A/I equipped with the three operations defined above is a $\mathbb{K}-$algebra which is often called *quotient algebra*. Then the quotient map ϕ is clearly a homomorphism. Moreover, if A is unital and I proper then also the quotient algebra A/I is unital. Indeed, as I is a proper ideal of A, the unit 1_A does not belong to I and so we have $\phi(1_A) \neq o$ and for all $x \in A$ we get $\phi(x)\phi(1_A) = \phi(x \cdot 1_A) = \phi(x) = \phi(1_A \cdot x) = \phi(1_A)\phi(x)$.

Suppose now that (A, ω) is a TA and I an ideal of A. Since A is in particular a topological space, we can endow it with the quotient topology w.r.t. the equivalence relation $\sim_I$. We already know that in this setting ϕ is a continuous homomorphism but actually the structure of TA on A guarantees also that it is open. Indeed, the following holds for any TVS and so for any TA:

Proposition 1.4.5. *For a linear subspace M of a t.v.s. X, the quotient mapping $\phi : X \to X/M$ is open (i.e. carries open sets in X to open sets in X/M) when X/M is endowed with the quotient topology.*

Proof.
Let V be open in X. Then we have

$$\phi^{-1}(\phi(V)) = V + M = \bigcup_{m \in M} (V + m).$$

Since X is a t.v.s, its topology is translation invariant and so $V + m$ is open for any $m \in M$. Hence, $\phi^{-1}(\phi(V))$ is open in X as union of open sets. By definition, this means that $\phi(V)$ is open in X/M endowed with the quotient topology. □

Theorem 1.4.6. *Let (A, ω) be a TA (resp. TA with continuous multiplication) and I an ideal of A. Then the quotient algebra A/I endowed with the quotient topology is a TA (resp. TA with continuous multiplication).*

Proof.
For convenience, in this proof we denote by $\mathbb{a}$ (resp. $\mathbb{m}$) the vector addition (resp. vector multiplication) in A/I and just by $+$ (resp. $\cdot$) the vector addition (resp. vector multiplication) in A. Let W be a neighbourhood of the origin o in A/I endowed with the quotient topology τ_Q. We first aim to prove that $\mathbb{a}^{-1}(W)$ is a neighbourhood of (o, o) in $A/I \times A/I$.

By definition of τ_Q, $\phi^{-1}(W)$ is a neighbourhood of the origin in (A, ω) and so, by Theorem 1.2.6-2 (we can apply the theorem because (A, ω) is a TA and so a TVS), there exists V neighbourhood of the origin in (A, ω) s.t. $V + V \subseteq \phi^{-1}(W)$. Hence, by the linearity of ϕ, we get

$$\mathbb{a}(\phi(V) \times \phi(V)) = \phi(V + V) \subseteq \phi(\phi^{-1}(W)) \subseteq W, \text{ i.e. } \phi(V) \times \phi(V) \subseteq \mathbb{a}^{-1}(W).$$

Since ϕ is also an open map, $\phi(V)$ is a neighbourhood of the origin o in $(A/I, \tau_Q)$ and so $\mathbb{a}^{-1}(W)$ is a neighbourhood of (o, o) in $A/I \times A/I$ endowed with the product topology given by τ_Q. A similar argument gives the continuity of the scalar multiplication. Hence, A/I endowed with the quotient topology is a TVS.

Furthermore, for any $\tilde{x} \in A/I$ and any W neighbourhood of the origin in $(A/I, \tau_Q)$, we know that $\tilde{x} = \phi(x)$ for some $x \in A$ and $\phi^{-1}(W)$ is a neighbourhood of the origin in (A, ω). Since (A, ω) is a TA, the multiplication $\cdot$ in A is separately continuous so there exist V_1, V_2 neighbourhoods of the origin in (A, ω) such that $x \cdot V_1 \subseteq \phi^{-1}(W)$ and $V_2 \cdot x \subseteq \phi^{-1}(W)$. Setting $N_1 := \phi(V_1)$ and $N_2 := \phi(V_2)$, we get $\mathbb{m}(\tilde{x} \times N_1) = \mathbb{m}(\phi(x) \times \phi(V_1)) = \phi(x \cdot V_1) \subseteq \phi(\phi^{-1}(W)) \subseteq W$ and similarly $\mathbb{m}(N_2 \times \tilde{x}) \subseteq \phi(\phi^{-1}(W)) \subseteq W$. This yields that $\mathbb{m}$ is separately continuous as the quotient map is open and so N_1, N_2 are both neighbourhoods of the origin in $(A/I, \tau_Q)$. $\square$

