2. *Let $S$ be a non-emptyset and $\mathbb{K}^S$ be the set of all functions from $S$ to $\mathbb{K}$ equipped with pointwise operations and the topology $\omega$ of pointwise convergence (or simple convergence), i.e. the topology generated by*

$$\mathcal{B} := \{W_\varepsilon(x_1, \dots, x_n) : n \in \mathbb{N}, x_1, \dots, x_n \in S, \varepsilon > 0\},$$

*where $W_\varepsilon(x_1, \dots, x_n) := \{f \in \mathbb{K}^S : f(x_i) \in B_\varepsilon(0), i = 1, \dots, n\}$ and $B_\varepsilon(0) = \{k \in \mathbb{K} : |k| \leq \varepsilon\}$. Then $(\mathbb{K}^S, \omega)$ is a TA with continuous multiplication. Indeed, for any $n \in \mathbb{N}, x_1, \dots, x_n \in S, \varepsilon > 0$ we have that*

$$\begin{aligned} W_{\sqrt{\varepsilon}}(x_1, \dots, x_n) W_{\sqrt{\varepsilon}}(x_1, \dots, x_n) &= \{fg : f(x_i), g(x_i) \in B_{\sqrt{\varepsilon}}(0), i = 1, \dots, n\} \\ &\subseteq \{h : h(x_i) \in B_\varepsilon(0), i = 1, \dots, n\} \\ &= W_\varepsilon(x_1, \dots, x_n). \end{aligned}$$

*As it is also easy to show that $(\mathbb{K}^S, \omega)$ is a TVS, the conclusion follows by Theorem 1.2.10.*

Two fundamental classes of TA are the following ones:

**Definition 1.2.12** (Normed Algebra)**.** *A* normed algebra *is a $\mathbb{K}-$algebra $A$ endowed with the topology induced by a submultiplicative norm $\|\cdot\|$, i.e. $\|xy\| \leq \|x\|\|y\|, \ \forall x, y \in A$.*

**Definition 1.2.13** (Banach Algebra)**.** *A normed algebra whose underlying space is Banach (i.e. complete normed space) is said to be a* Banach algebra*.*

**Proposition 1.2.14.** *Any normed algebra is a TA with continuous multiplication.*

*Proof.*
Let $(A, \|\cdot\|)$ be a normed algebra. It is easy to verify that the topology $\tau$ induced by the norm $\|\cdot\|$ (i.e. the topology generated by the collection $\mathcal{B} := \{B_\varepsilon(o) : \varepsilon > 0\}$, where $B_\varepsilon(o) := \{x \in A : \|x\| \leq \varepsilon\}$) makes $A$ into a TVS. Moreover, the submultiplicativity of the norm $\|\cdot\|$ ensures that for any $\varepsilon > 0$ we have: $B_{\sqrt{\varepsilon}}(o) B_{\sqrt{\varepsilon}}(o) \subseteq B_\varepsilon(o)$. Hence, $\mathcal{B}$ fulfills both a) and b') in Theorem 1.2.10 and so we get the desired conclusion. □

**Examples 1.2.15.**

1. *Let $n \in \mathbb{N}$. $\mathbb{K}^n$ equipped with the componentwise operations of addition, scalar and vector multiplication, and endowed with the supremum norm $\|x\| := \max_{i=1,\dots,n} |x_i|$ for all $x := (x_1, \dots, x_n) \in \mathbb{K}^n$ is a Banach algebra.*

2. *Let $n \in \mathbb{N}$. The algebra $\mathbb{R}^{n\times n}$ of all real square matrices of order $n$ equipped with the following norm is a Banach algebra:*

$$\|A\| := \sup_{x\in\mathbb{R}^n\setminus\{o\}} \frac{|Ax|}{|x|}, \forall A \in \mathbb{R}^{n\times n},$$

*where $|\cdot|$ is the usual euclidean norm on $\mathbb{R}^n$. Indeed, from the previous example it is easy to see that $(\mathbb{R}^{n\times n}, \|\cdot\|)$ is a Banach space. Also, for any $A, B \in \mathbb{R}^{n\times n}$ we have that:*

$$\|AB\| = \sup_{x\in\mathbb{R}^n\setminus\{o\}} \frac{|A(Bx)|}{|x|} \leq \|A\| \sup_{x\in\mathbb{R}^n\setminus\{o\}} \frac{|Bx|}{|x|} = \|A\|\|B\|.$$

