Let us conclude this section with a further very useful property of lmc algebras.

Proposition 2.2.14. *The topology of an lmc algebra can be always induced by a directed family of submultiplicative seminorms.*

Definition 2.2.15. *A family $\mathcal{Q} := \{q_j\}_{j\in J}$ of seminorms on a vector space X is said to be* directed *(or* fundamental *or* saturated*) if*

$$\forall\, n \in \mathbb{N}, j_1, \ldots, j_n \in J,\ \exists\, j \in J, C > 0 \ \text{ s.t. } \ Cq_j(x) \geq \max_{k=1,\ldots,n} q_{j_k}(x),\ \forall\, x \in X. \tag{2.4}$$

To prove Proposition 2.2.14 we need to recall an important criterion to compare topologies induced by families of seminorms.

Theorem 2.2.16.
Let $\mathcal{P} = \{p_i\}_{i\in I}$ and $\mathcal{Q} = \{q_j\}_{j\in J}$ be two families of seminorms on a $\mathbb{K}-$vector space X inducing respectively the topologies $\tau_{\mathcal{P}}$ and $\tau_{\mathcal{Q}}$. Then $\tau_{\mathcal{P}}$ is finer than $\tau_{\mathcal{Q}}$ (i.e. $\tau_{\mathcal{Q}} \subseteq \tau_{\mathcal{P}}$) iff

$$\forall q \in \mathcal{Q}\ \exists n \in \mathbb{N},\ i_1, \ldots, i_n \in I,\ C > 0 \ \text{ s.t. } \ Cq(x) \leq \max_{k=1,\ldots,n} p_{i_k}(x),\ \forall x \in X. \tag{2.5}$$

Proof.
Let us first recall that, by Theorem 2.2.12, we have that

$$\mathcal{B}_{\mathcal{P}} := \Big\{ \bigcap_{k=1}^{n} \varepsilon \mathring{U}_{p_{i_k}} : i_1, \ldots, i_n \in I, n \in \mathbb{N}, 0 < \varepsilon \leq 1 \Big\}$$

and

$$\mathcal{B}_{\mathcal{Q}} := \Big\{ \bigcap_{k=1}^{n} \varepsilon \mathring{U}_{q_{j_k}} : j_1, \ldots, j_n \in J, n \in \mathbb{N}, 0 < \varepsilon \leq 1 \Big\}.$$

are respectively bases of neighbourhoods of the origin for $\tau_{\mathcal{P}}$ and $\tau_{\mathcal{Q}}$.

By using Proposition 2.2.8, the condition (2.5) can be rewritten as

$$\forall q \in \mathcal{Q},\ \exists n \in \mathbb{N},\ i_1, \ldots, i_n \in I,\ C > 0 \ \text{ s.t. } \ C \bigcap_{k=1}^{n} \mathring{U}_{p_{i_k}} \subseteq \mathring{U}_q.$$

which means that

$$\forall q \in \mathcal{Q},\ \exists\ B_q \in \mathcal{B}_{\mathcal{P}} \ \text{ s.t. } \ B_q \subseteq \mathring{U}_q. \tag{2.6}$$

since $C \bigcap_{k=1}^{n} \mathring{U}_{p_{i_k}} \in \mathcal{B}_{\mathcal{P}}$.

Condition (2.6) means that for any $q \in \mathcal{Q}$ the set $\mathring{U}_q$ is a neighbourhood of the origin in $(X, \tau_{\mathcal{P}})$, which by Proposition 2.2.9 is equivalent to say that q is continuous w.r.t. $\tau_{\mathcal{P}}$. By definition of $\tau_{\mathcal{Q}}$, this gives that $\tau_{\mathcal{Q}} \subseteq \tau_{\mathcal{P}}$. $\square$

This theorem allows us to easily see that the topology induced by a family of seminorms on a vector space does not change if we close the family under taking the maximum of finitely many of its elements. Indeed, the following result holds.

