{k=1}^{m} \sigma(1)\overline{\rho}_k(f_k) = \sigma(1)\sum_{k=0}^{m} \overline{\rho}_k(f_k) = \sigma(1)\overline{\rho}(f).\end{aligned}$$

Hence, $\|\overline{\psi}\| \leq \sigma(1)$. □

Using the properties we have showed for the projective extension $\bar{\rho}$ of ρ to $S(V)$, we can easily pass to the case when V is endowed with a locally convex topology τ (generated by more than one seminorm) and to study how to extend this topology to $S(V)$ in a such a way that the latter becomes an lmc TA.

Let τ be any locally convex topology on a vector space V over $\mathbb{K}$ and let $\mathcal{P}$ be a directed family of seminorms generating τ. Denote by $\overline{\tau}$ the topology on $S(V)$ determined by the family of seminorms $\mathcal{Q} := \{\overline{n\rho} : \rho \in \mathcal{P},\ n \in \mathbb{N}\}$.

Proposition 4.4.5. *$\overline{\tau}$ is an lmc topology on $S(V)$ extending τ and is the finest lmc topology on $S(V)$ having this property.*

Proof. By definition of $\overline{\tau}$ and by Proposition 4.4.1, it is clear that $\mathcal{Q}$ is a directed family of submultiplicative seminorms and so that $\overline{\tau}$ is an lmc topology on $S(V)$ extending τ.It remains to show that τ is the finest lmc topology with extending τ to $S(V)$. Let μ an lmc topology on $S(V)$ s.t. $\mu \restriction_V = \tau$, i.e. μ extends τ to $S(V)$. Suppose that μ is finer than $\overline{\tau}$. Let $\mathcal{S}$ be a directed family of submultiplicative seminorms generating μ and consider the identity map id : $(V, \tau) \to (V, \mu \restriction_V)$. As by assumption $\mu \restriction_V = \tau$, we have that id is continuous and so by Theorem 4.6.3-TVS-I (applied for directed families of seminorms) we get that:

$$\forall\, s \in \mathcal{S}, \exists\, n \in \mathbb{N}, \exists\, \rho \in \mathcal{P} : \ s(v) = s((id(v)) \leq n\rho(v), \forall\, v \in V.$$

Consider the embedding $i : (V, n\rho) \to (S(V), q)$. Then $\|i\| \leq 1$ and so, by Proposition 4.4.4, the unique extension $\overline{i} : (S(V), \overline{n\rho}) \to (S(V), s)$ of i is continuous with $\|\overline{i}\| \leq q(1)$. This gives that

$$s(f) \leq s(1)\overline{n\rho}(f), \forall\, f \in S(V).$$

Hence, all $s \in \mathcal{F}$ are continuous w.r.t. $\overline{\tau}$ and so μ must be coarser than $\overline{\tau}$. □