3. *Let $(X,\tau)$ be a topological space and $\mathcal{C}_c(X)$ the set of all $\mathbb{K}-$valued continuous functions with compact support. If we equip $\mathcal{C}_c(X)$ with the pointwise operations and the supremum norm $\|f\| := \sup_{x\in X}|f(x)|$, then $(\mathcal{C}_c(X), \|\cdot\|)$ is a Banach algebra.*

Before coming back to general TA, let us observe a further nice property of normed and so of Banach algebras, which will allow us to assume w.l.o.g. that in a unital normed algebra the unit has always unitary norm.

**Proposition 1.2.16.** *If $(A,p)$ is a unital normed algebra with unit $1_A$, then there always exists a subultiplicative norm $q$ on $A$ equivalent to $p$ and such that $q(1_A) = 1$.*

*Proof.* Suppose that $p(1_A) \neq 1$ and define

$$q(a) := \sup_{x\in A\setminus\{o\}} \frac{p(ax)}{p(x)}, \ \forall a \in A.$$

Immediately from the definition, we see that $q(1_A) = 1$ and $p(ay) \leq q(a)p(y)$ for all $a, y \in A$. The latter implies at once that

$$p(a) = p(a1_A) \leq q(a)p(1_A), \ \forall a \in A \tag{1.1}$$

and

$$q(ab) = \sup_{x\in A\setminus\{o\}} \frac{p(abx)}{p(x)} \leq \sup_{x\in A\setminus\{o\}} \frac{q(a)p(bx)}{p(x)} = q(a)q(b), \ \forall a, b \in A. \tag{1.2}$$

Moreover, since $p$ is submultiplicative, we have that for all $a \in A$

$$q(a) \leq \sup_{x\in A\setminus\{o\}} \frac{p(a)p(x)}{p(x)} = p(a).$$

The latter together with (1.1) guarantees that $q$ is equivalent to $p$, while (1.2) its submultiplicativity. ☐

So far we have seen only examples of TA with continuous multiplication. In the following example, we will introduce a TA whose multiplication is separately continuous but not jointly continuous.

**Example 1.2.17.**
*Let $(H, \langle\cdot,\cdot,\rangle)$ be an infinite dimensional separable Hilbert space over $\mathbb{K}$. Denote by $\|\cdot\|_H$ the norm on $H$ defined as $\|x\|_H := \sqrt{\langle x, x\rangle}$ for all $x \in H$, and by $L(H)$ the set of all linear and continuous maps from $H$ to $H$. The set $L(H)$ equipped with the pointwise addition $\mathbb{a}$, the pointwise scalar multiplication $\mathbb{m}$ and the composition of maps $\circ$ as multiplication is a $\mathbb{K}-$algebra.*
*Let $\tau_w$ be the* ***weak operator topology*** *on $L(H)$, i.e. the coarsest topology on $L(H)$ such that all the maps $E_{x,y} : L(H) \to H, T \mapsto \langle Tx, y\rangle$ $(x, y \in H)$ are continuous. A basis of neighbourhoods of the origin in $(L(H), \tau_w)$ is given by:*

$$\mathcal{B}_w := \{V_\varepsilon(x_i, y_i, n) : \varepsilon > 0, n \in \mathbb{N}, x_1, \ldots, x_n, y_1, \ldots, y_n \in H\},$$

*where $V_\varepsilon(x_i, y_i, n) := \{W \in L(H) : |\langle Wx_i, y_i\rangle| < \varepsilon, i = 1, \ldots, n\}$.*

