Chapter 5

Short overview on the moment problem

In this chapter we are going to consider always *Radon measures* on Hausdorff topological spaces, i.e. non-negative Borel measures which are locally finite and inner regular.

5.1 The classical finite-dimensional moment problem

Let μ be a Radon measure on $\mathbb{R}$. We define the $n-th$ *moment of* μ as

$$m_n^{\mu} := \int_{\mathbb{R}} x^n \mu(dx)$$

If all moments of μ exist and are finite, then we can associate to μ the sequence of real numbers $(m_n^{\mu})_{n\in\mathbb{N}_0}$, which is said to be the *moment sequence* of μ. The moment problem exactly addresses the inverse question:

Problem 5.1.1 (The one-dimensional $K-$Moment Problem (KMP)).
Given a closed subset K of $\mathbb{R}$ and a sequence $m = (m_n)_{n\in\mathbb{N}_0}$ of real numbers, does there exist a Radon measure μ on $\mathbb{R}$ s.t. for any $n \in \mathbb{N}_0$ we have $m_n = m_n^{\mu}$ and μ is supported on K, i.e.

$$m_n = \underbrace{\int_{\mathbb{R}} x^n \mu(dx)}_{n\text{-}th\ moment\ of\ \mu},\ \forall\, n \in \mathbb{N}_0 \quad and \quad supp(\mu) \subseteq K\,?$$

If such a measure μ does exist we say that μ is a $K-$*representing measure* for m or that m is represented by μ on K.

Note that there is a bijective correspondence between the set $\mathbb{R}^{\mathbb{N}_0}$ of all sequences of real numbers and the set $(\mathbb{R}[x])^*$ of all linear functional from $\mathbb{R}[x]$ to $\mathbb{R}$.

$$
\begin{array}{lcclcl}
\mathbb{R}^{\mathbb{N}_0} & \to & (\mathbb{R}[x])^* & & & \\
(m_n)_{n\in\mathbb{N}_0} & \mapsto & L_m : & \mathbb{R}[x] & \to & \mathbb{R} \\
 & & & p(x) := \sum_j p_j x^j & \mapsto & L_m(p) := \sum_j p_j m_j. \\
(L(x^n))_{n\in\mathbb{N}_0} & \leftarrow\!\shortmid & L & & &
\end{array}
$$

In virtue of this correspondence, we can always reformulate the KMP in terms of linear functionals

Problem 5.1.2 (The one-dimensional $K-$Moment Problem (KMP))**.**
Given a closed subset K of $\mathbb{R}$ and a linear functional $L : \mathbb{R}[x] \to \mathbb{R}$, does there exists a Radon measure μ on $\mathbb{R}$ s.t.

$$L(p) = \int_{\mathbb{R}} p(x)\mu(dx),\ \forall p \in \mathbb{R}[x] \quad \textit{and} \quad supp(\mu) \subseteq K\,?$$

As before, if such a measure exists we say that μ is a *$K-$representing measure* for L and that it is a solution to the $K-$moment problem for L.

Clearly one can generalize the one-dimensional KMP to higher dimension by considering $\mathbb{R}[\mathbf{x}] := \mathbb{R}[x_1, \ldots, x_d]$ for some $d \in \mathbb{N}$ (see [15, Section 5.2.2]).

Problem 5.1.3 (The d-dimensional $K-$Moment Problem (KMP))**.**
Given a closed subset K of $\mathbb{R}^d$ and a linear functional $L : \mathbb{R}[\mathbf{x}] \to \mathbb{R}$, does there exists a Radon measure μ on $\mathbb{R}^d$ s.t.

$$L(p) = \int_{\mathbb{R}^d} p(\mathbf{x})\mu(d\mathbf{x}),\ \forall p \in \mathbb{R}[\mathbf{x}] \quad \textit{and} \quad supp(\mu) \subseteq K\,?$$

It is then very natural to ask the following:

Questions

- What if we have infinitely many variables, i.e. we consider $\mathbb{R}[x_i : i \in \Omega]$ where Ω is an infinite index set?
- What if instead of real variables we consider variables in a generic $\mathbb{R}-$vector space V (even infinite dimensional)?
- What if instead of the polynomial ring $\mathbb{R}[\mathbf{x}]$ we take any unital commutative $\mathbb{R}-$algebra A?

