Chapter 3

Further special classes of topological algebras

3.1 Metrizable and Fréchet algebras

Definition 3.1.1. *A* metrizable algebra X *is a TA which is in particular a metrizable TVS, i.e. a TVS whose topology is induced by a metric.*

We recall that a metric d on a set X is a mapping $d : X \times X \to \mathbb{R}^+$ with the following properties:

1. $d(x, y) = 0$ if and only if $x = y$ (identity of indiscernibles);
2. $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetry);
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (triangular inequality).

Saying that the topology of a TVS (X, τ) is induced by a metric d means that for any $x \in X$ the sets of all open (or equivalently closed) balls:

$$B_r(x) := \{y \in X : d(x, y) < r\}, \quad \forall r > 0$$

forms a basis of neighbourhoods of x for τ.

There exists a completely general characterization of metrizable TVS.

Theorem 3.1.2. *A TVS X is metrizable if and only if X is Hausdorff and has a countable basis of neighbourhoods of the origin.*

Note that one direction is quite straightforward. Indeed, suppose that X is a metrizable TVS and that d is a metric defining the topology of X, then the collection of all $B_{\frac{1}{n}}(o)$ with $n \in \mathbb{N}$ is a countable basis of neighbourhoods of the origin o in X. Moreover, the intersection of all these balls is just the singleton $\{o\}$, which proves that the TVS X is also Hausdorff (see Proposition 1.3.2) The other direction requires more work and we are not going to prove it in full generality as it would go beyond the aim of this course (see e.g. [16, Chapter I, Section 6.1] or [11, proof of Theorem 1.1] for a proof for locally convex TVS).

However, we are going to use Theorem 3.1.2 to give a characterization of all metrizable lmc algebras.

Theorem 3.1.3. *Let A be a $\mathbb{K}-$algebra. Then the following are equivalent:*

a) A is a metrizable lmc algebra

b) A is a TVS having a decreasing sequence of m-barrels with trivial intersection as a basis of neighbourhoods of the origin.

c) A is a TVS whose topology is generated by an increasing sequence of submultiplicative seminorms which form a separating family.

The obvious analogous statement is true for metrizable lc algebras.

Proof.

a)⇒b) Suppose that (A, τ) is a metrizable lmc algebra. Then in particular (A, τ) is a metrizable TVS and so by Theorem 3.1.2 it is Hausdorff and has a countable basis $\{U_n : n \in \mathbb{N}\}$ of neighbourhoods of the origin. As (A, τ) is an lmc algebra, by Theorem 2.1.11, we can assume w.l.o.g. that each U_n is an m-barrel. Now for each $n \in \mathbb{N}$ set $V_n := U_1 \cap \cdots \cap U_n$. Then one can easily verify that each V_n is still an m-barrel and clearly $V_{n+1} \subseteq V_n$. Hence, the decreasing sequence $\{V_n : n \in \mathbb{N}\}$ is a basis of neighbourhoods of the origin in (A, τ) consisting of m-barrels. The Hausdorfness of (A, τ) implies, by Proposition 1.3.2, that $\bigcap_{n\in\mathbb{N}} V_n = \{o\}$.

b)⇒c) Suppose that (A, τ) is a TVS and that $\{V_n : n \in \mathbb{N}\}$ is a basis of neighbourhoods of the origin such that, for any $n \in \mathbb{N}$, V_n is an m-barrel, $V_{n+1} \subseteq V_n$ and $\bigcap_{n\in\mathbb{N}} V_n = \{o\}$. Then Theorem 2.1.11 guarantees that (A, τ) is an lmc algebra and the family $\mathcal{S} := \{p_{V_n} : n \in \mathbb{N}\}$ is a family of submultiplicative seminorms generating τ (see proof of Theorem 2.2.12). Actually, $\mathcal{S}$ is an increasing sequence, since $V_{n+1} \subseteq V_n$ implies that $p_{V_n} \leq p_{V_{n+1}}$. Moreover, we have that $\{o\} \subseteq \bigcap_{n\in\mathbb{N}} \mathring{U}_{p_{V_n}} \subseteq \bigcap_{n\in\mathbb{N}} V_n = \{o\}$ and so $\bigcap_{n\in\mathbb{N},c>0} c\mathring{U}_{p_{V_n}} = \{o\}$, i.e. $\mathcal{S}$ is separating (c.f. (2.9)).

c)⇒a) Suppose that (A, τ) is a TVS and that $\mathcal{P} := \{p_n : n \in \mathbb{N}\}$ is a separating increasing sequence of submultiplicative seminorm generating τ. By Theorem 2.2.12 and Proposition 2.3.3, (A, τ) is a Hausdorff lmc algebra. W.l.o.g. we can assume that $\mathcal{P}$ is directed and so, by using Exercise 3 in Sheet 3, we have that $\{\frac{1}{n}U_{p_n} : n \in \mathbb{N}\}$ is a countable basis of neighbourhoods of the origin. Then Theorem 3.1.2 ensures that (A, τ) is also a metrizable TVS and, hence, a metrizable lmc algebra.