• *$\underline{(L(H), \tau_w) \text{ is a TA}}$.*
*For any $\varepsilon > 0$, $n \in \mathbb{N}$, $x_1, \ldots, x_n, y_1, \ldots, y_n \in H$, using the bilinearity of the inner product we easily have:*

$$\begin{aligned} V_{\frac{\varepsilon}{2}}(x_i, y_i, n) \times V_{\frac{\varepsilon}{2}}(x_i, y_i, n) &= \bigcap_{i=1}^{n} \left\{(T, S) : |\langle Tx_i, y_i\rangle| < \frac{\varepsilon}{2}, |\langle Sx_i, y_i\rangle| < \frac{\varepsilon}{2}\right\} \\ &\subseteq \bigcap_{i=1}^{n} \{(T, S) : |\langle (T + S)x_i, y_i\rangle| < \varepsilon\} \\ &= \{(T, S) : (T + S) \in V_\varepsilon(x_i, y_i, n)\} \\ &= \mathbb{a}^{-1}(V_\varepsilon(x_i, y_i, n)) \end{aligned}$$

$$\begin{aligned} B_1(0) \times V_\varepsilon(x_i, y_i, n) &= \bigcap_{i=1}^{n} \{(\lambda, T) \in \mathbb{K} \times L(H) : |\lambda| < 1, |\langle Tx_i, y_i\rangle| < \varepsilon\} \\ &\subseteq \bigcap_{i=1}^{n} \{(\lambda, T) : |\langle (\lambda T)x_i, y_i\rangle| < \varepsilon\} = \mathbb{m}^{-1}(V_\varepsilon(x_i, y_i, n)) \end{aligned}$$

*which prove that $\mathbb{a}$ and $\mathbb{m}$ are both continuous. Hence, $(L(H), \tau_w)$ is a TVS.*

*Furthermore, we can show that the multiplication in $(L(H), \tau_w)$ is separately continuous. For a fixed $T \in L(H)$ denote by $T^*$ the adjoint of $T$ and set $z_i := T^* y_i$ for $i = 1, \ldots, n$. Then*

$$\begin{aligned} T \circ V_\varepsilon(x_i, z_i, n) &= \{T \circ S : |\langle Sx_i, z_i\rangle| < \varepsilon, i = 1, \ldots, n\} \\ &\subseteq \{W \in L(H) : |\langle Wx_i, y_i\rangle| < \varepsilon, i = 1, \ldots, n\} = V_\varepsilon(x_i, y_i, n), \end{aligned}$$

*where in the latter inequality we used that*

$$|\langle (T\circ S)x_i, y_i\rangle| = |\langle T(Sx_i), y_i\rangle| = |\langle Sx_i, T^*y_i\rangle| = |\langle Sx_i, z_i\rangle| < \varepsilon.$$

*Similarly, we can show that* $V_\varepsilon(x_i, z_i, n)\circ T \subseteq V_\varepsilon(x_i, y_i, n)$. *Hence,* $\mathcal{B}_w$ *fulfills a) and b) in Theorem 1.2.9 and so we have that* $(L(H), \tau_w)$ *is a TA.*

- *the multiplication in* $(L(H), \tau_w)$ *is not jointly continuous.*

*Let us preliminarily observe that a sequence* $(W_j)_{j\in\mathbb{N}}$ *of elements in* $L(H)$ *converges to* $W\in L(H)$ *w.r.t.* $\tau_w$, *in symbols* $W_j \overset{\tau_w}{\to} W$, *if and only if for all* $x, y\in H$ *we have* $\langle W_j x, y\rangle \to \langle Wx, y\rangle$[3]. *As* $H$ *is separable, there exists a countable orthonormal basis* $\{e_k\}_{k\in\mathbb{N}}$ *for* $H$. *Define* $S\in L(H)$ *such that* $S(e_1) := o$ *and* $S(e_k) := e_{k-1}$ *for all* $k\in\mathbb{N}$ *with* $k\geq 2$. *Then for any* $n\in\mathbb{N}$, *the operator* $T_n := S^n = \Big(\underbrace{S\circ\cdots\circ S}_{n \text{ times}}\Big)$ *is such that* $T_n \overset{\tau_w}{\to} o$ *as* $n\to\infty$. *Indeed, for any* $x\in H$ *there exist unique* $\lambda_k\in\mathbb{K}$ *such that* $x = \sum_{k=1}^\infty \lambda_k e_k$ [4] *and so*

$$\begin{aligned}\|T_n x\| &= \left\|\sum_{k=1}^\infty \lambda_k T_n(e_k)\right\| = \left\|\sum_{k=n+1}^\infty \lambda_k T_n(e_k)\right\| = \left\|\sum_{k=n+1}^\infty \lambda_k e_{k-n}\right\| \\ &= \left\|\sum_{k=1}^\infty \lambda_{k+n} e_k\right\| \overset{4}{=} \sum_{k=1}^\infty |\lambda_{k+n}|^2 = \sum_{k=n+1}^\infty |\lambda_k|^2 \to 0, \text{ as } n\to\infty\end{aligned}$$

*which implies that* $\langle T_n x, y\rangle \to 0$ *as* $n\to\infty$ *since* $|\langle T_n x, y\rangle| \leq \|T_n x\|\|y\|$.