All these possible generalization of the moment problem usually go under the name of *infinite dimensional moment problem.*

5.2 Moment problem for commutative $\mathbb{R}-$algebras

In this section we are going to give a formulation of the moment problem general enough to encompass all the possible generalizations addressed in the previous section. Let us start by introducing some notation and terminology.

Given a unital commutative $\mathbb{R}-$algebra A, we denote by $\mathcal{X}(A)$ the *character space* of A (see Definition 2.4.6). For any $a \in A$, we define the *Gelfand transform* $\hat{a} : \mathcal{X}(A) \to \mathbb{R}$ as $\hat{a}(\alpha) := \alpha(a)$, $\forall \alpha \in \mathcal{X}(A)$. We endow the character space $\mathcal{X}(A)$ with the weakest topology $\tau_{\mathcal{X}(A)}$ s.t. all Gelfand transforms are continuous, i.e. $\hat{a}$ is continuous for all $a \in A$.

Remark 5.2.1. *$\mathcal{X}(A)$ can be seen as a subset of $\mathbb{R}^A$ via the embedding:*

$$\begin{array}{rccl} \pi: & \mathcal{X}(A) & \to & \mathbb{R}^A \\ & \alpha & \mapsto & \pi(\alpha) := (\alpha(a))_{a\in A} = (\hat{a}(\alpha))_{a\in A}. \end{array}$$

If we equip $\mathbb{R}^A$ with the product topology τ_{prod}, then it can be showed (see [19, Section 5.7]) that $\tau_{\mathcal{X}(A)}$ coincides with the topology induced by π on $\mathcal{X}(A)$ from $(\mathbb{R}^A, \tau_{prod})$, i.e.

$$\tau_{\mathcal{X}(A)} \equiv \left\{\pi^{-1}(O) : O \in \tau_{prod}\right\}.$$

The space $\left(\mathcal{X}(A), \tau_{\mathcal{X}(A)}\right)$ is therefore Hausdorff.

Problem 5.2.2 (The KMP for unital commutative $\mathbb{R}-$algebras)**.**
Given a closed subset $K \subseteq \mathcal{X}(A)$ and a linear functional $L : A \to \mathbb{R}$, does there exist a Radon measure μ on $\mathcal{X}(A)$ s.t. we have

$$L(a) = \int_{\mathcal{X}(A)} \hat{a}(\alpha)\mu(d\alpha), \forall a \in A \quad and \quad supp(\mu) \subseteq K?$$

lc

$$\begin{aligned} \alpha(p) &= \alpha\left(\sum_{\beta\in\mathbb{N}_0^d} p_\beta \mathbf{x}^\beta\right) = \sum_{\beta\in\mathbb{N}_0^d} \alpha(p_\beta)\alpha(x_1)^{\beta_1}\cdots\alpha(x_d)^{\beta_d} \\ &= \sum_{\beta\in\mathbb{N}_0^d} p_\beta \alpha(x_1)^{\beta_1}\cdots\alpha(x_d)^{\beta_d} = p\left(\alpha(x_1),\ldots,\alpha(x_d)\right). \end{aligned}$$

Conversely, for any $\mathbf{y} \in \mathbb{R}^d$ we can define the functional $\alpha_{\mathbf{y}} : \mathbb{R}[\mathbf{x}] \to \mathbb{R}$ by $\alpha_{\mathbf{y}}(p) := p(\mathbf{y})$ for any $p \in \mathbb{R}[\mathbf{x}]$, which is clearly a $\mathbb{R}-$algebra homomorphism. Hence, we have showed that $X(\mathbb{R}[\mathbf{x}]) \cong \mathbb{R}^d$ and via this isomorphism we have that, for any $p \in \mathbb{R}[\mathbf{x}]$, the Gelfand transform $\hat{p}$ is identified with the polynomial p itself. Using these identifications, we get that Problem 5.2.2 for $A = \mathbb{R}[\mathbf{x}]$ is nothing but Problem 5.1.3.

