Corollary 2.4.3. *Every $\mathbb{K}-$algebra endowed with the finest lmc topology is an m-barrelled algebra.*

Proof.
Let U be an m-barrel of (X, τ_{max}). Then U is closed absolutely m-convex and so it is a neighbourhood of the origin by the previous proposition. □

Using basically the same proofs, we could show the analogous results for the finest lc topology, namely:

The finest lc topology on a $\mathbb{K}-$vector space X is the topology induced by the family of all seminorms on X or equivalently the topology having the collection of all absorbing absolutely convex sets of X as a basis of neighbourhoods of the origin. Hence, every vector space endowed with the finest lc topology is a barrelled space.

Recall that

Definition 2.4.4. *A closed absorbing absolutely convex subset of a TVS is called a* barrel*. A TVS in which every barrel is a neighbourhood of the origin is called* barrelled space*.*

It is also important to remark that while the finest lc topology on a $\mathbb{K}-$vector space (and in particular on a $\mathbb{K}-$algebra) is always Hausdorff, the finest lmc topology on a $\mathbb{K}-$algebra does not have necessarily this property.

Proposition 2.4.5. *Any $\mathbb{K}-$vector space endowed with the finest lc topology is a Hausdorff TVS.*

Proof.
Let X be any non-empty $\mathbb{K}-$vector space and $\mathcal{S}$ the family of all seminorms on X. By Proposition 2.3.3-a), it is enough to show that $\mathcal{S}$ is separating. We will do that, by proving that there always exists a non-zero norm on X. In fact, let $\mathcal{B} = (b_i)_{i \in I}$ be an algebraic basis of X then for any $x \in X$ there exist a finite subset J of I and $\lambda_j \in \mathbb{K}$ for all $j \in J$ s.t. $x = \sum_{j \in J} \lambda_j b_j$ and so we can define $\|x\| := \max_{j \in J} |\lambda_j|$. Then it is easy to check that $\|\cdot\|$ is a norm on X and so $\|\cdot\| \in \mathcal{S}$. □

Note that if X is a $\mathbb{K}-$algebra, then the previous proof does not guaranteed the existence of a non-zero norm on X because, depending on the multiplication in X, the norm $\|\cdot\|$ might be or not submultiplicative. In fact, there exist algebras on which no submultiplicative norm can be defined. For instance, if the algebra $\mathcal{C}(Y)$ of all complex valued continuous functions on a topological space Y contains an unbounded function then it does not admit a submultiplicative norm (see Sheet 4). Actually, there exist algebras on which no non-zero submultiplicative seminorms can be defined, e.g. the algebra of all linear operator on an infinite dimensional complex vector space (see [13, Theorem 3]). The finest lmc topology on such algebras is the trivial topology which is obviously not Hausdorff.

We conclude this section with a nice further property of the finest lmc topology involving characters of an algebra.

Definition 2.4.6. *Let A be a $\mathbb{K}-$algebra. A* character *of A is a non-zero homomorpism of A into $\mathbb{K}$. The set of all characters is denoted by $\mathcal{X}(A)$.*

Proposition 2.4.7. *Every character on a $\mathbb{K}-$algebra A is continuous w.r.t. the finest lmc topology on A.*

Proof. Let $\alpha : A \to \mathbb{K}$ be a character on A. For any $\varepsilon > 0$, we denote by $B_\varepsilon(0)$ the open ball in $\mathbb{K}$ of radius ε and center $0 \in \mathbb{K}$, i.e. $B_\varepsilon(0) := \{k \in \mathbb{K} : |k| < \varepsilon\}$. Set $p(a) := |\alpha(a)|$ for all $a \in A$. Then p is a submultiplicative seminorm on A since for any $a, b \in A$ and $\lambda \in \mathbb{K} \setminus \{0\}$ we have that:

- $p(a + b) = |\alpha(a + b)| = |\alpha(a) + \alpha(b)| \leq |\alpha(a)| + |\alpha(b)| = p(a) + p(b)$
- $p(\lambda a) = |\alpha(\lambda a)| = |\lambda\alpha(a)| = |\lambda||\alpha(a)| = |\lambda|p(a)$
- $p(ab) = |\alpha(ab)| = |\alpha(a)\alpha(b)| = |\alpha(a)||\alpha(b)| = p(a)p(b)$.