Proposition 1.4.7. *Let A be a TA and I an ideal of A. Consider A/I endowed with the quotient topology. Then the two following properties are equivalent:*
a) I is closed
b) A/I is Hausdorff

Proof.
In view of Proposition 1.3.2, (b) is equivalent to say that the complement of the origin in A/I is open w.r.t. the quotient topology. But the complement of the origin in A/I is exactly the image under the canonical map ϕ of the complement of I in A. Since ϕ is an open continuous map, the image under ϕ of the complement of I in X is open in A/I iff the complement of I in A is open, i.e. (a) holds. $\square$

Corollary 1.4.8. *If A is a TA, then $A/\overline{\{o\}}$ endowed with the quotient topology is a Hausdorff TA. $A/\overline{\{o\}}$ is said to be the* Hausdorff TA associated *with A. When A is a Hausdorff TA, A and $A/\overline{\{o\}}$ are topologically isomorphic.*

Proof.
First of all, let us observe that $\overline{\{o\}}$ is a closed ideal of A. Indeed, since A is a TA, the multiplication is separately continuous and so for all $x, y \in A$ we have $x\overline{\{o\}} \subseteq \overline{\{x \cdot o\}} = \overline{\{o\}}$ and $\overline{\{o\}}y \subseteq \overline{\{o \cdot y\}} = \overline{\{o\}}$. Then, by Theorem 1.4.6 and Proposition 1.4.7, $A/\overline{\{o\}}$ is a Hausdorff TA. If in addition A is also Hausdorff, then Proposition 1.3.2 guarantees that $\overline{\{o\}} = \{o\}$ in A. Therefore, the quotient map $\phi : A \to A/\overline{\{o\}}$ is also injective because in this case $Ker(\phi) = \{o\}$.

Hence, ϕ is a topological isomorphism (i.e. bijective, continuous, open, linear) between A and $A/\overline{\{o\}}$ which is indeed $A/\{o\}$. $\square$

Let us finally focus on quotients of normed algebra. If $(A, \|\cdot\|)$ is a normed (resp. Banach) algebra and I a closed ideal of A, then Theorem 1.4.6 guarantees that A/I endowed with the quotient topology is a TA with continuous multiplication but, actually, the latter is also a normed (resp. Banach) algebra. Indeed, one can easily show that the quotient topology is generated by the so-called *quotient norm* defined by

$$q(\phi(x)) := \inf_{y \in I} \|x + y\|, \quad \forall x \in A$$

which has the nice property to be submultiplicative and so the following holds.

Proposition 1.4.9. *If $(A, \|\cdot\|)$ is a normed (resp. Banach) algebra and I a closed ideal of A, then A/I equipped with the quotient norm is a normed (resp. Banach) algebra.*

Proof. (Sheet 2) $\square$

Chapter 2

Locally multiplicative convex algebras

2.1 Neighbourhood definition of lmc algebras

In the study of locally multiplicative convex algebras a particular role will be played by multiplicative sets. Therefore, before starting the study of this class of topological algebras we are going to have a closer look to this concept.

Definition 2.1.1. *A subset U of a $\mathbb{K}-$algebra A is said to be a* multiplicative set *or* m-set *if $U \cdot U \subseteq U$. We call* m-convex *(resp.* m-balanced*) a multiplicative convex (resp. balanced) subset of A and* absolutely m-convex *a multiplicative subset of A which is both balanced and convex.*

The notions defined above are totally algebraic and so independent from the topological structure with which the algebra is endowed.

Example 2.1.2.

- *Any ideal of an algebra is an m-set.*
- *Fixed an element $a \neq o$ of an algebra, the set $\{a^n : n \in \mathbb{N}\}$ is an m-set.*
- *Given a normed algebra $(A, \|\cdot\|)$ and an integer $n \in \mathbb{N}$, the open and the closed ball centered at origin with radius $\frac{1}{n}$ are both examples of absolutely m-convex sets in A.*

The following proposition illustrates some operations under which the multiplicativity of a subset of an algebra is preserved.