Let us come back to the general KMP 5.2.2. Fixed a subset K of $\mathcal{X}(A)$, we denote by

$$\mathrm{Pos}(K) := \{a \in A : \hat{a} \geq 0 \text{ on } K\}.$$

A necessary condition for the existence of a solution to the KMP 5.2.2 is clearly that L is nonnegative on $\mathrm{Pos}(K)$. In fact, if there exists a $K-$representing measure μ for L then for all $a \in \mathrm{Pos}(K)$ we have

$$L(a) = \int_{\mathcal{X}(A)} \hat{a}(\alpha)\mu(d\alpha) \geq 0$$

since μ is nonnegative and supported on K and $\hat{a}$ is nonnegative on K.

It is then natural to ask if the nonnegativity of L on $\mathrm{Pos}(K)$ is also sufficient. For $A = \mathbb{R}[\mathbf{x}]$ a positive answer is provided by the so-called Riesz-Haviland theorem (see [15, Theorem 5.2.5]). An analogous result also holds in this general setting:

Theorem 5.2.3 (Generalized Riesz-Haviland Theorem)**.** *Let $K \subseteq \mathcal{X}(A)$ closed and $L : A \to \mathbb{R}$ linear. Suppose there exists $p \in A$ such that $\hat{p} \geq 0$ on K and for all $n \in \mathbb{N}$ the set $\{\alpha \in K : \hat{p}(\alpha) \leq n\}$ is compact. Then L has a $K-$representing measure if and only if $L(\mathrm{Pos}(K)) \subseteq [0, +\infty)$.*

This theorem provides a complete solution for the $K-$ moment problem 5.2.2 but it is somehow unpractical! In fact, it reduces the solvability of the $K-$moment problem to the problem of characterizing $\mathrm{Pos}(K)$. To approach to this problem we will try to approximate elements in $\mathrm{Pos}(K)$ with elements of A whose Gelfand transform is "more evidently" non-negative, e.g. sum of even powers of elements of A. In this spirit we consider $2d-$power modules of the algebra A for $d \in \mathbb{N}$.

Definition 5.2.4 ($2d-$power module)**.**
Let $d \in \mathbb{N}$. A $2d-$power module *of A is a subset M of A satisfying $1 \in M$, $M + M \subseteq M$ and $a^{2d}M \subseteq M$ for each $a \in A$.*

In the case $d = 1$, $2d-$power modules are referred to as *quadratic modules*. We denote by $\sum A^{2d}$ the set of all finite sums $\sum a_i^{2d}$, $a_i \in A$. $\sum A^{2d}$ is the smallest $2d-$power module of A.

Definition 5.2.5 (Generated $2d-$power module)**.**
Let $\{p_j\}_{j\in J}$ be an arbitrary subset of elements in A (J can have also infinite cardinality). The $2d-$power module of A generated by $\{p_j\}_{j\in J}$ is defined as

$$M := \{\sigma_0 + \sigma_1 p_{j_1} + \ldots + \sigma_s p_{j_s} : s \in \mathbb{N}, j_1, \ldots, j_s \in J, \sigma_0, \ldots, \sigma_s \in \sum A^{2d}\}.$$

For any subset M of A, we set

$$X_M := \{\alpha \in \mathcal{X}(A) : \hat{a}(\alpha) \geq 0,\ \forall a \in M\},$$

which is a closed subset of $\big(\mathcal{X}(A), \tau_{\mathcal{X}(A)}\big)$. If $M = \sum A^{2d}$ then $X_M = \mathcal{X}(A)$. If M is the $2d-$power module of A generated by $\{p_j\}_{j\in J}$ then $X_M := \{\alpha \in \mathcal{X}(A) : \hat{p_j}(\alpha) \geq 0,\ \forall\ j \in J\}$.

Given a $2d-$power module M, let us consider the X_M-moment problem for a linear functional $L : A \to \mathbb{R}$. If there exists a X_M-representing measure μ for L, then it is clear that $L(M) \subseteq [0, +\infty)$ since $M \subseteq \mathrm{Pos}(X_M)$. Under which assumptions does the converse hold?

The answer is positive when the module M is Archimedean. The main ingredient of the proof of this result is the the so-called *Jacobi Positivstellensatz*, which holds for Archimedean power modules and provides that $\mathrm{Pos}(X_M) \subseteq \overline{M}^{\varphi}$, where φ is the finest locally convex topology on A. This inclusion together with Proposition 2.4.8 allows to get the desired conclusion by applying of Hahn-Banach and Riesz-Haviland theorems.