Then $\alpha^{-1}(B_\varepsilon(0)) = \{a \in A : |\alpha(a)| < \varepsilon\} = \varepsilon \mathring{U}_p$, which is an absorbing absolutely m-convex subset of X and so, by Proposition 2.4.2, it is a neighbourhood of the origin in the finest lmc topology on X. Hence α is continuous at the origin and so continuous everywhere in A. □

With a proof similar to the previous one, we can deduce that

Proposition 2.4.8. *Every linear functional on a $\mathbb{K}-$vector space X is continuous w.r.t. the finest lc topology on X.*

2.5 Topological algebras admitting lmc topologies

In this section we will look for sufficient conditions on a TA to be an lmc algebra. More precisely, we would like to find out under which conditions a locally convex algebra (i.e. a TA which is a locally convex TVS) is in fact an lmc algebra. The main result in this direction was proved by Michael in 1952 (see [14, Proposition 4.3]) and it is actually a generalization of a well-known theorem by Gel'fand within the theory of Banach algebras (see [8]).

Theorem 2.5.1 (Michael's Theorem)**.** *Let A be a lc algebra. If*
a) A is m-barrelled, and

b) there exists a basis $\mathcal{M}$ of neighbourhoods of the origin in A such that

$$\forall a \in A, \forall U \in \mathcal{M}, \exists \lambda > 0 : aU \subseteq \lambda U, \tag{2.10}$$

then A is an lmc algebra.

Proof.
Let us first give the main proof structure and then proceed to show the more technical details.

Claim 1 W.l.o.g. we can assume that $\mathcal{M}$ consists of barrels.
Consider the unitization A_1 of A equipped with the product topology (see Definition 1.1.3 and Section 1.3). Denote by $\cdot$ the multiplication in A_1 and by $\overline{B}_1(0) := \{k \in \mathbb{K} : |k| \leq 1\}$. Then the family $\{\overline{B}_1(0) \times U : U \in \mathcal{M}\}$ is a basis of neighbourhoods of the origin $(0, o)$ in A_1 and the following holds.

Claim 2 For any $U \in \mathcal{M}$, $V(U) := \{x \in A : (0,x) \cdot (\overline{B}_1(0) \times U) \subseteq (\overline{B}_1(0) \times U)\}$ is an m-barrel subset of A.
Then the assumption a) ensures that each $V(U)$ is a neighbourhood of the origin in A. Moreover, for any $U \in \mathcal{M}$, $(1, o) \in (\overline{B}_1(0) \times U)$ and so

$$\forall x \in V(U), (0, x) = (0, x) \cdot (1, o) \in (\overline{B}_1(0) \times U),$$

which provides that $V(U) \subseteq U$. Hence, $\{V(U) : U \in \mathcal{M}\}$ is a basis of neighbourhoods of the origin in A consisting of m-barrels and so, by Theorem 2.1.11, A is an lmc algebra.

Proof. Claim 1
If $\mathcal{M}$ is not already consisting of all barrels, then we can always replace it by $\widetilde{M} := \{\overline{\text{conv}_b(U)} : U \in \mathcal{M}\}$, because $\widetilde{M}$ is a basis of neighbourhoods of the origin in A fulfilling (2.10).

In fact, since A is a lc TVS, then there exists a basis $\mathcal{N}$ of neighbourhoods of the origin in A consisting of barrels. Then, since also $\mathcal{M}$ is a basis of neighbourhoods of the origin in A, we have that:

$$\forall V \in \mathcal{N}, \exists U \in \mathcal{M} : U \subseteq V.$$

As $\overline{\text{conv}_b(U)}$ is the smallest closed convex balanced subset of A containing U and V has all such properties, we get that $\overline{\text{conv}_b(U)} \subseteq V$. Hence, $\widetilde{M}$ is a basis of neighbourhoods of the origin in A.

Moreover, let $a \in A$ and $U \in \mathcal{M}$. By assumption b), we know that there exists $\lambda > 0$ such that $aU \subseteq \lambda U$. Now recalling that $\text{conv}_b(U) = \text{conv}(\text{bal}(U))$, we can write any $x \in \text{conv}_b(U)$ as $x = \sum_{i=1}^{n} \mu_i \delta_i u_i$ for some $n \in \mathbb{N}$, $u_i \in U$,

$\mu_i \in [0,1]$ with $\sum_{i=1}^n \mu_i = 1$, and $\delta_i \in \mathbb{K}$ with $|\delta_i| \leq 1$. Then for each $i \in \{1,\ldots,n\}$ there exist $\tilde{u}_i \in U$ such that:

$$ax = \sum_{i=1}^n \mu_i \delta_i a u_i = \sum_{i=1}^n \mu_i \delta_i \lambda \tilde{u}_i = \lambda \sum_{i=1}^n \mu_i (\delta_i \tilde{u}_i)$$

and so $ax \in \lambda \cdot \mathrm{conv_b}(U)$. Hence, $a \cdot \mathrm{conv_b}(U) \subseteq \lambda \cdot \mathrm{conv_b}(U)$. This together with the separate continuity of the multiplication in A and the fact that the scalar multiplication is a homeomorphism imply that

$$a \cdot \overline{\mathrm{conv_b}(U)} \subseteq \overline{a \cdot \mathrm{conv_b}(U)} \subseteq \overline{\lambda \cdot \mathrm{conv_b}(U)} = \lambda \cdot \overline{\mathrm{conv_b}(U)}.$$

