From the last part of the proof we can immediately see that

Corollary 2.1.12. *If $\mathcal{M}$ is a basis for a filter on a $\mathbb{K}-$algebra A consisting of absorbing absolutely convex m-subsets, then there exists a unique topology τ on A both having $\widetilde{\mathcal{M}} := \{\lambda U : U \in \mathcal{M}, 0 < \lambda \leq 1\}$ as a basis of neighbourhoods of the origin and making (A, τ) an lmc algebra.*

Theorem 2.1.11 shows that in an lmc algebra every neighbourhood of the origin contains an m-barrel set. However, it is important to remark that not every m-barrel subset of a topological algebra, not even of an lmc algebra, is a neighbourhood of the origin (see Examples 2.2.19)! Topological algebras having this property are called *m-barrelled algebras*.

2.2 Seminorm characterization of lmc algebras

In this section we will investigate the intrinsic and very useful connection between lmc algebras and seminorms. Therefore, let us briefly recall this concept and focus in particular on submultiplicative seminorms.

Definition 2.2.1. *Let X be a $\mathbb{K}-$vector space. A function $p : X \to \mathbb{R}$ is called a* seminorm *if it satisfies the following conditions:*

1. *p is subadditive: $\forall x, y \in X$, $p(x + y) \leq p(x) + p(y)$.*
2. *p is positively homogeneous: $\forall x \in X$, $\forall \lambda \in \mathbb{K}, p(\lambda x) = |\lambda| p(x)$.*

A seminorm on a $\mathbb{K}-$algebra X is called submultiplicative *if*

$$\forall x, y \in X,\ p(xy) \leq p(x)p(y).$$

Definition 2.2.2. *A seminorm p on a vector space X is a* norm *if $p(x) = 0$ implies $x = o$ (i.e. if $p^{-1}(\{0\}) = \{o\}$).*

The following properties are an easy consequence of Definition 2.2.1.

Proposition 2.2.3. *Let p be a seminorm on a vector space X. Then:*

- *p is symmetric, i.e. $p(x) = p(-x)$, $\forall\, x \in X$.*
- *$p(o) = 0$.*
- *$|p(x) - p(y)| \leq p(x - y)$, $\forall\, x, y \in X$.*
- *$p(x) \geq 0$, $\forall\, x \in X$.*
- *$\ker(p)$ is a linear subspace of X.*

Examples 2.2.4.

a) *Suppose $X = \mathbb{R}^n$ is equipped with the componentwise operations of addition, scalar and vector multiplication. Let M be a linear subspace of X. For any $x \in X$, set*

$$q_M(x) := \inf_{m \in M} \|x - m\|,$$

where $\|\cdot\|$ *is the Euclidean norm on* $\mathbb{R}^n$, *i.e.* $q_M(x)$ *is the distance from the point* x *to* M *in the usual sense. If* $\dim(M) \geq 1$ *then* q_M *is a submultiplicative seminorm but not a norm (*M *is exactly the kernel of* q_M*). When* $M = \{o\}$, $p_M(\cdot)$ *and* $\|\cdot\|$ *coincide.*

b) *Let* $\mathcal{C}(\mathbb{R})$ *be the vector space of all real valued continuous functions on the real line equipped with the pointwise operations of addition, multiplication and scalar multiplication. For any* $a \in \mathbb{R}^+$, *we define*

$$p_a(f) := \sup_{-a \leq t \leq a} |f(t)|, \ \forall f \in \mathcal{C}(\mathbb{R}).$$

Then p_a *is a submultiplicative seminorm but is never a norm because it might be that* $f(t) = 0$ *for all* $t \in [-a, a]$ *(and so that* $p_a(f) = 0$*) but* $f \not\equiv 0$.

c) *Let* $n \geq 2$ *be an integer and consider the algebra* $\mathbb{R}^{n\times n}$ *of real square matrices of order* n. *Then*

$$q(A) := \max_{i,j=1,\ldots,n} |A_{ij}|, \quad \forall A = (A_{ij})_{i,j=1}^n \in \mathbb{R}^{n\times n}$$

is a norm (so in particular a seminorm) but it is not submultiplicative because for example if A *is the matrix with all entries equal to* 1 *then it is easy to check that* $\|A^2\| > \|A\|$.