Theorem 5.2.6. *Let M be an archimedean $2d-$power module of A and $L : A \to \mathbb{R}$ a linear functional. L has a X_M-representing measure if and only if $L(M) \subseteq [0, +\infty)$.*

Proof. See [13, Corollary 2.6]. The conclusion can be also obtained as a consequence of [11, Theorem 5.5]. □

A $2d-$power module M in A is said to be *archimedean* if for each $a \in A$ there exists an integer N such that $N \pm a \in M$. If M is a $2d-$power module of A which is archimedean then X_M is compact. The converse is false in general (see [19, Section 7.3]).

Does Theorem 5.2.6 still hold when M is not Archimedean? Can we find other topologies τ rather than the finest lc topology φ on A such that $\mathrm{Pos}(X_M) \subseteq \overline{M}^{\tau}$ so that we can get a similar result for $\tau-$continuous linear functionals on A? In order to attack those questions we are going to investigate the KMP for linear functionals on some special kind of topological $\mathbb{R}-$algebras.

5.3 Moment problem for submultiplicative seminormed $\mathbb{R}-$algebras

In this section we are going to present some results about Problem 5.2.2 when A a submultiplicative seminormed $\mathbb{R}-$algebra (for more details see [12]).

Let A be a unital commutative $\mathbb{R}-$algebra and σ be a submultiplicative seminorm on an $\mathbb{R}-$algebra A, i.e. $\sigma(a \cdot b) \leq \sigma(a)\sigma(b)$ for all $a, b \in A$ ($\cdot$ denotes the multiplication in A). The algebra A together with such a σ is called a *submultiplicative seminormed* $\mathbb{R}-$*algebra* and is denoted by (A, σ).

We denote the set of all $\sigma-$continuous $\mathbb{R}-$algebra homomorphisms from A to $\mathbb{R}$ by $\mathfrak{sp}(\sigma)$, which we refer to as the *Gelfand spectrum* of (A, σ), i.e.

$$\mathfrak{sp}(\sigma) := \{\alpha \in \mathcal{X}(A) : \alpha \text{ is } \sigma - \text{continuous}\}.$$

We endow $\mathfrak{sp}(\sigma)$ with the subspace topology induced by $\left(\mathcal{X}(A), \tau_{\mathcal{X}(A)}\right)$. Then one can show the following two results (see [12] for a proof).

Lemma 5.3.1.
For any submultiplicative seminormed $\mathbb{R}-$algebra (A, σ) we have:

$$\mathfrak{sp}(\sigma) = \{\alpha \in \mathcal{X}(A) : |\alpha(a)| \leq \sigma(a) \textit{ for all } a \in A\}.$$

Corollary 5.3.2. *The Gelfand spectrum of any submultiplicative seminormed $\mathbb{R}-$algebra (A, σ) is compact.*

An important closure result useful for the Problem 5.2.2 when A a submultiplicative seminormed $\mathbb{R}-$algebra was proved by M. Ghasemi, S. Kuhlmann and M. Marshall in [12, Theorem 3.7]. We just state it here but we show in details how this result helps to get better conditions than the ones provided by the Generalized Riesz-Haviland theorem.

Theorem 5.3.3. *Let (A, σ) be a submultiplicative seminormed $\mathbb{R}-$algebra and M is a $2d-$power module of A (not necessarily Archimedean). Then*

$$\overline{M}^{\rho} = \mathrm{Pos}(X_M \cap \mathfrak{sp}(\sigma)).$$

Corollary 5.3.4. *Let (A, σ) be a submultiplicative seminormed $\mathbb{R}-$algebra, M is a $2d-$power module of A and $L : A \to \mathbb{R}$ a linear functional. L has a representing measure supported on $X_M \cap \mathfrak{sp}(\sigma)$ if and only if L is $\sigma-$continuous and $L(M) \subseteq [0, +\infty)$.*