Proposition 2.2.17. *Let $\mathcal{P} := \{p_i\}_{i\in I}$ be a family of seminorms on a $\mathbb{K}-$vector space (resp. submultiplicative seminorms on a $\mathbb{K}-$algebra) X. Then we have that $\mathcal{Q} := \left\{ \max_{i\in B} p_i : \emptyset \neq B \subseteq I \text{ with } B \text{ finite} \right\}$ is a directed family of seminorms (resp. submultiplicative seminorms) and $\tau_{\mathcal{P}} = \tau_{\mathcal{Q}}$, where $\tau_{\mathcal{P}}$ and $\tau_{\mathcal{Q}}$ denote the topology induced on X by $\mathcal{P}$ and $\mathcal{Q}$, respectively.*

Proof.
First of all let us note that, by Proposition 2.2.8-d), $\mathcal{Q}$ is a family of seminorms. On the one hand, since $\mathcal{P} \subseteq \mathcal{Q}$, by definition of induced topology we have $\tau_{\mathcal{P}} \subseteq \tau_{\mathcal{Q}}$. On the other hand, for any $q \in \mathcal{Q}$ we have $q = \max\limits_{i\in B} p_i$ for some $\emptyset \neq B \subseteq I$ finite. Then (2.5) is fulfilled for $n = |B|$ (where $|B|$ denotes the cardinality of the finite set B), $i_1, \ldots, i_n$ being the n elements of B and for any $0 < C \leq 1$. Hence, by Theorem 2.2.16, $\tau_{\mathcal{Q}} \subseteq \tau_{\mathcal{P}}$. If X is a $\mathbb{K}-$algebra and $\mathcal{P}$ consists of submultiplicative seminorms, then $\mathcal{Q}$ consists of submultiplicative seminorms by the second part of Proposition 2.2.8-d).

We claim that $\mathcal{Q}$ is directed.

Let $n \in \mathbb{N}$ and $q_1, \ldots, q_n \in \mathcal{Q}$. Then for each $j \in \{1, \ldots, n\}$ we have $q_j = \max\limits_{i\in B_j} p_i$ for some non-empty finite subset B_j of I. Let us define $B := \bigcup_{j=1}^{n} B_j$ and $q := \max\limits_{i\in B} p_i$. Then $q \in \mathcal{Q}$ and for any $C \geq 1$ we have that (2.4) is satisfied, because we get that for any $x \in X$

$$Cq(x) \geq \max_{i\in B} p_i(x) = \max_{j=1,\ldots,n} \left(\max_{i\in B_j} p_i(x) \right) = \max_{j=1,\ldots,n} q_j(x).$$

Hence, $\mathcal{Q}$ is directed.

□

We are ready now to show Proposition 2.2.14.

Proof. of Proposition 2.2.14
Let (X, τ) be an lmc algebra. By Theorem 2.2.11, we have that there exists a family of submultiplicative seminorms $\mathcal{P} := \{p_i\}_{i\in I}$ on X s.t. $\tau = \tau_{\mathcal{P}}$. Let us define $\mathcal{Q}$ as the collection obtained by forming the maximum of finitely many elements of $\mathcal{P}$, i.e. $\mathcal{Q} := \left\{ \max\limits_{i\in B} p_i : \emptyset \neq B \subseteq I \text{ with } B \text{ finite} \right\}$. By

Proposition 2.2.17, $\mathcal{Q}$ is a directed family of submultiplicative seminorms and we have that $\tau_{\mathcal{P}} = \tau_{\mathcal{Q}}$. □

It is possible to show (Sheet 3) that a basis of neighbourhoods of the origin for the lmc topology $\tau_{\mathcal{Q}}$ induced by a directed family of submultiplicative seminorms $\mathcal{Q}$ is given by:

$$\mathcal{B}_d := \{rU_q : q \in \mathcal{Q}, 0 < r \leq 1\}. \tag{2.7}$$

Remark 2.2.18. *The proof of Proposition 2.2.14 can be easily adapted to show that the topology of a lc tvs can be always induced by a directed family of seminorms $\tau_{\mathcal{Q}}$ and that the corresponding (2.7) is basis of neighbourhoods of the origin for $\tau_{\mathcal{Q}}$.*