Seminorms on vector spaces are strongly related to a special kind of functionals, i.e. *Minkowski functionals*. Let us investigate more in details such a relation. Note that we are still in the realm of vector spaces with no topology!

Definition 2.2.5. *Let* X *be a vector space and* V *a non-empty subset of* X. *We define the* Minkowski functional (or gauge) *of* V *to be the mapping:*

$$\begin{array}{rccl} p_V: & X & \to & \mathbb{R} \\ & x & \mapsto & p_V(x) := \inf\{\lambda > 0 : x \in \lambda V\} \end{array}$$

(where $p_V(x) = \infty$ *if the set* $\{\lambda > 0 : x \in \lambda V\}$ *is empty).*

It is then natural to ask whether there exists a class of subsets for which the associated Minkowski functionals are actually seminorms, and in particular submultiplicative seminorms in the context of algebras. The answer is positive in both cases as established in the following lemma.

Notation 2.2.6. *Let* X *be a vector space and* p *a seminorm on* X. *The sets*

$$\mathring{U}_p = \{x \in X : p(x) < 1\} \text{ and } U_p = \{x \in X : p(x) \leq 1\}.$$

are said to be, respectively, the open and the closed unit semiball *of* p.

Lemma 2.2.7. *Let X be a $\mathbb{K}-$vector space (resp. $\mathbb{K}-$algebra).*
a) If V is a non-empty subset of X which is absorbing and absolutely convex (resp. absolutely m-convex), then the associated Minkowski functional p_V is a seminorm (resp. submultiplicative seminorm) and $\mathring{U}_{p_V} \subseteq V \subseteq U_{p_V}$.
b) If q is a seminorm (resp. submultiplicative seminorm) on X then both $\mathring{U}_q$ and U_q are absorbing absolutely convex sets [resp. absolutely m-convex] and for any absorbing absolutely convex (resp. absolutely m-convex) V such that $\mathring{U}_q \subseteq V \subseteq U_q$ we have $q = p_V$.

Proof.
a) Let V be a non-empty subset of X which is absorbing and absolutely convex and denote by p_V the associated Minkowski functional. We want to show that p_V is a seminorm.

- First of all, note that $p_V(x) < \infty$ for all $x \in X$ because V is absorbing. Indeed, for any $x \in X$ there exists $\rho_x > 0$ s.t. for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq \rho_x$ we have $\lambda x \in V$ and so the set $\{\lambda > 0 : x \in \lambda V\}$ is never empty, i.e. p_V has only finite nonnegative values. Moreover, since $o \in V$, we also have that $o \in \lambda V$ for any $\lambda \in \mathbb{K}$ and so $p_V(o) = \inf\{\lambda > 0 : o \in \lambda V\} = 0$.
- The balancedness of V implies that p_V is positively homogeneous. Since we have already showed that $p_V(o) = 0$ it remains to prove the positive homogeneity of p_V for non-zero scalars. Since V is balanced we have that for any $x \in X$ and for any $\xi, \lambda \in \mathbb{K}$ with $\xi \neq 0$ the following holds:

$$\xi x \in \lambda V \text{ if and only if } x \in \frac{\lambda}{|\xi|}V. \tag{2.2}$$

Indeed, V balanced guarantees that $\xi V = |\xi| V$ and so $x \in \frac{\lambda}{|\xi|}V$ is equivalent to $\xi x \in \lambda \frac{\xi}{|\xi|}V = \lambda V$. Using (2.2), we get that for any $x \in X$ and for any $\xi \in \mathbb{K}$ with $\xi \neq 0$:

$$\begin{aligned} p_V(\xi x) &= \inf\{\lambda > 0 : \xi x \in \lambda V\} \\ &= \inf\left\{\lambda > 0 : x \in \frac{\lambda}{|\xi|}V\right\} \\ &= \inf\left\{|\xi|\frac{\lambda}{|\xi|} > 0 : x \in \frac{\lambda}{|\xi|}V\right\} \\ &= |\xi| \inf\{\mu > 0 : x \in \mu V\} = |\xi| p_V(x). \end{aligned}$$