Proof.
($\Leftarrow$) By our hypothesis and Theorem 5.3.3, L is nonnegative on $\mathrm{Pos}(X_M \cap \mathfrak{sp}(\sigma))$. Hence, by applying Theorem 5.2.3, L has a $(X_M \cap \mathfrak{sp}(\sigma))-$representing measure.[1]

[1] Note that we can apply the Generalized Riesz-Haviland Theorem since $X_M \cap \mathfrak{sp}(\sigma)$ is compact in $\left(\mathcal{X}(A), \tau_{\mathcal{X}(A)}\right)$. (This is a direct consequence of Corollary 5.3.2 and of the fact that X_M is a closed subset of $\left(\mathcal{X}(A), \tau_{\mathcal{X}(A)}\right)$).

($\Rightarrow$) Suppose that L has a representing measure μ supported on $X_M \cap \mathfrak{sp}(\sigma)$. Then for all $b \in M$ we have

$$L(b) = \int_{X_M \cap \mathfrak{sp}(\sigma)} \hat{b}(\alpha)\mu(d\alpha) \geq 0$$

since μ is nonnegative and supported on a subset of X_M. Therefore, we have got $L(M) \subseteq [0, +\infty)$. Also, we have that for all $a \in A$:

$$\begin{aligned} |L(a)| &\leq \int_{X_M \cap \mathfrak{sp}(\sigma)} |\hat{a}(\alpha)|\mu(d\alpha) \\ &= \int_{X_M \cap \mathfrak{sp}(\sigma)} |\alpha(a)|\mu(d\alpha) \\ &\overset{\text{Lemma 5.3.1}}{\leq} \int_{X_M \cap \mathfrak{sp}(\sigma)} \sigma(a)\mu(d\alpha) = \sigma(a)\mu\left(X_M \cap \mathfrak{sp}(\sigma)\right) \end{aligned}$$

Note that $\mu\left(X_M \cap \mathfrak{sp}(\sigma)\right)$ is finite since μ is Radon and $X_M \cap \mathfrak{sp}(\sigma)$ compact. Hence, L is $\sigma-$continuous. $\square$

5.4 Moment problem for symmetric algebras of lc spaces

In this section we are going to present some results about Problem 5.2.2 when A is the symmetric algebra $S(V)$ of a locally convex space V over $\mathbb{R}$ (for more details see [14]).

Let us start with the simplest case, i.e. when V is a $\mathbb{R}-$vector space endowed with a seminorm ρ. In Section 5.2, we have showed how to extend the seminorm ρ to a seminorm $\overline{\rho}$ on $S(V)$, which we proved to be submultiplicative by Proposition 4.4.1. Therefore, $(S(V), \overline{\rho})$ is a submultiplicative seminormed $\mathbb{R}-$algebra and so we can apply Corollary 5.3.4, obtaining the following result.

Proposition 5.4.1. *Let (V, ρ) be a seminormed $\mathbb{R}$-vector space, M a $2d-$power module of $S(V)$ and $L : S(V) \to \mathbb{R}$ a linear functional. L is $\overline{\rho}$-continuous and $L(M) \subseteq [0, +\infty)$ if and only if $\exists!\, \mu$ on V^*: $L(f) = \int_{V^*} \hat{f}(\alpha)\mu(d\alpha)$ and $\operatorname{supp}\mu \subseteq X_M \cap \overline{B}_1^{\|\cdot\|_\rho}$, where $\|\cdot\|_\rho$ denotes the operator norm on V^*, i.e. $\|\beta\|_\rho := \sup_{\substack{v \in V \\ \rho(v) \leq 1}} |\beta(v)|$ and $\overline{B}_1^{\|\cdot\|_\rho} := \{\beta \in V^* : \|\beta\|_\rho \leq 1\}$.*

Proof.
By Proposition 4.4.1, we can apply Corollary 5.3.4 to $(S(V), \overline{\rho})$ and obtain that: L is $\overline{\rho}$-continuous and $L(M) \subseteq [0, +\infty)$ if and only if $\exists!\, \mu$ on $X(S(V))$:

$L(f) = \int\limits_{X(S(V))} \hat{f}(\alpha)\mu(d\alpha)$ and $\operatorname{supp}\mu \subseteq X_M \cap \mathfrak{sp}(\overline{\rho})$. Now by Corollary 4.3.5 we know that $\operatorname{Hom}(S(V), \mathbb{R}) \cong V^*$, i.e. $X(S(V)) \cong V^*$. Using this isomorphism we can get that $\mathfrak{sp}(\overline{\rho}) \cong \overline{B}_1^{\|\cdot\|_\rho}$ and so the desired conclusion.

