So far we have seen only examples of TA with continuous multiplication. In the following example, we will introduce a TA whose multiplication is separately continuous but not jointly continuous.

**Example 1.2.17.**
*Let $(H, \langle \cdot, \cdot, \rangle)$ be an infinite dimensional separable Hilbert space over $\mathbb{K}$. Denote by $\|\cdot\|_H$ the norm on $H$ defined as $\|x\|_H := \sqrt{\langle x, x\rangle}$ for all $x \in H$, and by $L(H)$ the set of all linear and continuous maps from $H$ to $H$. The set $L(H)$ equipped with the pointwise addition $\mathbb{a}$, the pointwise scalar multiplication $\mathbb{m}$ and the composition of maps $\circ$ as multiplication is a $\mathbb{K}-$algebra.*
*Let $\tau_w$ be the **weak operator topology** on $L(H)$, i.e. the coarsest topology on $L(H)$ such that all the maps $E_{x,y} : L(H) \to H, T \mapsto \langle Tx, y\rangle$ $(x, y \in H)$ are continuous. A basis of neighbourhoods of the origin in $(L(H), \tau_w)$ is given by:*

$$\mathcal{B}_w := \{V_\varepsilon(x_i, y_i, n) : \varepsilon > 0, n \in \mathbb{N}, x_1, \ldots, x_n, y_1, \ldots, y_n \in H\},$$

*where $V_\varepsilon(x_i, y_i, n) := \{W \in L(H) : |\langle W x_i, y_i\rangle| < \varepsilon, i = 1, \ldots, n\}$.*

• *$\underline{(L(H), \tau_w) \text{ is a TA}}$.*
*For any $\varepsilon > 0$, $n \in \mathbb{N}$, $x_1, \ldots, x_n, y_1, \ldots, y_n \in H$, using the bilinearity of the inner product we easily have:*

$$\begin{aligned} V_{\frac{\varepsilon}{2}}(x_i, y_i, n) \times V_{\frac{\varepsilon}{2}}(x_i, y_i, n) &= \bigcap_{i=1}^{n} \left\{(T, S) : |\langle T x_i, y_i\rangle| < \frac{\varepsilon}{2}, |\langle S x_i, y_i\rangle| < \frac{\varepsilon}{2}\right\} \\ &\subseteq \bigcap_{i=1}^{n} \{(T, S) : |\langle (T + S)x_i, y_i\rangle| < \varepsilon\} \\ &= \{(T, S) : (T + S) \in V_\varepsilon(x_i, y_i, n)\} \\ &= \mathbb{a}^{-1}(V_\varepsilon(x_i, y_i, n)), \end{aligned}$$

$$\begin{aligned} B_1(0) \times V_\varepsilon(x_i, y_i, n) &= \bigcap_{i=1}^{n} \{(\lambda, T) \in \mathbb{K} \times L(H) : |\lambda| < 1, |\langle T x_i, y_i\rangle| < \varepsilon\} \\ &\subseteq \bigcap_{i=1}^{n} \{(\lambda, T) : |\langle (\lambda T)x_i, y_i\rangle| < \varepsilon\} = \mathbb{m}^{-1}(V_\varepsilon(x_i, y_i, n)) \end{aligned}$$

*which prove that $\mathbb{a}$ and $\mathbb{m}$ are both continuous. Hence, $(L(H), \tau_w)$ is a TVS.*

*Furthermore, we can show that the multiplication in $(L(H), \tau_w)$ is separately continuous. For a fixed $T \in L(H)$ denote by $T^*$ the adjoint of $T$ and set $z_i := T^* y_i$ for $i = 1, \ldots, n$. Then*

$$\begin{aligned} T \circ V_\varepsilon(x_i, z_i, n) &= \{T \circ S : |\langle S x_i, z_i\rangle| < \varepsilon, i = 1, \ldots, n\} \\ &\subseteq \{W \in L(H) : |\langle W x_i, y_i\rangle| < \varepsilon, i = 1, \ldots, n\} = V_\varepsilon(x_i, y_i, n), \end{aligned}$$

*where in the latter inequality we used that*

$$|\langle (T\circ S)x_i, y_i\rangle| = |\langle T(Sx_i), y_i\rangle| = |\langle Sx_i, T^*y_i\rangle| = |\langle Sx_i, z_i\rangle| < \varepsilon.$$

*Similarly, we can show that* $V_\varepsilon(x_i, z_i, n)\circ T \subseteq V_\varepsilon(x_i, y_i, n)$. *Hence,* $\mathcal{B}_w$ *fulfills a) and b) in Theorem 1.2.9 and so we have that* $(L(H), \tau_w)$ *is a TA.*