Proposition 2.1.3. *Let A be a $\mathbb{K}-$algebra and $U \subset A$ multiplicative, then*
a) The convex hull of U is an m-convex set in A.
b) The balanced hull of U is an m-balanced set in A.
c) The convex balanced hull of U is an absolutely m-convex set in A.
d) Any direct or inverse image via a homomorphism is a m-set.

Proof. (Sheet 2) $\square$

Recall that

Definition 2.1.4. *Let S be any subset of a vector space X over $\mathbb{K}$. The convex (resp. balanced) hull of S, denoted by* conv(S) *(resp.* bal(S)*) is the smallest convex (resp. balanced) subset of X containing S, i.e. the intersection of all convex (resp. balanced) subsets of X containing S. Equivalently,*

$$\operatorname{conv}(S) := \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in S, \lambda_i \in [0,1], \sum_{i=1}^{n} \lambda_i = 1, n \in \mathbb{N} \right\}$$

and the balanced hull *of S, denoted by* bal(S) *as*

$$\operatorname{bal}(S) := \bigcup_{\lambda \in \mathbb{K}, |\lambda| \leq 1} \lambda S.$$

The convex balanced hull of S, denoted by conv$_b(S)$*, is defined as the smallest convex and balanced subset of X containing S and it can be easily proved that* $\operatorname{conv}_b(S) = \operatorname{conv}(\operatorname{bal}(S))$.

Let us come back now to topological algebras.

Proposition 2.1.5. *In any topological algebra, the operation of closure preserves the multiplicativity of a subset as well as its m-convexity and absolute m-convexity.*

Proof.
First of all let us show that the following property holds in any TA (A, τ):

$$\forall\, V, W \subseteq A, \; \overline{V} \cdot \overline{W} \subseteq \overline{VW}. \tag{2.1}$$

where the closure in A is here clearly intended w.r.t. the topology τ. Let $x \in \overline{V}$, $y \in \overline{W}$ and $O \in \mathcal{F}(o)$ where $\mathcal{F}(o)$ denotes the filter of neighbourhoods of the origin in A. As A is in particular a TVS, Theorem 1.2.6-2 ensures that there exists $N \in \mathcal{F}(o)$ s.t. $N + N \subseteq O$. Then for each $a \in A$, by Theorem 1.2.9, there exist $N_1, N_2 \in \mathcal{F}(o)$ such that $N_1 a \subseteq N$ and $a N_2 \subseteq N$. Moreover, since $x \in \overline{V}$ and $y \in \overline{W}$, there exist $v \in V$ and $w \in W$ s.t. $v \in x + N_1$ and $w \in y + N_2$. Putting all together, we have that

$$\begin{aligned} vw \in (x + N_1)w &= xw + N_1 w \subseteq xw + N \subseteq x(y + N_2) + N \\ &= xy + xN_2 + N \subseteq xy + N + N \subseteq xy + O. \end{aligned}$$

Hence, $(xy + O) \cap VW \neq \emptyset$, which proves that $xy \in \overline{VW}$. Therefore, if U is an m-set in A then by (2.1) we get $\overline{U} \cdot \overline{U} \subseteq \overline{U \cdot U} \subseteq \overline{U}$, which proves that $\overline{U}$ is an m-set.

Suppose now that U is m-convex. The first part of the proof guarantees that $\overline{U}$ is an m-set. Moreover, using that A is in particular a TVS, we have that for any $\lambda \in [0,1]$ the mapping

$$\begin{array}{rccl} \varphi_\lambda : & A \times A & \to & A \\ & (x,y) & \mapsto & \lambda x + (1-\lambda) y \end{array}$$

is continuous and so $\varphi_\lambda(\overline{U \times U}) \subseteq \overline{\varphi_\lambda(U \times U)}$. Since U is also convex, for any $\lambda \in [0,1]$ we have that $\varphi_\lambda(U \times U) \subseteq U$ and so $\overline{\varphi_\lambda(U \times U)} \subseteq \overline{U}$. Putting all together, we can conclude that $\varphi_\lambda(\overline{U} \times \overline{U}) = \varphi_\lambda(\overline{U \times U}) \subseteq \overline{U}$, i.e. $\overline{U}$ is convex. Hence, $\overline{U}$ is an m-convex set.