- The convexity of V ensures the subadditivity of p_V. Take $x, y \in X$. By the definition of Minkowski functional, for every $\varepsilon > 0$ there exist $\lambda, \mu > 0$ s.t.

$$\lambda < p_V(x) + \frac{\varepsilon}{2} \text{ and } x \in \lambda V$$

and

$$\mu < p_V(y) + \frac{\varepsilon}{2} \text{ and } y \in \mu V.$$

Then, by the convexity of V, we obtain that $\frac{\lambda}{\lambda+\mu}V + \frac{\mu}{\lambda+\mu}V \subseteq V$, i.e. $\lambda V + \mu V \subseteq (\lambda+\mu)V$, and therefore $x + y \in (\lambda+\mu)V$. Hence:

$$p_V(x+y) = \inf\{\delta > 0 : x + y \in \delta V\} \leq \lambda + \mu < p_V(x) + p_V(y) + \varepsilon$$

which proves the subadditivity of p_V since ε is arbitrary.

We can then conclude that p_V is a seminorm. Furthermore, we have the following inclusions:

$$\mathring{U}_{p_V} \subseteq V \subseteq U_{p_V}.$$

In fact, if $x \in \mathring{U}_{p_V}$ then $p_V(x) < 1$ and so there exists $0 < \lambda < 1$ s.t. $x \in \lambda V$. Since V is balanced, for such λ we have $\lambda V \subseteq V$ and therefore $x \in V$. On the other hand, if $x \in V$ then clearly $1 \in \{\lambda > 0 : x \in \lambda V\}$ which gives $p_V(x) \leq 1$ and so $x \in U_{p_V}$.

If X is a $\mathbb{K}-$algebra and V an absorbing absolutely m-convex subset of X, then the previous part of the proof guarantees that p_V is a seminorm and $\mathring{U}_{p_V} \subseteq V \subseteq U_{p_V}$. Moreover, for any $a, b \in X$, the multiplicativity of V implies that $\{\lambda > 0 : a \in \lambda V\}\{\mu > 0 : b \in \mu V\} \subseteq \{\delta > 0 : ab \in \delta V\}$ and so

$$\begin{aligned} p_V(a)p_V(b) &= \inf\left(\{\lambda > 0 : a \in \lambda V\}\{\mu > 0 : b \in \mu V\}\right) \\ &\geq \inf\{\delta > 0 : ab \in \delta V\} = p_V(ab). \end{aligned}$$

Hence, p_V is a submultiplicative seminorm.

b) Let us take any seminorm q on X. Let us first show that $\mathring{U}_q$ is absorbing and absolutely convex.

- $\mathring{U}_q$ is absorbing.
  Let x be any point in X. If $q(x) = 0$ then clearly $x \in \mathring{U}_q$. If $q(x) > 0$, we can take $0 < \rho < \frac{1}{q(x)}$ and then for any $\lambda \in \mathbb{K}$ s.t. $|\lambda| \leq \rho$ the positive homogeneity of q implies that $q(\lambda x) = |\lambda| q(x) \leq \rho q(x) < 1$, i.e. $\lambda x \in \mathring{U}_q$.
- $\mathring{U}_q$ is balanced.
  For any $x \in \mathring{U}_q$ and for any $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$, again by the positive homogeneity of q, we get: $q(\lambda x) = |\lambda| q(x) \leq q(x) < 1$ i.e. $\lambda x \in \mathring{U}_q$.
- $\mathring{U}_q$ is convex.
  For any $x, y \in \mathring{U}_q$ and any $t \in [0, 1]$, by both the properties of seminorm, we have that $q(tx + (1-t)y) \leq tq(x) + (1-t)q(y) < t + 1 - t = 1$ i.e. $tx + (1-t)y \in \mathring{U}_q$.

The proof above easily adapts to show that U_q is absorbing and absolutely convex. Also, it is easy to check that

$$p_{\mathring{U}_q}(x) = q(x) = p_{U_q}(x),\ \forall x \in X. \tag{2.3}$$

Since for any absorbing absolutely convex subset V of X s.t. $\mathring{U}_q \subseteq V \subseteq U_q$ and for any $x \in X$ we have that

$$p_{U_q}(x) \leq p_V(x) \leq p_{\mathring{U}_q}(x),$$

by (2.3) we can conclude that $p_V(x) = q(x)$.