This shows that $\widetilde{M}$ fulfills (2.10). □

Proof. Claim 2
Let $U \in \mathcal{M}$ and $V(U) := \{x \in A : (0,x) \cdot (\overline{B}_1(0) \times U) \subseteq (\overline{B}_1(0) \times U)\}$. Then:

• <u>$V(U)$ is multiplicative.</u>
For any $a, b \in V(U)$ we have
$(0,ab)\cdot(\overline{B}_1(0)\times U) = (0,a)\cdot(0,b)\cdot(\overline{B}_1(0)\times U) \subseteq (0,a)\cdot(\overline{B}_1(0)\times U) \subseteq (\overline{B}_1(0)\times U)$,
i.e. $ab \in V(U)$.

• <u>$V(U)$ is closed.</u>
Let us show that $A \setminus V(U)$ is open, i.e. that for any $x \in A \setminus V(U)$ there exists $N \in \mathcal{M}$ such that $x + N \subseteq A \setminus V(U)$. If $x \in A \setminus V(U)$, then there exist $t \in \overline{B}_1(0)$ and $u \in U$ such that $(0,x)\cdot(t,u) \notin (\overline{B}_1(0)\times U)$, i.e. $tx+ux \in A \setminus U$. As U is closed, $A \setminus U$ is open and so there exists $W \in \mathcal{M}$ s.t.

$$tx + ux + W \subseteq A \setminus U. \tag{2.11}$$

Take $N \in \mathcal{M}$ s.t. $uN \subseteq \frac{1}{2}W$ and $N \subseteq \frac{1}{2}W$ (this exists because left multiplication is continuous and $\mathcal{M}$ is basis of neighbourhoods of the origin). Then $x + N \subseteq A \setminus U$, because otherwise there would exists $n \in N$ such that $x + n \in V(U)$ and so $(0, n+x)\cdot(t,u) \in (\overline{B}_1(0)\times U)$ that is $nt+nu+xt+xu \in U$, which in turns implies $xt+xu \in U - tN - uN \subseteq U - N - \frac{1}{2}W \subseteq U - \frac{1}{2}W - \frac{1}{2}W \subseteq U - W$, i.e. $xt + xu + W \subseteq U$ which contradicts (2.11).

• <u>$V(U)$ is absorbing.</u>
Let $a \in A$. Then (2.10) ensures that there exists $\lambda > 0$ s.t. $aU \subseteq \lambda U$. Also, since U is absorbing, there exists $\mu > 0$ such that $a \in \mu U$. Take $\rho := \frac{1}{\lambda+\mu}$. Then for all $k \in \mathbb{K}$ with $|k| \leq \rho$ and for any $(t,u) \in (\overline{B}_1(0) \times U)$ we get that $kta + kau \in kt\mu U + k\lambda U \subseteq k\mu U + k\lambda U = k(\mu+\lambda)U \subseteq U$ where in both inclusions we have used that U is balanced together with $|t| \leq 1$ in the first and $|k(\mu+\lambda)| \leq \rho|\mu+\lambda| = 1$ in the second. Hence, we have obtained that $(0,ka)\cdot(t,u) = (0, kta+kau) \in (\overline{B}_1(0) \times U)$ which gives that $ka \in V(U)$.

• $V(U)$ is balanced.
Let $a \in V(U)$ and $k \in \mathbb{K}$ with $|k| \leq 1$. Then

$$(0, ka) \cdot (\overline{B}_1(0) \times U) = k(0, a) \cdot (\overline{B}_1(0) \times U) \subseteq (k\overline{B}_1(0) \times kU) \subseteq (\overline{B}_1(0) \times U)$$

where in the last inclusion we used that both $\overline{B}_1(0)$ and U are balanced.
• $V(U)$ is convex.
Let $a, b \in V(U)$ and $\mu \in [0, 1]$. Then for any $(t, u) \in (\overline{B}_1(0) \times U)$ we know that $(0, a) \cdot (t, u) \in (\overline{B}_1(0) \times U)$ and $(0, b) \cdot (t, u) \in (\overline{B}_1(0) \times U)$, which give in turn that $at + au \in U$ and $bt + bu \in U$. Therefore, the convexity of U implies that $\mu(at + au) + (1 - \mu)(bt + bu) \in U$ and so we obtain

$$(0, \mu a + (1 - \mu)b) \cdot (t, u) = (0, \mu(at + au) + (1 - \mu)(bt + bu)) \in (\overline{B}_1(0) \times U),$$

i.e. $\mu a + (1 - \mu)b \in V(U)$. □

□

Let us present now a stronger version of Michael's theorem, which has however the advantage of providing a less technical and so more manageable sufficient condition for a topology to be lmc. This more convenient condition actually identifies an entire class of TA: the so-called *A-convex algebras* introduced by Cochran, Keown and Williams in the early seventies [3].