- *the multiplication in* $(L(H), \tau_w)$ *is not jointly continuous.*

*Let us preliminarily observe that a sequence* $(W_j)_{j\in\mathbb{N}}$ *of elements in* $L(H)$ *converges to* $W \in L(H)$ *w.r.t.* $\tau_w$, *in symbols* $W_j \overset{\tau_w}{\to} W$, *if and only if for all* $x, y \in H$ *we have* $\langle W_j x, y\rangle \to \langle Wx, y\rangle$[3]. *As* $H$ *is separable, there exists a countable orthonormal basis* $\{e_k\}_{k\in\mathbb{N}}$ *for* $H$. *Define* $S \in L(H)$ *such that* $S(e_1) := o$ *and* $S(e_k) := e_{k-1}$ *for all* $k \in \mathbb{N}$ *with* $k \geq 2$. *Then the operator*

$$T_n := S^n = \Big(\underbrace{S\circ\cdots\circ S}_{n \text{ times}}\Big), \quad n\in\mathbb{N} \tag{1.3}$$

*is s.t.* $T_n \overset{\tau_w}{\to} o$ *as* $n\to\infty$. *Indeed,* $\forall x\in H$, $\exists!\,\lambda_k\in\mathbb{K}: x = \sum_{k=1}^\infty \lambda_k e_k$ [4] *so*

$$\begin{aligned}\|T_n x\| &= \left\|\sum_{k=1}^\infty \lambda_k T_n(e_k)\right\| = \left\|\sum_{k=n+1}^\infty \lambda_k T_n(e_k)\right\| = \left\|\sum_{k=n+1}^\infty \lambda_k e_{k-n}\right\| \\ &= \left\|\sum_{k=1}^\infty \lambda_{k+n} e_k\right\| \overset{4}{=} \sum_{k=1}^\infty |\lambda_{k+n}|^2 = \sum_{k=n+1}^\infty |\lambda_k|^2 \to 0, \text{ as } n\to\infty\end{aligned}$$

*which implies that* $\langle T_n x, y\rangle \to 0$ *as* $n\to\infty$ *since* $|\langle T_n x, y\rangle| \leq \|T_n x\|\|y\|$.

*Moreover, the adjoint of* $S$ *is the continuous linear operator* $S^*: H\to H$ *such that* $S^*(e_k) = e_{k+1}$ *for all* $k\in\mathbb{N}$. *Hence, for any* $n\in\mathbb{N}$ *we have that* $T_n^* = (S^n)^* = (S^*)^n$ *and we can easily show that also* $T_n^* \overset{\tau_w}{\to} o$. *In fact, for any* $x, y\in H$*we have that* $|\langle T_n^* x, y\rangle| = |\langle x, T_n y\rangle| \leq \|x\|\|T_n y\| \to 0$ *as* $n\to\infty$. *However, we have* $S^*S = I$ *where* $I$ *denotes the identity map on* $H$, *which gives in turn that* $T_n^*\circ T_n = I$ *for any* $n\in\mathbb{N}$. *Hence, for any* $n\in\mathbb{N}$ *and any* $x, y\in H$ *we have that* $\langle (T_n^*\circ T_n)x, y\rangle = \langle x, y\rangle$ *and so that* $T_n^*\circ T_n \overset{\tau_w}{\not\to} o$ *as* $n\to\infty$, *which proves that* $\circ$ *is not jointly continuous.*

[3]Indeed, we have

$$\begin{aligned} W_j \overset{\tau_w}{\to} W &\iff \forall\varepsilon>0, n\in\mathbb{N}, x_i, y_i\in H, \exists\bar{j}\in\mathbb{N}: \forall j\geq\bar{j}, W_j - W\in V_\varepsilon(x_i, y_i, n)\\ &\iff \forall\varepsilon>0, n\in\mathbb{N}, x_i, y_i\in H, \exists\bar{j}\in\mathbb{N}: \forall j\geq\bar{j}, |\langle (W_j-W)x_i, y_i\rangle| < \varepsilon\\ &\iff \forall n\in\mathbb{N}, x_i, y_i\in H, \langle (W_j-W)x_i, y_i\rangle \to 0, \text{ as } j\to\infty\\ &\iff \forall x, y\in H, \langle (W_j-W)x, y\rangle\to 0, \text{ as } j\to\infty.\end{aligned}$$

[4]Recall that if $\{h_i\}_{i\in I}$ is an orthonormal basis of a Hilbert space $H$ then for each $y\in H$ $y = \sum_{i\in I}\langle y, h_i\rangle h_i$ and $\|y\|^2 = \sum_{i\in I}|\langle y, h_i\rangle|^2$ (see e.g. [13, Theorem II.6] for a proof)

*Let $\tau_s$ be the* ***strong operator topology*** *or* topology of pointwise convergence *on $L(H)$, i.e. the coarsest topology on $L(H)$ such that all the maps $E_x : L(H) \to H, T \mapsto Tx$ ($x \in H$) are continuous. A basis of neighbourhoods of the origin in $(L(H), \tau_s)$ is given by:*

$$\mathcal{B}_s := \{U_\varepsilon(x_i, n) : \varepsilon > 0, n \in \mathbb{N}, x_1, \ldots, x_n \in H\},$$