If X is a $\mathbb{K}-$algebra and q is submultiplicative, then the previous part of the proof of b) applies but in addition we get that both $\mathring{U}_q$ and U_q are multiplicative sets. Indeed, for any $a, b \in \mathring{U}_q$ we have $q(ab) \leq q(a)q(b) < 1$, i.e. $ab \in \mathring{U}_q$ and similarly for U_q. □

In a nutshell this lemma says that: *a real-valued functional on a $\mathbb{K}-$vector space X (resp. a $\mathbb{K}-$algebra) is a seminorm (resp-submultiplicative seminorm) if and only if it is the Minkowski functional of an absorbing absolutely convex (resp. absolutely m-convex) non-empty subset of X.*

Let us collect some interesting properties of semiballs in a vector space, which we will repeatedly use in the following.

Proposition 2.2.8. *Let X be a $\mathbb{K}-$vector space and p a seminorm on X. Then:*

a) $\forall\, r > 0$, $r\mathring{U}_p = \{x \in X : p(x) < r\} = \mathring{U}_{\frac{1}{r}p}$.

b) $\forall\, x \in X$, $x + \mathring{U}_p = \{y \in X : p(y - x) < 1\}$.

c) If q is also a seminorm on X, then $p \leq q$ if and only if $\mathring{U}_q \subseteq \mathring{U}_p$.

d) If $n \in \mathbb{N}$ and $s_1, \ldots, s_n$ are seminorms on X, then their maximum *s defined as $s(x) := \max\limits_{i=1,\ldots,n} s_i(x)$, $\forall x \in X$ is also seminorm on X and $\mathring{U}_s = \bigcap_{i=1}^n \mathring{U}_{s_i}$. In particular, if X is a $\mathbb{K}-$algebra and all s_i's are submultiplicative seminorms, then $s(x)$ is also submultiplicative.*

All the previous properties also hold for closed semballs.

Proof. (Sheet 3) □

Let us start to put some topological structure on our space and so to consider continuous seminorms on it. The following result holds in any TVS and so in particular in any TA.

Proposition 2.2.9. *Let X be a TVS and p a seminorm on X. Then the following conditions are equivalent:*

a) The open unit semiball $\mathring{U}_p$ of p is an open neighbourhood of the origin and coincides with the interior of U_p.

b) p is continuous at the origin.

c) The closed unit semiball U_p of p is a closed neighbourhood of the origin and coincides with the closure of $\mathring{U}_p$.

d) p is continuous at every point.

Proof.

a) $\Rightarrow$ *b)* Suppose that $\mathring{U}_p$ is open in the topology on X. Then for any $\varepsilon > 0$ we have that $p^{-1}(]-\varepsilon, \varepsilon[) = \{x \in X : p(x) < \varepsilon\} = \varepsilon \mathring{U}_p$ is an open neighbourhood of the origin in X. This is enough to conclude that $p : X \to \mathbb{R}^+$ is continuous at the origin.

b) $\Rightarrow$ *c)* Suppose that p is continuous at the origin, then $U_p = p^{-1}([0,1])$ is a closed neighbourhood of the origin. Also, by definition $\mathring{U}_p \subseteq U_p$ and so $\overline{\mathring{U}_p} \subseteq \overline{U_p} = U_p$. To show the converse inclusion, we consider $x \in X$ s.t. $p(x) = 1$ and take $(\lambda_n)_{n\in\mathbb{N}} \subset (0,1)$ s.t. $\lim_{n\to\infty} \lambda_n = 1$. Then $\lambda_n x \in \mathring{U}_p$ and $\lim_{n\to\infty} \lambda_n x = x$ since the scalar multiplication is continuous. Hence, $x \in \overline{\mathring{U}_p}$ which completes the proof of c).

c) $\Rightarrow$ *d)* Assume that c) holds and fix $x \in X$. Using Proposition 2.2.8 and Proposition 2.2.3, we get that for any $\varepsilon > 0$: $p^{-1}([-\varepsilon + p(x), p(x) + \varepsilon]) = \{y \in X : |p(y) - p(x)| \leq \varepsilon\} \supseteq \{y \in X : p(y - x) \leq \varepsilon\} = x + \varepsilon U_p$, which is a closed neighbourhood of x since X is a TVS and by the assumption *c)*. Hence, p is continuous at x.