*Moreover, the adjoint of* $S$ *is the continuous linear operator* $S^* : H\to H$ *such that* $S^*(e_k) = e_{k+1}$ *for all* $k\in\mathbb{N}$. *Hence, for any* $n\in\mathbb{N}$ *we have that* $T_n^* = (S^n)^* = (S^*)^n$ *and we can easily show that also* $T_n^* \overset{\tau_w}{\to} o$. *In fact, for any* $x, y\in H$ *we have that* $|\langle T_n^* x, y\rangle| = |\langle x, T_n y\rangle| \leq \|x\|\|T_n y\| \to 0$ *as* $n\to\infty$. *However, we have* $S^*S = I$ *where* $I$ *denotes the identity map on* $H$, *which gives in turn that* $T_n^*\circ T_n = I$ *for any* $n\in\mathbb{N}$. *Hence, for any* $n\in\mathbb{N}$ *and any* $x, y\in H$ *we have that* $\langle (T_n^*\circ T_n)x, y\rangle = \langle x, y\rangle$ *and so that* $T_n^*\circ T_n \overset{\tau_w}{\not\to} o$ *as* $n\to\infty$, *which proves that* $\circ$ *is not jointly continuous.*

[3] Indeed, we have

$$\begin{aligned} W_j \overset{\tau_w}{\to} W &\iff \forall\varepsilon>0, n\in\mathbb{N}, x_i, y_i\in H, \exists \bar{j}\in\mathbb{N} : \forall j\geq\bar{j}, W_j - W\in V_\varepsilon(x_i, y_i, n) \\ &\iff \forall\varepsilon>0, n\in\mathbb{N}, x_i, y_i\in H, \exists \bar{j}\in\mathbb{N} : \forall j\geq\bar{j}, |\langle (W_j - W)x_i, y_i\rangle| < \varepsilon \\ &\iff \forall n\in\mathbb{N}, x_i, y_i\in H, \langle (W_j - W)x_i, y_i\rangle \to 0, \text{ as } j\to\infty \\ &\iff \forall x, y\in H, \langle (W_j - W)x, y\rangle \to 0, \text{ as } j\to\infty.\end{aligned}$$

[4] Recall that if $\{h_i\}_{i\in I}$ is an orthonormal basis of a Hilbert space $H$ then for each $y\in H$ $y = \sum_{i\in I}\langle y, h_i\rangle h_i$ and $\|y\|^2 = \sum_{i\in I}|\langle y, h_i\rangle|^2$ (see e.g. [13, Theorem II.6] for a proof)

*Let* $\tau_s$ *be the* ***strong operator topology*** *or* topology of pointwise convergence *on* $L(H)$*, i.e. the coarsest topology on* $L(H)$ *such that all the maps* $E_x : L(H) \to H, T \mapsto Tx$ $(x \in H)$ *are continuous. A basis of neighbourhoods of the origin in* $(L(H), \tau_s)$ *is given by:*

$$\mathcal{B}_s := \{U_\varepsilon(x_i, n) : \varepsilon > 0, n \in \mathbb{N}, x_1, \dots, x_n \in H\},$$

*where* $U_\varepsilon(x_i, n) := \{T \in L(H) : \|Tx_i\|_H < \varepsilon, i = 1, \dots, n\}$*.*