*where $U_\varepsilon(x_i, n) := \{T \in L(H) : \|Tx_i\|_H < \varepsilon, i = 1, \ldots, n\}$.*

• *$(L(H), \tau_s)$ is a TA.*
*For any $r > 0$, denote by $B_r(o)$ (resp. $B_r(0)$) the open unit ball centered at $o$ in $H$ (resp. at $0$ in $\mathbb{K}$). Then for any $\varepsilon > 0$, $n \in \mathbb{N}$, $x_1, \ldots, x_n \in H$ we have:*

$$\begin{aligned} U_{\frac{\varepsilon}{2}}(x_i, n) \times U_{\frac{\varepsilon}{2}}(x_i, n) &= \left\{(T, S) : Tx_i, Sx_i \in B_{\frac{\varepsilon}{2}}(o), i = 1, \ldots, n\right\} \\ &\subseteq \{(T, S) : \|(T + S)x_i\|_H < \varepsilon, i = 1, \ldots, n\} \\ &= \{(T, S) : (T + S) \in U_\varepsilon(x_i, n)\} = \mathrm{a}^{-1}(U_\varepsilon(x_i, n)) \end{aligned}$$

$$\begin{aligned} B_1(0) \times U_\varepsilon(x_i, n) &= \{(\lambda, T) \in \mathbb{K} \times L(H) : |\lambda| < 1, \|Tx_i\|_H < \varepsilon, i = 1, \ldots, n\} \\ &\subseteq \{(\lambda, T) : \|(\lambda T)x_i\|_H < \varepsilon, i = 1, \ldots, n\} = \mathrm{m}^{-1}(U_\varepsilon(x_i, n)) \end{aligned}$$

*which prove that $\mathrm{a}$ and $\mathrm{m}$ are both continuous.*

*Furthermore, we can show that the multiplication in $(L(H), \tau_s)$ is separately continuous. Fixed $T \in L(H)$, its continuity implies that $T^{-1}(B_\varepsilon(o))$ is a neighbourhood of $o$ in $H$ and so that there exists $\eta > 0$ such that $B_\eta(o) \subseteq T^{-1}(B_\varepsilon(o))$. Therefore, we get:*

$$\begin{aligned} T \circ U_\eta(x_i, n) &= \{T \circ S : S \in L(H) \text{ with } Sx_i \in B_\eta(o), i = 1, \ldots, n\} \\ &\subseteq \{W \in L(H) : Wx_i \in B_\varepsilon(o), i = 1, \ldots, n\} \\ &= U_\varepsilon(x_i, n), \end{aligned}$$

*where in the latter inequality we used that*

$$(T \circ S)x_i = T(Sx_i) \in T(B_\eta(o)) \subseteq T(T^{-1}(B_\varepsilon(o))) \subseteq B_\varepsilon(o).$$

*Similarly, we can show that $U_\eta(x_i, n) \circ T \subseteq U_\varepsilon(x_i, n)$. Hence, $\mathcal{B}_s$ fulfills a) and b) in Theorem 1.2.9 and so we have that $(L(H), \tau_s)$ is a TA.*

• *the multiplication in $(L(H), \tau_s)$ is not jointly continuous*
*It is enough to show that there exists a neighbourhood of the origin in $(L(H), \tau_s)$ which does not contain the product of any other two such neighbourhoods. More precisely, we will show $\exists\, \varepsilon > 0$, $\exists\, x_0 \in H$ s.t. $\forall \varepsilon_1, \varepsilon_2 > 0$, $\forall p, q \in \mathbb{N}$,*

*$\forall x_1, \dots, x_p, y_1, \dots, y_q \in H$ we have $U_{\varepsilon_1}(x_i, p) \circ U_{\varepsilon_2}(y_i, q) \not\subseteq U_\varepsilon(x_0)$, i.e. there exist $A \in U_{\varepsilon_1}(x_i, p)$ and $B \in U_{\varepsilon_2}(y_i, q)$ with $B \circ A \notin U_\varepsilon(x_0)$.*

*Choose $0 < \varepsilon < 1$ and $x_0 \in H$ s.t. $\|x_0\| = 1$. For any $\varepsilon_1, \varepsilon_2 > 0, p, q \in \mathbb{N}$, $x_1, \dots, x_p, y_1, \dots, y_q \in H$, take*

$$0 < \delta < \frac{\varepsilon_2}{\max\limits_{i=1,\dots,q} \|y_i\|} \tag{1.4}$$

*and $n \in \mathbb{N}$ such that*

$$\|T_n(x_k)\| < \delta\varepsilon_1, \text{for } k = 1, \dots, p, \tag{1.5}$$

*where $T_n$ is defined as in (1.3). (Note that we can choose such an $n$ as we showed above that $\|T_j x\| \to 0$ as $j \to \infty$). Setting $A := \frac{1}{\delta} T_n$ and $B := \delta T_n^*$ we get that:*

$$\|Ax_k\| = \frac{1}{\delta}\|T_n x_k\| \overset{(1.5)}{<} \varepsilon_1, \text{ for } k = 1, \dots, p$$