Example 2.2.19. *Let $\mathcal{C}_b(\mathbb{R})$ the set of all real-valued bounded continuous functions on the real line endowed with the pointwise operations of addition, multiplication and scalar multiplication and endowed with the topology $\tau_{\mathcal{Q}}$ induced by the family $\mathcal{Q} := \{p_a : a > 0\}$, where $p_a(f) := \sup_{-a\leq t\leq a} |f(t)|$, $\forall f \in \mathcal{C}_b(\mathbb{R})$. Since each p_a is a submultiplicative seminorm (see Example 2.2.4-d)), the algebra $(\mathcal{C}_b(\mathbb{R}), \tau_{\mathcal{Q}})$ is lmc.*

Note that $\mathcal{Q}$ is directed since for any $n \in \mathbb{N}$ and any positive real numbers $a_1, \ldots, a_n$ we have that $\max_{i=1,\ldots,n} p_{a_i}(f) = \sup_{t\in[-b,b]} |f(t)| = p_b(f)$, where $b := \max_{i=1,\ldots,n} a_i$, and so (2.4) is fulfilled. Hence, $\mathcal{B}_d$ as in (2.7) is a basis of neighbourhoods of the origin for the lmc topology $\tau_{\mathcal{Q}}$.

The algebra $(\mathcal{C}_b(\mathbb{R}), \tau_{\mathcal{Q}})$ is not m-barrelled, because for instance the set $M := \{f \in \mathcal{C}_b(\mathbb{R}) : \sup_{t\in\mathbb{R}} |f(t)| \leq 1\}$ is an m-barrel but not a neighbourhood of the origin in $(\mathcal{C}_b(\mathbb{R}), \tau_{\mathcal{Q}})$. Indeed, no elements of the basis $\mathcal{B}_d$ of neighbourhoods of the origin is entirely contained in M, because for any $a > 0$ and any $0 < r \leq 1$ the set rU_{p_a} also contains continuous functions bounded by r on $[-a, a]$ but bounded by $C > 1$ on the whole $\mathbb{R}$ and so not belonging to M.

2.3 Hausdorff lmc algebras

In Section 1.3, we gave some characterization of Hausdorff TVS which can of course be applied to establish whether an lmc algebra is Hausdorff or not. However, in this section we aim to provide necessary and sufficient conditions bearing only on the family of seminorms generating an lmc topology for being a Hausdorff topology.

Definition 2.3.1.
A family of seminorms $\mathcal{P} := \{p_i\}_{i\in I}$ on a vector space X is said to be separating *if*
$$\forall x \in X \setminus \{o\}, \exists\ i \in I\ s.t.\ p_i(x) \neq 0. \tag{2.8}$$

Note that the separation condition (2.8) is equivalent to

$$p_i(x) = 0,\ \forall i \in I\ \Rightarrow x = o$$

which by using Proposition 2.2.8 can be rewritten as

$$\bigcap_{i\in I, c>0} c\mathring{U}_{p_i} = \{o\}, \tag{2.9}$$

since $p_i(x) = 0$ is equivalent to say that $p_i(x) < c$, for all $c > 0$.

It is clear that if any of the elements in a family of seminorms is actually a norm, then the the family is separating.

Lemma 2.3.2. *Let $\tau_{\mathcal{P}}$ be the topology induced by a separating family of seminorms $\mathcal{P} := (p_i)_{i\in I}$ on a vector space X. Then $\tau_{\mathcal{P}}$ is a Hausdorff topology.*