- *$(L(H), \tau_s)$ is a TA.*

*For any* $r > 0$*, denote by* $B_r(o)$ *(resp.* $B_r(0)$*) the open unit ball centered at* $o$ *in* $H$ *(resp. at* $0$ *in* $\mathbb{K}$*). Then for any* $\varepsilon > 0$*,* $n \in \mathbb{N}$*,* $x_1, \dots, x_n \in H$ *we have:*

$$\begin{aligned} U_{\frac{\varepsilon}{2}}(x_i, n) \times U_{\frac{\varepsilon}{2}}(x_i, n) &= \left\{(T, S) : Tx_i, Sx_i \in B_{\frac{\varepsilon}{2}}(o), i = 1, \dots, n\right\} \\ &\subseteq \{(T, S) : \|(T+S)x_i\|_H < \varepsilon, i = 1, \dots, n\} \\ &= \{(T, S) : (T+S) \in U_\varepsilon(x_i, n)\} = \mathbb{a}^{-1}(U_\varepsilon(x_i, n)) \end{aligned}$$

$$\begin{aligned} B_1(0) \times U_\varepsilon(x_i, n) &= \{(\lambda, T) \in \mathbb{K} \times L(H) : |\lambda| < 1, \|Tx_i\|_H < \varepsilon, i = 1, \dots, n\} \\ &\subseteq \{(\lambda, T) : \|(\lambda T)x_i\|_H < \varepsilon, i = 1, \dots, n\} = \mathbb{m}^{-1}(U_\varepsilon(x_i, n)) \end{aligned}$$

*which prove that* $\mathbb{a}$ *and* $\mathbb{m}$ *are both continuous.*

*Furthermore, we can show that the multiplication in* $(L(H), \tau_s)$ *is separately continuous. Fixed* $T \in L(H)$*, its continuity implies that* $T^{-1}(B_\varepsilon(o))$ *is a neighbourhood of* $o$ *in* $H$ *and so that there exists* $\eta > 0$ *such that* $B_\eta(o) \subseteq T^{-1}(B_\varepsilon(o))$*. Therefore, we get:*

$$\begin{aligned} T \circ U_\eta(x_i, n) &= \{T \circ S : S \in L(H) \text{ with } Sx_i \in B_\eta(o), i = 1, \dots, n\} \\ &\subseteq \{W \in L(H) : Wx_i \in B_\varepsilon(o), i = 1, \dots, n\} \\ &= U_\varepsilon(x_i, n), \end{aligned}$$

*where in the latter inequality we used that*

$$(T \circ S)x_i = T(Sx_i) \in T(B_\eta(o)) \subseteq T(T^{-1}(B_\varepsilon(o))) \subseteq B_\varepsilon(o).$$

*Similarly, we can show that* $U_\eta(x_i, n) \circ T \subseteq U_\varepsilon(x_i, n)$*. Hence,* $\mathcal{B}_s$ *fulfills a) and b) in Theorem 1.2.9 and so we have that* $(L(H), \tau_s)$ *is a TA.*

- *the multiplication in* $(L(H), \tau_s)$ *is not jointly continuous*

*(proof in next lecture!)*

*Note that* $L(H)$ *endowed with the* ***operator norm*** $\|\cdot\|$ *is instead a normed algebra and so has jointly continuous multiplication. Recall that the operator norm is defined by* $\|T\| := \sup_{x \in H \setminus \{o\}} \frac{\|Tx\|_H}{\|x\|_H}$*,* $\forall T \in L(H)$*.*

## 1.3 Hausdorffness and unitizations of a TA

Topological algebras are in particular topological spaces so their Hausdorfness can be established just by verifying the usual definition of Hausdorff topological space.

**Definition 1.3.1.** *A topological space $X$ is said to be* Hausdorff *or* (T2) *if any two distinct points of $X$ have neighbourhoods without common points; or equivalently if two distinct points always lie in disjoint open sets.*

However, a TA is more than a mere topological space but it is also a TVS. This provides TAs with the following characterization of their Hausdorfness which holds in general for any TVS.

**Proposition 1.3.2.** *For a TVS $X$ the following are equivalent:*
*a) $X$ is Hausdorff.*
*b) $\{o\}$ is closed in $X$.*
*c) The intersection of all neighbourhoods of the origin $o$ is just $\{o\}$.*
*d) $\forall\, o \neq x \in X$, $\exists\, U \in \mathcal{F}(o)$ s.t. $x \notin U$.*

Since the topology of a TVS is translation invariant, property (d) means that the TVS is a (T1)[5]topological space. Recall for general topological spaces (T2) always implies (T1), but the converse does not always hold (c.f. Example 1.1.41-4 in [9]). However, Proposition 1.3.2 ensures that for TVS and so for TAs the two properties are equivalent.