d) $\Rightarrow$ *a)* If p is continuous on X then a) holds because $\mathring{U}_p = p^{-1}(]-1,1[)$ and the preimage of an open set under a continuous function is open. Also, by definition $\mathring{U}_p \subseteq U_p$ and so $\mathring{U}_p = \operatorname{int}\left(\mathring{U}_p\right) \subseteq \operatorname{int}(U_p)$. To show the converse inclusion, we consider $x \in \operatorname{int}(U_p)$. Then $p(x) \leq 1$ but, since $p(x) = p_{\mathring{U}_p}(x)$, we also have that for any $\varepsilon > 0$ there exists $\lambda > 0$ s.t. $x \in \lambda \mathring{U}_p$ and $\lambda < p(x) + \varepsilon$. This gives that $p(x) < \lambda < 1 + \varepsilon$ and so $p(x) < 1$, i.e. $x \in \mathring{U}_p$ which completes the proof of a). $\square$

Definition 2.2.10. *Let X be a vector space and $\mathcal{P} := \{p_i\}_{i\in I}$ a family of seminorms on X. The coarsest topology $\tau_{\mathcal{P}}$ on X s.t. each p_i is continuous is said to be the* topology induced or generated by the family of seminorms *$\mathcal{P}$.*

We are now ready to see the connection between submultiplicative seminorms and locally convex multiplicative algebras.

Theorem 2.2.11. *Let X be a $\mathbb{K}-$algebra and $\mathcal{P} := \{p_i\}_{i\in I}$ a family of submultiplicative seminorms. Then the topology induced by the family $\mathcal{P}$ is the unique topology both making X into an lmc algebra and having as a basis of neighbourhoods of the origin the following collection:*

$$\mathcal{B} := \Big\{\{x \in X : p_{i_1}(x) \leq \varepsilon, \dots, p_{i_n}(x) \leq \varepsilon\} : i_1, \dots, i_n \in I, n \in \mathbb{N}, 0 < \varepsilon \leq 1\Big\}.$$

Viceversa, the topology of an arbitrary lmc algebra is always induced by a family of submultiplicative seminorms (often called generating*).*

Proof.
Let us first observe that

$$\mathcal{B} = \Big\{\bigcap_{j=1}^{n} \varepsilon U_{p_{i_j}} : n \in \mathbb{N}, i_1, \dots, i_n \in I, 0 < \varepsilon \leq 1\Big\}$$

and is a basis for a filter on X as it is closed under finite intersections. Moreover, by Proposition 2.2.8-a) and Lemma 2.2.7-b), we have that for any $i \in I$ the semiball εU_{p_i} is absorbing and absolutely m-convex. Therefore, any element in $\mathcal{B}$ is an absorbing absolutely m-convex subset of X as finite intersection of sets having such properties. Hence, Corollary 2.1.12 guarantees that there exists a unique topology τ having $\mathcal{B}$ as a basis of neighbourhoods of the origin and s.t. (X, τ) is an lmc algebra.

Since for any $i \in I$ we have $U_{p_i} \in \mathcal{B}$, U_{p_i} is a neighbourhood of the origin in (X, τ), then by Proposition 2.2.9, the seminorm p_i is $\tau-$continuous. Therefore, the topology $\tau_{\mathcal{P}}$ induced by the family $\mathcal{P}$ is by definition coarser than τ. On the other hand, each p_i is also $\tau_{\mathcal{P}}-$continuous and so U_{p_i} is a closed neighbourhood of the origin in $(X, \tau_{\mathcal{P}})$. Then $\mathcal{B}$ consists of neighbourhoods of the origin in $(X, \tau_{\mathcal{P}})$ which implies that τ is coarser than $\tau_{\mathcal{P}}$. Hence, $\tau \equiv \tau_{\mathcal{P}}$.