*and*

$$\|By_i\| = \delta\|T_n^* y_i\| \overset{(4)}{=} \delta\|y_i\| \overset{(1.4)}{<} \varepsilon_2, \text{ for } i = 1, \dots, q.$$

*Hence, $A \in U_{\varepsilon_1}(x_i, p)$ and $B \in U_{\varepsilon_2}(y_i, q)$ but $B \circ A \notin U_\varepsilon(x_0)$ because*

$$\|(B \circ A)x_0\| = \|(T_n^* T_n)x_0\| = \|x_0\| = 1 > \varepsilon.$$

*Note that $L(H)$ endowed with the **operator norm** $\|\cdot\|$ is instead a normed algebra and so has jointly continuous multiplication. Recall that the operator norm is defined by $\|T\| := \sup\limits_{x \in H \setminus \{o\}} \frac{\|Tx\|_H}{\|x\|_H}$, $\forall T \in L(H)$.*

## 1.3 Hausdorffness and unitizations of a TA

Topological algebras are in particular topological spaces so their Hausdorffness can be established just by verifying the usual definition of Hausdorff topological space.

**Definition 1.3.1.** *A topological space $X$ is said to be* Hausdorff *or* (T2) *if any two distinct points of $X$ have neighbourhoods without common points; or equivalently if two distinct points always lie in disjoint open sets.*

However, a TA is more than a mere topological space but it is also a TVS. This provides TAs with the following characterization of their Hausdorffness which holds in general for any TVS.

**Proposition 1.3.2.** *For a TVS $X$ the following are equivalent:*
*a) $X$ is Hausdorff.*
*b) $\{o\}$ is closed in $X$.*
*c) The intersection of all neighbourhoods of the origin $o$ is just $\{o\}$.*
*d) $\forall\, o \neq x \in X$, $\exists\, U \in \mathcal{F}(o)$ s.t. $x \notin U$.*

Since the topology of a TVS is translation invariant, property (d) means that the TVS is a (T1)[5]topological space. Recall for general topological spaces (T2) always implies (T1), but the converse does not always hold (c.f. Example 1.1.41-4 in [9]). However, Proposition 1.3.2 ensures that for TVS and so for TAs the two properties are equivalent.

*Proof.*
Let us just show that (d) implies (a) (for a complete proof see [9, Proposition 2.2.3, Corollary 2.2.4] or even better try it yourself!).

Suppose that (d) holds and let $x, y \in X$ with $x \neq y$, i.e. $x - y \neq o$. Then there exists $U \in \mathcal{F}(o)$ s.t. $x - y \notin U$. By (2) and (5) of Theorem 1.2.6, there exists $V \in \mathcal{F}(o)$ balanced and s.t. $V + V \subset U$. Since $V$ is balanced $V = -V$ then we have $V - V \subset U$. Suppose now that $(V + x) \cap (V + y) \neq \emptyset$, then there exists $z \in (V + x) \cap (V + y)$, i.e. $z = v + x = w + y$ for some $v, w \in V$. Then $x - y = w - v \in V - V \subset U$ and so $x - y \in U$ which is a contradiction. Hence, $(V + x) \cap (V + y) = \emptyset$ and by Proposition 1.2.4 we know that $V + x \in \mathcal{F}(x)$ and $V + y \in \mathcal{F}(y)$. Hence, $X$ is Hausdorff. □

We have already seen that a $\mathbb{K}-$algebra can be always embedded in a unital one, called unitization see Definition 1.1.3-4) . In the rest of this section, we will discuss about which topologies on the unitization of a $\mathbb{K}-$algebra makes it into a TA. To start with, let us look at normed algebras.

**Proposition 1.3.3.** *If $A$ is a normed algebra, then there always exists a norm on its unitization $A_1$ making both $A_1$ into a normed algebra and the canonical embedding an isometry. Such a norm is called a* unitization norm*.*

*Proof.*
Let $(A, \|\cdot\|)$ be a normed algebra and $A_1 = \mathbb{K} \times A$ its unitization. Define

$$\|(k, a)\|_1 := |k| + \|a\|,\ \forall k \in \mathbb{K}, a \in A.$$

[5] A topological space $X$ is said to be *(T1)* if, given two distinct points of $X$, each lies in a neighborhood which does not contain the other point; or equivalently if, for any two distinct points, each of them lies in an open subset which does not contain the other point.