*Proof.*
Let us just show that (d) implies (a) (for a complete proof see [9, Proposition 2.2.3, Corollary 2.2.4] or even better try it yourself!).

Suppose that (d) holds and let $x, y \in X$ with $x \neq y$, i.e. $x - y \neq o$. Then there exists $U \in \mathcal{F}(o)$ s.t. $x - y \notin U$. By (2) and (5) of Theorem 1.2.6, there exists $V \in \mathcal{F}(o)$ balanced and s.t. $V + V \subset U$. Since $V$ is balanced $V = -V$ then we have $V - V \subset U$. Suppose now that $(V + x) \cap (V + y) \neq \emptyset$, then there exists $z \in (V + x) \cap (V + y)$, i.e. $z = v + x = w + y$ for some $v, w \in V$. Then $x - y = w - v \in V - V \subset U$ and so $x - y \in U$ which is a contradiction. Hence, $(V + x) \cap (V + y) = \emptyset$ and by Proposition 1.2.4 we know that $V + x \in \mathcal{F}(x)$ and $V + y \in \mathcal{F}(y)$. Hence, $X$ is Hausdorff. □

[5] A topological space $X$ is said to be *(T1)* if, given two distinct points of $X$, each lies in a neighborhood which does not contain the other point; or equivalently if, for any two distinct points, each of them lies in an open subset which does not contain the other point.

We have already seen that a $\mathbb{K}-$algebra can be always embedded in a unital one, called unitization see Definition 1.1.3-4) . In the rest of this section, we will discuss about which topologies on the unitization of a $\mathbb{K}-$algebra makes it into a TA. To start with, let us look at normed algebras.

**Proposition 1.3.3.** *If $A$ is a normed algebra, then there always exists a norm on its unitization $A_1$ making both $A_1$ into a normed algebra and the canonical embedding an isometry. Such a norm is called a* unitization norm*.*

*Proof.*
Let $(A, \|\cdot\|)$ be a normed algebra and $A_1 = \mathbb{K} \times A$ its unitization. Define

$$\|(k, a)\|_1 := |k| + \|a\|, \ \forall k \in \mathbb{K}, a \in A.$$

Then $\|(1, o)\|_1 = 1$ and it is straightforward that $\|\cdot\|_1$ is a norm on $A_1$ since $|\cdot|$ is a norm on $\mathbb{K}$ and $\|\cdot\|$ is a norm on $A$. Also, for any $\lambda, k \in \mathbb{K}, a, b \in A$ we have:

$$\begin{aligned}
\|(k,a)(\lambda,b)\|_1 &= \|(k\lambda, ka + \lambda b + ab)\|_1 = |k\lambda| + \|ka + \lambda b + ab\| \\
&\leq |k||\lambda| + k\|a\| + \lambda\|b\| + \|a\|\|b\| = |k|(|\lambda| + \|b\|) + \|a\|(|\lambda| + \|b\|) \\
&= (|k| + \|a\|)(|\lambda| + \|b\|) = \|(k,a)\|_1\|(\lambda,b)\|_1.
\end{aligned}$$

This proves that $(A_1, \|\cdot\|_1)$ is a unital normed algebra. Moreover, the canonical embedding $\mathbb{e} : A \to A_1, a \mapsto (0, a)$ is an isometry because $\|\mathbb{e}(a)\|_1 = |0| + \|a\| = \|a\|$ for all $a \in A$. This in turn gives that $\mathbb{e}$ is continuous and so a topological embedding. □

**Remark 1.3.4.** *Note that $\|\cdot\|_1$ induces the product topology on $A_1$ given by $(\mathbb{K}, |\cdot|)$ and $(A, \|\cdot\|)$ but there might exist other unitization norms on $A_1$ not necessarily equivalent to $\|\cdot\|_1$ (see Sheet 1).*

The latter remark suggests the following generalization of Proposition 1.3.3 to any TA.