Finally, assume that U is absolutely m-convex. As U is in particular m-convex, by the previous part of the proof, we can conclude immediately that $\overline{U}$ is an m-convex set. Furthermore, since U is balanced and A has the TVS structure, we can conclude that $\overline{U}$ is also balanced. Indeed, in any TVS the closure of a balanced set is still balanced because the multiplication by scalar is continuous and so for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$ we have $\lambda \overline{U} \subseteq \overline{\lambda U} \subseteq U$. □

Definition 2.1.6. *A closed absorbing absolutely convex multiplicative subset of a TA is called a* m-barrel.

Proposition 2.1.7. *Every multiplicative neighbourhood of the origin in a TA is contained in a neighbourhood of the origin which is an m-barrel.*

Proof.
Let U be a multiplicative neighbourhood of the origin and define $T(U) := \overline{\mathrm{conv}_b(U)}$. Clearly, $U \subseteq T(U)$. Therefore, $T(U)$ is a neighbourhood of the origin and so it is absorbing by Theorem 1.2.6-4). By Proposition 2.1.3-c), $\mathrm{conv}_b(U)$ is an absolutely m-convex set as U is an m-set. Hence, Proposition 2.1.5 ensures that $T(U)$ is closed and absolutely m-convex, i.e. an m-barrel. □

Note that the converse inclusion in Proposition 2.1.7 does not hold in general. Indeed, in any TA not every neighbourhood of the origin (not even every multiplicative one) contains another one which is a m-barrel. This means that not every TA has a basis of neighbourhoods consisting of m-barrels. However, this is true for any lmc TA.

Definition 2.1.8. *A TA is said to be* locally multiplicative convex (lmc) *if it has a basis of neighbourhoods of the origin consisting of m-convex sets.*

It is then easy to show that

Proposition 2.1.9. *A locally multiplicative convex algebra is a TA with continuous multiplication.*
Proof.
Let (A, τ) be an lmc algebra and let $\mathcal{B}$ denote a basis of neighbourhoods of the origin in (A, τ) consisting of m-convex sets. Then (A, τ) is in particular a TVS and for any $U \in \mathcal{B}$ we have $U \cdot U \subset U$. Hence, both conditions of Theorem 1.2.10 are fulfilled by $\mathcal{B}$, which proves that (A, τ) is a TA with continuous multiplication. $\square$

Note that any lmc algebra is in particular a *locally convex TVS*, i.e. a TVS having a basis of neighbourhoods of the origin consisting of convex sets. Hence, in the study of this class of TAs we can make use of all the powerful results about locally convex TVS. To this aim let us recall that the class of locally convex TVS can be characterized in terms of absorbing absolutely convex neighbourhoods of the origin.

Theorem 2.1.10. *If X is a lc TVS then there exists a basis $\mathcal{B}$ of neighbourhoods of the origin consisting of absorbing absolutely convex subsets s.t.*
a) $\forall U, V \in \mathcal{B}$, $\exists W \in \mathcal{B}$ *s.t.* $W \subseteq U \cap V$
b) $\forall U \in \mathcal{B}$, $\forall \rho > 0$, $\exists W \in \mathcal{B}$ *s.t.* $W \subseteq \rho U$
Conversely, if $\mathcal{B}$ is a collection of absorbing absolutely convex subsets of a vector space X s.t. a) and b) hold, then there exists a unique topology compatible with the linear structure of X s.t. $\mathcal{B}$ is a basis of neighbourhoods of the origin in X for this topology (which is necessarily locally convex).

Proof.
Let N be a neighbourhood of the origin in the lc TVS (X, τ). The local convexity ensures that there exists W convex neighbourhood of the origin in (X, τ) s.t. $W \subseteq N$. Moreover, by Theorem 1.2.6-5), there exists U balanced neighbourhood of the origin in X s.t. $U \subseteq W$. Then, using that W is a convex set containing U, we get $\mathrm{conv}(U) \subseteq W \subseteq N$. Now $\mathrm{conv}(U)$ is convex by definition, balanced because U is balanced and it is also a neighbourhood of the origin (and so an absorbing set) since $U \subseteq \mathrm{conv}(U)$. Hence, the collection $\mathcal{B} := \{\mathrm{conv}(U) : U \in \mathcal{B}_b\}$ is a basis of absorbing absolutely convex neighbourhoods of the origin in (X, τ); here $\mathcal{B}_b$ denotes a basis of balanced neighbourhoods of the origin in (X, τ). Observing that for any $U, W \in \mathcal{B}_b$ and any $\rho > 0$ we have $\mathrm{conv}(U \cap W) \subseteq \mathrm{conv}(U) \cap \mathrm{conv}(W)$ and $\mathrm{conv}(\rho U) \subseteq \rho \mathrm{conv}(U)$, we see that $\mathcal{B}$ fulfills both a) and b).