Proof. [2]
Let $x, y \in X$ be such that $x \neq y$. Since $\mathcal{P}$ is separating, we have that $\exists\ i \in I$ with $p_i(x-y) \neq 0$. Then $\exists\ \varepsilon > 0$ s.t. $p_i(x-y) = 2\varepsilon$. Take $V_x := x + \varepsilon\mathring{U}_{p_i}$ and $V_y := y + \varepsilon\mathring{U}_{p_i}$. Since Theorem 2.2.12 guarantees that $(X, \tau_{\mathcal{P}})$ is a TVS where the set $\varepsilon\mathring{U}_{p_i}$ is a neighbourhood of the origin, V_x and V_y are neighbourhoods of x and y, respectively. They are clearly disjoint. Indeed, if there would exist $u \in V_x \cap V_y$ then $p_i(x-y) = p_i(x-u+u-y) \leq p_i(x-u) + p_i(u-y) < 2\varepsilon$, which is a contradiction. $\square$

Proposition 2.3.3.
a) A locally convex TVS is Hausdorff if and only if its topology can be induced by a separating family of seminorms.
b) An lmc algebra is Hausdorff if and only if its topology can be induced by a separating family of submultiplicative seminorms.

[2] Alternative proof By Theorem 2.2.12, we know that $(X, \tau_{\mathcal{P}})$ is a TVS and that $\mathcal{B}_{\mathcal{P}} := \left\{ \bigcap_{k=1}^{n} \varepsilon\mathring{U}_{p_{i_k}} : i_1, \ldots, i_n \in I, n \in \mathbb{N}, 0 < \varepsilon \leq 1 \right\}$ is a basis of neighbourhoods of the origin. Then $\bigcap_{B\in\mathcal{B}_{\mathcal{P}}} B = \bigcap_{i\in I, \varepsilon>0} \varepsilon\mathring{U}_{p_i} \overset{(2.9)}{=} \{o\}$ and so Proposition 1.3.2 gives that $(X, \tau_{\mathcal{P}})$ is Hausdorff.

Proof.

a) Let (X, τ) be a locally convex TVS. Then we know that τ is induced by a directed family $\mathcal{P}$ of seminorms on X and that $\mathcal{B}_d := \{rU_p : p \in \mathcal{Q}, 0 < r \leq 1\}$ (see Remark 2.2.18).

Suppose that (X, τ) is also Hausdorff. Then Proposition 1.3.2 ensures that for any $x \in X$ with $x \neq o$ there exists a neighbourhood V of the origin in X s.t. $x \notin V$. This implies that there exists at least $B \in \mathcal{B}_d$ s.t. $x \notin B$,[3] i.e. there exist $p \in \mathcal{P}$ and $0 < r \leq 1$ s.t. $x \notin rU_p$. Hence, $p(x) > r > 0$ and so $p(x) \neq 0$, i.e. $\mathcal{P}$ is separating.

Conversely, if τ is induced by a separating family of seminorms $\mathcal{P}$, i.e. $\tau = \tau_{\mathcal{P}}$, then Lemma 2.3.2 ensures that X is Hausdorff.

b) A Hausdorff lmc algebra (X, τ) is in particular a Hausdorff lc tvs, so by a) there exists a separating family $\mathcal{P}$ of seminorms s.t. $\tau = \tau_{\mathcal{P}}$. Since (X, τ) is an lmc algebra, Theorem 2.2.11 ensures that there exists $\mathcal{Q}$ family of submultiplicative seminorms s.t. $\tau = \tau_{\mathcal{Q}}$. Hence, we have got $\tau_{\mathcal{P}} = \tau_{\mathcal{Q}}$ which gives in turn that for any $p \in \mathcal{P}$ there exist $q_1, q_2 \in \mathcal{Q}$ and $C_1, C_2 > 0$ s.t. $C_1 q_1(x) \leq p(x) \leq C_2 q_2(x), \forall x \in X$. This gives in turn that if $q(x) = 0$ for all $q \in \mathcal{Q}$ then we have $p(x) = 0$ for all $p \in \mathcal{P}$ which implies $x = 0$ because $\mathcal{P}$ is separating. This shows that $\mathcal{Q}$ is a separating family of submultiplicative seminorms. Conversely, if τ is induced by a separating family of submultiplicative seminorms $\mathcal{P}$, i.e. $\tau = \tau_{\mathcal{P}}$, then Lemma 2.3.2 ensures that X is Hausdorff and Theorem 2.2.11 that it is an lmc algebra. □

Examples 2.3.4.