Definition 2.5.2. *A $\mathbb{K}-$algebra X is called A-convex if it is endowed with the topology induced by an absorbing family of seminorms on X.*

Definition 2.5.3. *A seminorm p on a $\mathbb{K}-$algebra X is called:*

- left absorbing *if $\forall a \in X, \exists \lambda > 0$ s.t. $p(ax) \leq \lambda p(x), \forall x \in X$.*
- right absorbing *if $\forall a \in X, \exists \lambda > 0$ s.t. $p(xa) \leq \lambda p(x), \forall x \in X$.*
- absorbing *if it is both left and right absorbing.*

Proposition 2.5.4. *Every A-convex algebra is a lc algebra.*

Proof.
Let (X, τ) be an A-convex algebra. Then by definition $\tau = \tau_{\mathcal{P}}$ where $\mathcal{P}$ is a family of absorbing seminorm. Hence, by Theorem 2.2.12, (X, τ) is an lc TVS. It remains to show that it is a TA. Let $a \in X$ and consider the left multiplication $\ell_a : X \to X, x \mapsto ax$. Since any $p \in \mathcal{P}$ is left absorbing, we have that there exists $\lambda > 0$ such that $p(ax) \leq \lambda p(x)$ for all $x \in X$ and so that $\frac{1}{\lambda} U_p \subseteq \ell_a^{-1}(U_p)$. Hence, ℓ_a is $\tau-$continuous. Similarly, one can prove the continuity of the right multiplication. We can then conclude that (X, τ) is an lc algebra. □

Note that not every lc algebra is A-convex (see Sheet 4) but every lmc algebra is A-convex as the submultiplicativity of the generating seminorms implies that they are absorbing. Let us focus now on the inverse question of establishing when an A-convex algebra is lmc.

Theorem 2.5.5. *Every m-barrelled A-convex algebra is an lmc algebra.*

Proof.
Let (X, τ) be an m-barrelled A-convex algebra. By the previous proposition, we have that (X, τ) is an lc algebra. Denote by $\mathcal{P} := \{p_i : i \in I\}$ a family of absorbing seminorm generating τ. Then, by Proposition 2.2.17, $\mathcal{Q} := \left\{ \max_{i\in B} p_i : \emptyset \neq B \subseteq I \text{ with } B \text{ finite} \right\}$ is a directed family of seminorms such that $\tau = \tau_{\mathcal{Q}}$. Also, each $q \in \mathcal{Q}$ is absorbing. Indeed, $q = \max_{i\in B} p_i$ for some $\emptyset \neq B \subseteq I$ with B finite and so for any $i \in B$ and any $a \in X$ we have that there exists $\lambda_i > 0$ such that $p_i(ax) \leq \lambda_i p(x)$ for all $x \in X$. Hence, for any $a \in X$ we get

$$q(ax) = \max_{i\in B} p_i(ax) \leq \max_{i\in B} \lambda_i p_i(x) \leq \lambda \max_{i\in B} p_i(x) = q(x), \forall x \in X,$$

where $\lambda := \max_{i\in B} \lambda_i$. Then $\mathcal{M} := \{\varepsilon U_q : q \in \mathcal{Q}, 0 < \varepsilon \leq 1\}$ is a basis of neighbourhoods of the origin for (X, τ) and for each $a \in A, q \in \mathcal{Q}$ and $0 < \varepsilon \leq 1$ we have that if $x \in a\varepsilon U_q$ then $x = a\varepsilon y$ for some $y \in U_q$ and so $q(x) = q(a\varepsilon y) \leq \varepsilon q(ay) \leq \lambda\varepsilon q(y) \leq \lambda\varepsilon$ i.e. $x \in \lambda\varepsilon U_q$. Hence, we proved that $\forall a \in A, \forall q \in \mathcal{Q}, \forall 0 < \varepsilon \leq 1, a\varepsilon U_q \subseteq \lambda\varepsilon U_q$ which means that $\mathcal{M}$ fulfills condition b) in Theorem 2.5.1. Then (X, τ) satisfies all the assumptions of Theorem 2.5.1 which guarantees that it is an lmc algebra. □

To conclude this section let us just restate the result by Gel'fand mentioned in the beginning in one of the many formulation which reveals the analogy with Michael's theorem.

Theorem 2.5.6. *If X is a $\mathbb{K}-$algebra endowed with a norm which makes it into a Banach space and a TA, then there exists an equivalent norm on X which makes it into a Banach algebra.*