Then $\|(1,o)\|_1 = 1$ and it is straightforward that $\|\cdot\|_1$ is a norm on $A_1$ since $|\cdot|$ is a norm on $\mathbb{K}$ and $\|\cdot\|$ is a norm on $A$. Also, for any $\lambda, k \in \mathbb{K}, a, b \in A$ we have:

$$\begin{aligned}
\|(k,a)(\lambda,b)\|_1 &= \|(k\lambda, ka+\lambda b+ab)\|_1 = |k\lambda| + \|ka+\lambda b+ab\| \\
&\leq |k||\lambda| + k\|a\| + \lambda\|b\| + \|a\|\|b\| = |k|(|\lambda|+\|b\|) + \|a\|(|\lambda|+\|b\|) \\
&= (|k|+\|a\|)(|\lambda|+\|b\|) = \|(k,a)\|_1\|(\lambda,b)\|_1.
\end{aligned}$$

This proves that $(A_1, \|\cdot\|_1)$ is a unital normed algebra. Moreover, the canonical embedding $\mathrm{e} : A \to A_1, a \mapsto (0,a)$ is an isometry because $\|\mathrm{e}(a)\|_1 = |0| + \|a\| = \|a\|$ for all $a \in A$. This in turn gives that $\mathrm{e}$ is continuous and so a topological embedding. □

**Remark 1.3.4.** *Note that $\|\cdot\|_1$ induces the product topology on $A_1$ given by $(\mathbb{K}, |\cdot|)$ and $(A, \|\cdot\|)$ but there might exist other unitization norms on $A_1$ not necessarily equivalent to $\|\cdot\|_1$ (see Sheet 1, Exercise 3).*

The latter remark suggests the following generalization of Proposition 1.3.3 to any TA.

**Proposition 1.3.5.** *Let $A$ be a TA. Its unitization $A_1$ equipped with the corresponding product topology is a TA and $A$ is topologically embedded in $A_1$. Note that $A_1$ is Hausdorff if and only if $A$ is Hausdorff.*

*Proof.* Suppose $(A,\tau)$ is a TA. By Proposition 1.1.4, we know that the unitization $A_1$ of $A$ is a $\mathbb{K}-$algebra. Moreover, since $(\mathbb{K}, |\cdot|)$ and $(A,\tau)$ are both TVS, we have that $A_1 := \mathbb{K} \times A$ endowed with the corresponding product topology $\tau_{prod}$ is also a TVS. Then the definition of multiplication in $A_1$ together with the fact that the multiplication in $A$ is separately continuous imply that the multiplication in $A_1$ is separately continuous, too. Hence, $(A_1, \tau_{prod})$ is a TA.

The canonical embedding $\mathrm{e}$ of $A$ in $A_1$ is then a continuous monomorphism, since for any $U$ neighbourhood of $(0,o)$ in $(A_1, \tau_{prod})$ there exist $\varepsilon > 0$ and a neighbourhood $V$ of $o$ in $(A,\tau)$ such that $B_\varepsilon(0) \times V \subseteq U$ and so $V = \mathrm{e}^{-1}(B_\varepsilon(0) \times V) \subseteq \mathrm{e}^{-1}(U)$. Hence, $(A,\tau)$ is topologically embedded in $(A_1, \tau_{prod})$.

Finally, recall that the cartesian product of topological spaces endowed with the corresponding product topology is Hausdorff iff each of them is Hausdorff. Then, as $(\mathbb{K}, |\cdot|)$ is Hausdorff, it is clear that $(A_1, \tau_{prod})$ is Hausdorff iff $(A,\tau)$ is Hausdorff. [6] □

[6] Alternative proof:
$A$ Hausdorff $\overset{1.3.2}{\Longleftrightarrow}$ $\{o\}$ closed in $A \overset{\{0\}\text{closed in } \mathbb{K}}{\Longleftrightarrow}$ $\{(0,o)\}$ closed in $A_1$ $\overset{1.3.2}{\Longleftrightarrow}$ $(A_1, \tau_{prod})$ Hausdorff.

If $A$ is a TA with continuous multiplication, then $A_1$ endowed with the corresponding product topology is also a TA with continuous multiplication. Moreover, from Remark 1.3.4, it is clear that the product topology is not the unique one making the unitization of a TA into a TA itself.

## 1.4 Subalgebras and quotients of a TA

In this section we are going to see some methods which allow us to construct new TAs from a given one. In particular, we will see under which conditions the TA structure is preserved under taking subalgebras and quotients.

Let us start with an immediate application of Theorem 1.2.9.

**Proposition 1.4.1.** *Let $X$ be a $\mathbb{K}-$algebra, $(Y, \omega)$ a TA (resp. TA with continuous multiplication) over $\mathbb{K}$ and $\varphi : X \to Y$ a homomorphism. Denote by $\mathcal{B}_\omega$ a basis of neighbourhoods of the origin in $(Y, \omega)$. Then the collection $\mathcal{B} := \{\varphi^{-1}(U) : U \in \mathcal{B}_\omega\}$ is a basis of neighbourhoods of the origin for a topology $\tau$ on $X$ such that $(X, \tau)$ is a TA (resp. TA with continuous multiplication).*

The topology $\tau$ constructed in the previous proposition is usually called *initial topology* or *inverse image topology* induced by $\varphi$.

*Proof.*
We first show that $\mathcal{B}$ is a basis for a filter in $X$.