**Proposition 1.3.5.** *Let $A$ be a TA. Its unitization $A_1$ equipped with the corresponding product topology is a TA and $A$ is topologically embedded in $A_1$. Note that $A_1$ is Hausdorff if and only if $A$ is Hausdorff.*

*Proof.* Suppose $(A, \tau)$ is a TA. By Proposition 1.1.4, we know that the unitization $A_1$ of $A$ is a $\mathbb{K}-$algebra. Moreover, since $(\mathbb{K}, |\cdot|)$ and $(A, \tau)$ are both TVS, we have that $A_1 := \mathbb{K} \times A$ endowed with the corresponding product topology $\tau_{prod}$ is also a TVS. Then the definition of multiplication in $A_1$ together with the fact that the multiplication in $A$ is separately continuous imply that the multiplication in $A_1$ is separately continuous, too. Hence, $(A_1, \tau_{prod})$ is a TA.

The canonical embedding $\mathbb{e}$ of $A$ in $A_1$ is then a continuous monomorphism, since for any $U$ neighbourhood of $(0, o)$ in $(A_1, \tau_{prod})$ there exist $\varepsilon > 0$ and a neighbourhood $V$ of $o$ in $(A, \tau)$ such that $B_\varepsilon(0) \times V \subseteq U$ and so $V = \mathbb{e}^{-1}(B_\varepsilon(0) \times V) \subseteq \mathbb{e}^{-1}(U)$. Hence, $(A, \tau)$ is topologically embedded in $(A_1, \tau_{prod})$.

Finally, recall that the cartesian product of topological spaces endowed with the corresponding product topology is Hausdorff iff each of them is Hausdorff. Then, as $(\mathbb{K}, |\cdot|)$ is Hausdorff, it is clear that $(A_1, \tau_{prod})$ is Hausdorff iff $(A, \tau)$ is Hausdorff. [6] □

If $A$ is a TA with continuous multiplication, then $A_1$ endowed with the corresponding product topology is also a TA with continuous multiplication. Moreover, from Remark 1.3.4, it is clear that the product topology is not the unique one making the unitization of a TA into a TA itself.

## 1.4 Subalgebras and quotients of a TA

In this section we are going to see some methods which allow us to construct new TAs from a given one. In particular, we will see under which conditions the TA structure is preserved under taking subalgebras and quotients.

Let us start with an immediate application of Theorem 1.2.9.

**Proposition 1.4.1.** *Let $X$ be a $\mathbb{K}-$algebra, $(Y, \omega)$ a TA (resp. TA with continuous multiplication) over $\mathbb{K}$ and $\varphi : X \to Y$ a homomorphism. Denote by $\mathcal{B}_\omega$ a basis of neighbourhoods of the origin in $(Y, \omega)$. Then the collection $\mathcal{B} := \{\varphi^{-1}(U) : U \in \mathcal{B}_\omega\}$ is a basis of neighbourhoods of the origin for a topology $\tau$ on $X$ such that $(X, \tau)$ is a TA (resp. TA with continuous multiplication).*

The topology $\tau$ constructed in the previous proposition is usually called *initial topology* or *inverse image topology* induced by $\varphi$.

**Corollary 1.4.2.** *Let $(A, \omega)$ be a TA (resp. TA with continuous multiplication) and $M$ a subalgebra of $A$. If we endow $M$ with the relative topology $\tau_M$ induced by $A$, then $(M, \tau_M)$ is a TA.*

[6] Alternative proof:
$A$ Hausdorff $\overset{1.3.2}{\Longleftrightarrow}$ $\{o\}$ closed in $A$ $\overset{\{0\}\text{closed in } \mathbb{K}}{\Longleftrightarrow}$ $\{(0, o)\}$ closed in $A_1$ $\overset{1.3.2}{\Longleftrightarrow}$ $(A_1, \tau_{prod})$ Hausdorff.

*Proof.*
Consider the identity map $id : M \to A$ and let $\mathcal{B}_\omega$ a basis of neighbourhoods of the origin in $(A, \omega)$ Clearly, $id$ is a homomorphism and the initial topology induced by $id$ on $M$ is nothing but the relative topology $\tau_M$ induced by $A$ since

$$\{id^{-1}(U) : U \in \mathcal{B}_\omega\} = \{U \cap M : U \in \mathcal{B}_\omega\} = \tau_M.$$

Hence, Proposition 1.4.1 ensures that $(M, \tau_M)$ is a TA. □

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