The converse direction is left as an exercise for the reader. $\square$

This theorem will be a handful tool in the proof of the following characterization of lmc algebras in terms of neighbourhood basis.

Theorem 2.1.11. *Let A be a $\mathbb{K}-$algebra. Then the following are equivalent:*
a) A is an lmc algebra
b) A is a TVS having a basis of neighbourhoods consisting of m-barrels.
c) There exists a basis for a filter on A consisting of absorbing absolutely m-convex subsets.

Proof.
a) $\Rightarrow$ *b)* If A is an lmc algebra, then we have already observed that it is a lc TVS. Let $\mathcal{F}(o)$ be the filter of neighbourhoods of the origin in A and let $N \in \mathcal{F}(o)$. The TVS structure ensures that there exists $V \in \mathcal{F}(o)$ closed s.t. $V \subseteq N$[1] and the local convexity allows to apply Theorem 2.1.10 which guarantees that we can always find $M \in \mathcal{F}(o)$ absolutely convex s.t. $M \subseteq V$. Finally, since A is an lmc algebra, we know that there exists $C \in \mathcal{F}(o)$ m-convex s.t. $C \subseteq M$. Using the previous inclusions we have that

$$T(C) := \overline{\mathrm{conv}_b(C)} \subseteq \overline{M} \subseteq \overline{V} = V \subseteq N.$$

(Note that the first inclusion follows from the fact that M is a convex and balanced subset containing C.) Hence, the conclusion holds because $T(C)$ is an m-barrel set as C is a multiplicative neighbourhood of the origin (see last part of proof of Proposition 2.1.7).

b) $\Rightarrow$ *c)* This is clear because every m-barrelled neighbourhood of the origin is an absorbing absolutely m-convex subsets of A.

c) $\Rightarrow$ *a)* Suppose that $\mathcal{M}$ is a basis for a filter on A consisting of absorbing absolutely convex m-subsets. Then it is easy to verify that the collection $\widetilde{\mathcal{M}} := \{\lambda U : U \in \mathcal{M}, 0 < \lambda \leq 1\}$ also consists of absorbing absolutely m-convex subsets of A. Moreover, for any $U, V \in \mathcal{M}$ we know that there exists $W \in \mathcal{M}$ s.t. $W \subseteq U \cap W$ and so for any $0 < \lambda, \mu \leq 1$ we have that $\delta W \subseteq \delta(U \cap V) = \delta U \cap \delta V \subseteq \lambda U \cap \mu V$ where $\delta := \min\{\lambda, \mu\}$. As $\delta W \in \widetilde{\mathcal{M}}$ we have that a) of Theorem 2.1.10. Also b) of this same theorem is satisfied because for any $\rho > 0$, $0 < \lambda \leq 1$ and $U \in \mathcal{M}$ we easily get that there exists $M \in \widetilde{\mathcal{M}}$ s.t. $M \subseteq \rho(\lambda U)$ by choosing $M = \rho(\lambda U)$ when $0 < \rho \leq 1$ and $M = \lambda U$ when $\rho > 1$. Hence, $\widetilde{M}$ fulfills all the assumptions of the second part of Theorem 2.1.10 and so it is a basis of neighbourhoods of the origin for a uniquely defined topology τ on A making (A, τ) a lc TVS. As every set in $\widetilde{\mathcal{M}}$ is m-convex, (A, τ) is in fact a lmc algebra. $\square$

[1] Every TVS has basis of closed neighbourhoods of the origin.
Proof.
Let $\mathcal{F}(o)$ be the filter of neighbourhoods of the origin in a TVS X and $N \in \mathcal{F}(o)$. Then Theorem 1.2.6 guarantees that there exists $V \in \mathcal{F}(o)$ balanced such that $V - V \subseteq N$. If $x \in \overline{V}$ then $(V + x) \cap V \neq \emptyset$ and so there exist $u, v \in V$ s.t. $u + x = v$, which gives $x = v - u \in V - V \subseteq N$. Hence, $\overline{V} \subseteq N$. $\square$