For any $B_1, B_2 \in \mathcal{B}$, we have $B_1 = \varphi^{-1}(U_1)$ and $B_2 = \varphi^{-1}(U_2)$ for some $U_1, U_2 \in \mathcal{B}_\omega$. Since $\mathcal{B}_\omega$ is a basis of the filter of neighbourhoods of the origin in $(Y, \omega)$, there exists $U_3 \in \mathcal{B}_\omega$ such that $U_3 \subseteq U_1 \cap U_2$ and so $B_3 := \varphi^{-1}(U_3) \subseteq \varphi^{-1}(U_1) \cap \varphi^{-1}(U_2) = B_1 \cap B_2$ and clearly $B_3 \in \mathcal{B}$.

Now consider the filter $\mathcal{F}$ generated by $\mathcal{B}$. For any $M \in \mathcal{F}$, there exists $U \in \mathcal{B}_\omega$ such that $\varphi^{-1}(U) \subseteq M$ and so we have the following:

1. $o_Y \in U$ and so $o_X \in \varphi^{-1}(o_Y) \in \varphi^{-1}(U) = M$.
2. by Theorem 1.2.6-2 applied to the TVS $(Y, \omega)$, we have that there exists $V \in \mathcal{B}_\omega$ such that $V + V \subseteq U$. Hence, setting $N := \varphi^{-1}(V) \in \mathcal{F}$ we have $N + N \subseteq \varphi^{-1}(V + V) \subseteq \varphi^{-1}(U) = M$.
3. by Theorem 1.2.6-3 applied to the TVS $(Y, \omega)$, we have that for any $\lambda \in \mathbb{K} \setminus \{0\}$ there exists $V \in \mathcal{B}_\omega$ such that $V \subseteq \lambda U$. Therefore, setting $N := \varphi^{-1}(V) \in \mathcal{B}$ we have $N \subseteq \varphi^{-1}(\lambda U) = \lambda\varphi^{-1}(U) \subseteq \lambda M$, and so $\lambda M \in \mathcal{F}$.
4. For any $x \in X$ there exists $y \in Y$ such that $x = \varphi^{-1}(y)$. As $U$ is absorbing (by Theorem 1.2.6-4 applied to the TVS $(Y, \omega)$), we have that there exists $\rho > 0$ such that $\lambda y \in U$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq \rho$.

This yields $\lambda x = \lambda\varphi^{-1}(y) = \varphi^{-1}(\lambda y) \in \varphi^{-1}(U) = M$ and hence, $M$ is absorbing in $X$.

5. by Theorem 1.2.6-5 applied to the TVS $(Y, \omega)$, we have that there exists $V \in \mathcal{B}_\omega$ balanced such that $V \subseteq U$. By the linearity of $\varphi$ also $\varphi^{-1}(V)$ is balanced and so, setting $N := \varphi^{-1}(V)$ we have $N \subseteq \varphi^{-1}(U) = M$.

Therefore, we have showed that $\mathcal{F}$ fulfills itself all the 5 properties of Theorem 1.2.6 and so it is a filter of neighbourhoods of the origin for a topology $\tau$ making $(X, \tau)$ a TVS.

Furthermore, for any $x \in X$ and any $B \in \mathcal{B}$ we have that there exist $y \in Y$ and $U \in \mathcal{B}_\omega$ such that $x = \varphi^{-1}(y)$ and $B = \varphi^{-1}(U)$. Then, as $(Y, \omega)$ is a TA, Theorem 1.2.9 guarantees that there exist $V_1, V_2 \in \mathcal{B}_\omega$ such that $yV_1 \subseteq U$ and $V_2 y \subseteq U$. Setting $N_1 := \varphi^{-1}(V_1)$ and $N_2 := \varphi^{-1}(V_2)$, we obtain that $N_1, N_2 \in \mathcal{B}$ and $xN_1 = \varphi^{-1}(y)\varphi^{-1}(V_1) = \varphi^{-1}(yV_1) \subseteq \varphi^{-1}(U) = B$ and $xN_2 = \varphi^{-1}(y)\varphi^{-1}(V_2) = \varphi^{-1}(yV_2) \subseteq \varphi^{-1}(U) = B$. (Similarly, if $(Y, \omega)$ is a TA with continuous multiplication, then one can show that for any $B \in \mathcal{B}$ there exists $N \in \mathcal{B}$ such that $NN \subseteq B$.)