1. *Every normed algebra is a Hausdorff lmc algebra, since every submultiplicative norm is a submultiplicative seminorm satisfying the separation property. Therefore, every Banach algebra is a complete Hausdorff lmc algebra.*
2. *Every family of submultiplicative seminorms on a vector space containing a submultiplicative norm induces a Hausdorff llmc topology.*
3. *Given an open subset Ω of $\mathbb{R}^d$ with the euclidean topology, the space $\mathcal{C}(\Omega)$ of real valued continuous functions on Ω with the so-called* topology of uniform convergence on compact sets *is a lmc algebra. This topology is defined by the family $\mathcal{P}$ of all the submultiplicative seminorms on $\mathcal{C}(\Omega)$ given by*
$$p_K(f) := \max_{x \in K} |f(x)|, \ \forall K \subset \Omega \ \textit{compact}.$$

[3] Since $\mathcal{B}_d$ is a basis of neighbourhoods of the origin, $\exists\, B \in \mathcal{B}_d$ s.t. $B \subseteq V$. If x would belong to all elements of the basis then in particular it would be $x \in B$ and so also $x \in V$, contradiction.

Moreover, $(\mathcal{C}(\Omega), \tau_{\mathcal{P}})$ is Hausdorff, because the family $\mathcal{P}$ is clearly separating. In fact, if $p_K(f) = 0$, $\forall K$ compact subsets of Ω then in particular $p_{\{x\}}(f) = |f(x)| = 0$, $\forall x \in \Omega$, which implies $f \equiv 0$ on Ω.

2.4 The finest lmc topology

In the previous sections we have seen how to generate topologies on an algebra which makes it into an lmc algebra. Among all of them, there is the finest one (i.e. the one having the largest number of open sets).

Proposition 2.4.1. *The finest lmc topology on an algebra X is the topology induced by the family of all submultiplicative seminorms on X.*

Proof.
Let us denote by $\mathcal{S}$ the family of all submultiplicative seminorms on the vector space X. By Theorem 2.2.11, we know that the topology $\tau_{\mathcal{S}}$ induced by $\mathcal{S}$ makes X into an lmc algebra. We claim that $\tau_{\mathcal{S}}$ is the finest lmc topology. In fact, if there was a finer lmc topology τ (i.e. $\tau_{\mathcal{S}} \subseteq \tau$ with (X, τ) lmc algebra) then Theorem 2.2.11 would give that τ is also induced by a family $\mathcal{P}$ of submultiplicative seminorms. But then $\mathcal{P} \subseteq \mathcal{S}$ and so $\tau = \tau_{\mathcal{P}} \subseteq \tau_{\mathcal{S}}$ by definition of induced topology. Hence, $\tau = \tau_{\mathcal{S}}$. □

An alternative way of describing the finest lmc topology on an algebra without using the seminorms is the following:

Proposition 2.4.2. *The collection of all absorbing absolutely m-convex sets of an algebra X is a basis of neighbourhoods of the origin for the finest lmc topology on X.*

Proof.
Let τ_{max} be the finest lmc topology on X and $\mathcal{M}$ the collection of all absorbing absolutely m-convex sets of X. Since $\mathcal{M}$ fulfills all the properties required in Corollary 2.1.12, there exists a unique topology τ which makes X into an lmc algebra having as basis of neighbourhoods of the origin $\mathcal{M}$. Hence, by definition of finest lmc topology, $\tau \subseteq \tau_{max}$. On the other hand, (X, τ_{max}) is itself an lmc algebra and so Theorem 2.2.11 ensures that has a basis $\mathcal{B}_{max}$ of neighbourhoods of the origin consisting of absorbing absolutely m-convex subsets of X. Then clearly $\mathcal{B}_{max}$ is contained in $\mathcal{M}$ and, hence, $\tau_{max} \subseteq \tau$. □

This result can be proved also using Proposition 2.4.1 and the correspondence between Minkowski functionals and absorbing absolutely convex subsets of X introduced in the Section 2.2.