Let us prove that $\mathfrak{sp}(\overline{\rho}) \cong \overline{B}_1^{\|\cdot\|_\rho}$. Suppose that $\alpha \in \mathfrak{sp}(\overline{\rho})$. Then by Lemma 5.3.1 we have that $|\alpha(f)| \leq \overline{\rho}(f)\ \forall\ f \in S(V)$. Clearly this implies that $|\alpha(v)| \leq \rho(v)\ \forall\ v \in V$, so $\|\alpha \restriction_V \|_\rho \leq 1$, i.e. $\alpha \in \overline{B}_1^{\|\cdot\|_\rho}$. Conversely, suppose that $\beta \in V^*$ s.t. $\|\beta\|_\rho \leq 1$. Denote by $\overline{\beta}$ the unique extension of β to an $\mathbb{R}$-algebra homomorphism $\overline{\beta} : S(V) \to \mathbb{R}$. Then, by Proposition 4.4.4, we get that $\|\overline{\beta}\|_{\overline{\rho}} \leq 1$ and so that $|\overline{\beta}(f)| \leq \overline{\rho}(f)\ \forall\ f \in S(V)$. Thus $\overline{\beta} \in \mathfrak{sp}(\overline{\rho})$. $\square$

We can generalize this result to (V, τ) locally convex TVS over $\mathbb{R}$ by using Proposition 4.4.5, which provides an extension of τ to an lmc topology $\overline{\tau}$ to $S(V)$.

Proposition 5.4.2. *Let (V, τ) be a lmc TVS over $\mathbb{R}$ whose topology is generated by a directed family of seminorms $\mathcal{P}$. Let M be a $2d-$power module of $S(V)$ and $L : S(V) \to \mathbb{R}$ a linear functional. L is $\overline{\tau}$-continuous and $L(M) \subseteq [0, +\infty)$ if and only if $\exists!\,\mu$ on V^*: $L(f) = \int\limits_{V^*} \hat{f}(\alpha)\mu(d\alpha)$ and $\operatorname{supp}\mu \subseteq X_M \cap \overline{B}_n^{\|\cdot\|_\rho}$, for some $n \in \mathbb{N}$ and $\rho \in \mathcal{P}$.*

Proof.
By Proposition 4.4.5, we know that $\overline{\tau}$ is a lmc topology on $S(V)$ generated by the family $\mathcal{Q} := \{\overline{n\rho} : \rho \in \mathcal{P},\ n \in \mathbb{N}\}$. Then Proposition 4.6.1 in TVS-I guarantees that L is $\overline{\tau}$-continuous if and only if there exists $q \in \mathcal{Q}$ s.t. L is $q-$continuous, i.e. there exists $n \in \mathbb{N}$ and $\rho \in \mathcal{P}$ s.t. L is $\overline{n\rho}-$continuous. Thus we reduced to the case of one single seminorm and so we can apply Proposition 5.4.1 and get that: L is $\overline{\tau}$-continuous and $L(M) \subseteq [0, +\infty)$ if and only if $\exists!\,\mu$ on V^*: $L(f) = \int\limits_{V^*} \hat{f}(\alpha)\mu(d\alpha)$ and $\operatorname{supp}\mu \subseteq X_M \cap \overline{B}_1^{\|\cdot\|_{n\rho}}$. This yields the conclusion as $\overline{B}_1^{\|\cdot\|_{n\rho}} = \overline{B}_n^{\|\cdot\|_\rho}$. $\square$

What happens when the assumption of continuity of L is weakened? Can we get results for this moment problem for measures which are not compactly supported? Some results in this direction have been obtained in [17] for the case when $V = \mathcal{C}_c^\infty(\mathbb{R}^d)$ endowed with the projective topology introduced in Section 1.4. However, there are still many open questions concerning the moment problem in this general framework and we are still far from a complete understanding of the infinite dimensional moment problem.