Hence, by Theorem 1.2.9 (resp. Theorem 1.2.10) , $(X, \tau)$ is a TA (resp. TA with continuous multiplication). □

**Corollary 1.4.2.** *Let $(A, \omega)$ be a TA (resp. TA with continuous multiplication) and $M$ a subalgebra of $A$. If we endow $M$ with the relative topology $\tau_M$ induced by $A$, then $(M, \tau_M)$ is a TA (resp. TA with continuous multiplication).*

*Proof.*
Consider the identity map $id : M \to A$ and let $\mathcal{B}_\omega$ a basis of neighbourhoods of the origin in $(A, \omega)$ Clearly, $id$ is a homomorphism and the initial topology induced by $id$ on $M$ is nothing but the relative topology $\tau_M$ induced by $A$ since

$$\{id^{-1}(U) : U \in \mathcal{B}_\omega\} = \{U \cap M : U \in \mathcal{B}_\omega\} = \tau_M.$$

Hence, Proposition 1.4.1 ensures that $(M, \tau_M)$ is a TA (resp. TA with continuous multiplication). □

With similar techniques to the ones used in Proposition 1.4.1 one can show:

**Proposition 1.4.3.** *Let $(X, \omega)$ be a TA (resp. TA with continuous multiplication) over $\mathbb{K}$, $Y$ a $\mathbb{K}-$algebra and $\varphi : X \to Y$ a surjective homomorphism. Denote by $\mathcal{B}_\omega$ a basis of neighbourhoods of the origin in $(X, \omega)$. Then $\mathcal{B} := \{\varphi(U) : U \in \mathcal{B}_\omega\}$ is a basis of neighbourhoods of the origin for a topology $\tau$ on $Y$ such that $(Y, \tau)$ is a TA (resp. TA with continuous multiplication).*

*Proof.* (Sheet 2) □

Using the latter result one can show that the quotient of a TA over an ideal endowed with the quotient topology is a TA (Sheet 2). However, in the following we are going to give a direct proof of this fact without making use of bases. Before doing that, let us briefly recall the notion of quotient topology.

Given a topological space $(X, \omega)$ and an equivalence relation $\sim$ on $X$. The *quotient set* $X/\sim$ is defined to be the set of all equivalence classes w.r.t. to $\sim$. The map $\phi : X \to X/\sim$ which assigns to each $x \in X$ its equivalence class $\phi(x)$ w.r.t. $\sim$ is called the *canonical map* or *quotient map*. Note that $\phi$ is surjective. The*quotient topology* on $X/\sim$ is the collection of all subsets $U$ of $X/\sim$ such that $\phi^{-1}(U) \in \omega$. Hence, the quotient map $\phi$ is continuous and actually the quotient topology on $X/\sim$ is the finest topology on $X/\sim$ such that $\phi$ is continuous.

Note that the quotient map $\phi$ is not necessarily open or closed.

**Example 1.4.4.** *Consider $\mathbb{R}$ with the standard topology given by the modulus and define the following equivalence relation on $\mathbb{R}$:*

$$x \sim y \Leftrightarrow (x = y \vee \{x, y\} \subset \mathbb{Z}).$$

*Let $\mathbb{R}/\sim$ be the quotient set w.r.t $\sim$ and $\phi : \mathbb{R} \to \mathbb{R}/\sim$ the correspondent quotient map. Let us consider the quotient topology on $\mathbb{R}/\sim$. Then $\phi$ is not an open map. In fact, if $U$ is an open proper subset of $\mathbb{R}$ containing an integer, then $\phi^{-1}(\phi(U)) = U \cup \mathbb{Z}$ which is not open in $\mathbb{R}$ with the standard topology. Hence, $\phi(U)$ is not open in $\mathbb{R}/\sim$ with the quotient topology.*

For an example of not closed quotient map see e.g. [9, Example 2.3.3].

Let us consider now a $\mathbb{K}-$algebra $A$ and an ideal $I$ of $A$. We denote by $A/I$ the quotient set $A/\sim_I$, where $\sim_I$ is the equivalence relation on $A$ defined by $x \sim_I y$ iff $x - y \in I$. The canonical (or quotient) map $\phi : A \to A/I$ which assigns to each $x \in A$ its equivalence class $\phi(x)$ w.r.t. the relation $\sim_I$ is clearly surjective.

Using the fact that $I$ is an ideal of the algebra $A$ (see Definition 1.1.3-2), it is easy to check that:

1. if $x \sim_I y$, then $\forall\, \lambda \in \mathbb{K}$ we have $\lambda x \sim_I \lambda y$.
2. if $x \sim_I y$, then $\forall\, z \in A$ we have $x + z \sim_I y + z$.
3. if $x \sim_I y$, then $\forall\, z \in A$ we have $xz \sim_I yz$ and $zx \sim_I zy$ .

These three properties guarantee that the following operations are well-defined on $A/I$:

- vector addition: $\forall\, \phi(x), \phi(y) \in A/I$, $\phi(x) + \phi(y) := \phi(x + y)$
- scalar multiplication: $\forall\, \lambda \in \mathbb{K}$, $\forall\, \phi(x) \in A/I$, $\lambda\phi(x) := \phi(\lambda x)$
- vector multiplication: $\forall\, \phi(x), \phi(y) \in A/I$, $\phi(x) \cdot \phi(y) := \phi(xy)$

$A/I$ equipped with the three operations defined above is a $\mathbb{K}-$algebra which is often called *quotient algebra*. Then the quotient map $\phi$ is clearly a homomorphism.