Viceversa, let us assume that (X, τ) is an lmc algebra. Then by Theorem 2.1.11 there exists a basis $\mathcal{N}$ of neighbourhoods of the origin in (X, τ) consisting of m-barrels. Consider now the family $\mathcal{S} := \{p_N : N \in \mathcal{N}\}$. By Lemma 2.2.7-a), we know that each p_N is a submultiplicative seminorm and that $\mathring{U}_{p_N} \subseteq N \subseteq U_{p_N}$. Now each p_N is $\tau-$continuous because $U_{p_N} \supseteq N \in \mathcal{N}$ and hence, $\tau_{\mathcal{S}} \subseteq \tau$. Moreover, each p_N is clearly $\tau_{\mathcal{S}}-$continuous and so, by Proposition 2.2.9, $\mathring{U}_{p_N}$ is open in $(X, \tau_{\mathcal{S}})$. Since $\mathring{U}_{p_N} \subseteq N$, we have that $\mathcal{N}$ consists of neighbourhoods of the origin in $(X, \tau_{\mathcal{S}})$, which implies $\tau \subseteq \tau_{\mathcal{S}}$. $\square$

Historically the following more general result holds for locally convex tvs and the previous theorem could be also derived as a corollary of:

Theorem 2.2.12. *Let X be a vector space and $\mathcal{P} := \{p_i\}_{i\in I}$ a family of seminorms. Then the topology induced by the family $\mathcal{P}$ is the unique topology both making X into a locally convex TVS and having as a basis of neighbourhoods of the origin the following collection:*

$$\mathcal{B} := \Big\{ \{x \in X : p_{i_1}(x) \leq \varepsilon, \dots, p_{i_n}(x) \leq \varepsilon\} : i_1, \dots, i_n \in I, n \in \mathbb{N}, 0 < \varepsilon \leq 1 \Big\}.$$

Viceversa, the topology of an arbitrary locally convex TVS is always induced by a family of seminorms (often called generating*).*

Coming back to lmc algebras, Theorem 2.2.11 allows us to give another characterization of such a class, namely: *A TA is lmc if and only if its topology is induced by a family of submultiplicative seminorms.* This is very helpful in establishing whether a given topological algebras is lmc or not as we will see from the following examples.

Examples 2.2.13.

1. *Normed algebras are clearly lmc algebras.*
2. *A seminormed algebra, i.e. a $\mathbb{K}-$algebra endowed with the topology generated by a submultiplicative seminorm, is lmc.*
3. *The weak and the strong operator topologies on the space $L(H)$ introduced in Example 1.2.17 both make $L(H)$ into a locally convex algebra which is not lmc. Indeed, the weak operator topology τ_w is generated by the family of seminorms $\{p_{x,y} : x, y \in H\}$ where $p_{x,y}(T) := |\langle Tx, y\rangle|$, while the strong operator topology τ_s is generated by the family of seminorms $\{p_x : x \in H\}$ where $p_x(T) := \|Tx\|$. If $(L(H), \tau_w)$ and $(L(H), \tau_s)$ were lmc algebras, then by Proposition 2.1.9 the multiplication should have been jointly continuous in both of them but this is not the case as we have already showed in Example 1.2.17.*
4. *Consider $L^\omega([0,1]) := \bigcap_{p\geq 1} L^p([0,1])$, where for each $p \geq 1$ we define $L^p([0,1])$ to be the space of all equivalence classes of functions $f : [0,1] \to \mathbb{R}$ such that $\|f\|_p := \left(\int_0^1 |f(t)|^p dt\right)^{\frac{1}{p}} < \infty$ which agree almost everywhere. The set $L^\omega([0,1])$ endowed with the pointwise operations is a real algebra since for any $q, r \geq 1$ such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ we have*

   $$\|fg\|_p \leq \|f\|_q \|g\|_r, \forall f, g \in L^\omega([0,1]).$$

   The algebra $L^\omega([0,1])$ endowed with the topology induced by the family $\mathcal{P} := \{\|\cdot\|_p : p \geq 1\}$ of seminorms is a locally convex algebra. However, $(L^\omega([0,1]), \tau_{\mathcal{P}})$ is not an lmc algebra because any m-convex subset U is open in $(L^\omega([0,1]), \tau_{\mathcal{P}})$ if and only if $U = L^\omega([0,1])$ (Sheet 3).