Moreover, if $A$ is unital and $I$ proper then also the quotient algebra $A/I$ is unital. Indeed, as $I$ is a proper ideal of $A$, the unit $1_A$ does not belong to $I$ and so we have $\phi(1_A) \neq o$ and for all $x \in A$ we get $\phi(x)\phi(1_A) = \phi(x \cdot 1_A) = \phi(x) = \phi(1_A \cdot x) = \phi(1_A)\phi(x)$.

Suppose now that $(A, \omega)$ is a TA and $I$ an ideal of $A$. Since $A$ is in particular a topological space, we can endow it with the quotient topology w.r.t. the equivalence relation $\sim_I$. We already know that in this setting $\phi$ is a continuous homomorphism but actually the structure of TA on $A$ guarantees also that it is open. Indeed, the following holds for any TVS and so for any TA:

**Proposition 1.4.5.** *For a linear subspace $M$ of a t.v.s. $X$, the quotient mapping $\phi : X \to X/M$ is open (i.e. carries open sets in $X$ to open sets in $X/M$) when $X/M$ is endowed with the quotient topology.*

*Proof.* Let $V$ be open in $X$. Then we have

$$\phi^{-1}(\phi(V)) = V + M = \bigcup_{m \in M} (V + m).$$

Since $X$ is a t.v.s, its topology is translation invariant and so $V + m$ is open for any $m \in M$. Hence, $\phi^{-1}(\phi(V))$ is open in $X$ as union of open sets. By definition, this means that $\phi(V)$ is open in $X/M$ endowed with the quotient topology. □

**Theorem 1.4.6.** *Let $(A, \omega)$ be a TA (resp. TA with continuous multiplication) and $I$ an ideal of $A$. Then the quotient algebra $A/I$ endowed with the quotient topology is a TA (resp. TA with continuous multiplication).*

*Proof.*
(in the next lecture!) □

**Proposition 1.4.7.** *Let $A$ be a TA and $I$ an ideal of $A$. Consider $A/I$ endowed with the quotient topology. Then the two following properties are equivalent:*
*a) $I$ is closed*
*b) $A/I$ is Hausdorff*

*Proof.*
In view of Proposition 1.3.2, (b) is equivalent to say that the complement of the origin in $A/I$ is open w.r.t. the quotient topology. But the complement of the origin in $A/I$ is exactly the image under the canonical map $\phi$ of the complement of $I$ in $A$. Since $\phi$ is an open continuous map, the image under $\phi$ of the complement of $I$ in $X$ is open in $A/I$ iff the complement of $I$ in $A$ is open, i.e. (a) holds. $\square$

**Corollary 1.4.8.** *If $A$ is a TA, then $A/\overline{\{o\}}$ endowed with the quotient topology is a Hausdorff TA. $A/\overline{\{o\}}$ is said to be the* Hausdorff TA associated *with $A$. When $A$ is a Hausdorff TA, $A$ and $A/\overline{\{o\}}$ are topologically isomorphic.*

*Proof.*
First of all, let us observe that $\overline{\{o\}}$ is a closed ideal of $A$. Indeed, since $A$ is a TA, the multiplication is separately continuous and so for all $x, y \in A$ we have $x\overline{\{o\}} \subseteq \overline{\{x \cdot o\}} = \overline{\{o\}}$ and $\overline{\{o\}}y \subseteq \overline{\{o \cdot y\}} = \overline{\{o\}}$ is a linear subspace of $X$. Then, by Theorem 1.4.6 and Proposition 1.4.7, $A/\overline{\{o\}}$ is a Hausdorff TA. If in addition $A$ is also Hausdorff, then Proposition 1.3.2 guarantees that $\overline{\{o\}} = \{o\}$ in $A$. Therefore, the quotient map $\phi : A \to A/\overline{\{o\}}$ is also injective because in this case $Ker(\phi) = \{o\}$. Hence, $\phi$ is a topological isomorphism (i.e. bijective, continuous, open, linear) between $A$ and $A/\overline{\{o\}}$ which is indeed $A/\{o\}$. $\square$

Let us finally focus on quotients of normed algebra. If $(A, \|\cdot\|)$ is a normed (resp. Banach) algebra and $I$ an ideal of $A$, then Theorem 1.4.6 guarantees that $A/I$ endowed with the quotient topology is a TA with continuous multiplication but, actually, the latter is also a normed (resp. Banach) algebra. Indeed, one can easily show that the quotient topology is generated by the so-called *quotient norm* defined by

$$q(\phi(x)) := \inf_{y \in I} \|x + y\|, \quad \forall x \in A$$

which has the nice property to be submultiplicative (Sheet 2).

**Proposition 1.4.9.** *If $(A, \|\cdot\|)$ is a normed (resp. Banach) algebra and $I$ a closed ideal of $A$, then $A/I$ equipped with the quotient norm is a normed (resp. Banach) algebra.*

*Proof.* (Sheet 2) $\square$

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