□

A special class of metrizable algebras are the so-called *Fréchet algebras.*

Definition 3.1.4. *A* Fréchet algebra *is a TA which is in particular a Fréchet TVS, i.e. a complete metrizable lc TVS.*

It is clear that every Fréchet algebra is a Hausdorff complete lc algebra whose topology is induced by an increasing family of seminorms, but these are not necessarily submultiplicative. If this is the case, we speak of *Fréchet lmc algebras.*

Definition 3.1.5. *A* Fréchet lmc algebra *is a complete metrizable lmc algebra.*

As completeness is fundamental to understand the structure of a Fréchet algebra, let us recall here some of the most important properties of complete TVS (for a more detailed exposition about complete TVS see e.g. [10, Section 2.5] or [17, Part I, Section 5]).

Definition 3.1.6.
A TVS X is said to be complete *if every Cauchy filter on X converges to a point x of A.*

It is important to distinguish between completeness and sequentially completeness.

Definition 3.1.7.
A TVS X is said to be sequentially complete *if any Cauchy sequence in X converges to a point in A.*

Clearly, a TA is complete (resp. sequentially complete) if it is in particular a complete (resp. sequentially complete) TVS. Remind that

Definition 3.1.8. *A filter $\mathcal{F}$ on a TVS (X, τ) is said to be a* Cauchy filter *if*

$$\forall\, U \in \mathcal{F}(o) \, in \, X, \, \exists\, M \in \mathcal{F} : \, M - M \subset U, \tag{3.1}$$

where $\mathcal{F}(o)$ denotes the filter of neighbourhoods of the origin o in (X, τ).

Definition 3.1.9. *A sequence $S := \{x_n\}_{n\in\mathbb{N}}$ of points in a TVS (X, τ) is said to be a* Cauchy sequence *if*

$$\forall\, U \in \mathcal{F}(o) \, in \, X, \, \exists\, N \in \mathbb{N} : \, x_m - x_n \in U, \, \forall m, n \geq N, \tag{3.2}$$

where $\mathcal{F}(o)$ denotes the filter of neighbourhoods of the origin o in (X, τ).

Proposition 3.1.10.
The filter associated with a Cauchy sequence in a TVS X is a Cauchy filter.

Proof.
Let S be a Cauchy sequence. Then, recalling that the collection $\mathcal{B} := \{S_m : m \in \mathbb{N}\}$ with $S_m := \{x_n \in S : n \geq m\}$ is a basis of the filter $\mathcal{F}_S$ associated with S, we can easily rewrite (3.2) as

$$\forall\, U \in \mathcal{F}(o) \,\text{in}\, X,\, \exists\, N \in \mathbb{N} :\, S_N - S_N \subset U.$$

This immediately gives that $\mathcal{F}_S$ fulfills (3.2) and so that it is a Cauchy filter. □

It is then not hard to prove that

Proposition 3.1.11.
If a TVS X is completem then A is sequentially complete.

Proof.
Let $S := \{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence of points in A. Then Proposition 3.1.10 guarantees that the filter $\mathcal{F}_S$ associated to S is a Cauchy filter in A. By the completeness of A we get that there exists $x \in A$ such that $\mathcal{F}_S$ converges to x. This is equivalent to say that the sequence S is convergent to $x \in A$ (see [10, Proposition 1.1.29]). Hence, A is sequentially complete. □

The converse is false in general (see [10, Example 2.5.9]). However, the two notions coincide in metrizable TVS, and so we have that

Proposition 3.1.12. *A metrizable lc algebra is a Fréchet algebra if and only if it is sequentially complete.*

Another important property of Fréchet algebras is that they are Baire spaces, i.e. topological spaces in which the union of any countable family of closed sets, none of which has interior points, has no interior points itself (or, equivalently, the intersection of any countable family of everywhere dense open sets is an everywhere dense set). This is actually a consequence of the following more general result:

Proposition 3.1.13. *A complete metrizable TVS X is a Baire space.*

Proof. (see [11, Proposition 1.1.8]) □

Example 3.1.14. *An example of Baire space is $\mathbb{R}$ with the euclidean topology. Instead $\mathbb{Q}$ with the subset topology given by the euclidean topology on $\mathbb{R}$ is not a Baire space. Indeed, for any $q \in \mathbb{Q}$ the subset $\{q\}$ is closed and has empty* *interior in $\mathbb{Q}$, but $\cup_{q\in\mathbb{Q}}\{q\} = \mathbb{Q}$ which has interior points in $\mathbb{Q}$ (actually its interior is the whole $\mathbb{Q}$).*

Corollary 3.1.15. *Every Fréchet TVS is barrelled. In particular, every Fréchet algebra is m-barrelled.*

Proof.
Let (X,τ) be a Fréchet TVS and V a barrelled subset of X. Then V is absorbing and closed, so $X = \bigcup_{n\in\mathbb{N}} nV$ is a countable union of closed sets. Hence, as Proposition 3.1.13 ensures that (X,τ) is a Baire space, we have that there exists $k \in \mathbb{N}$ such that $(k\mathring{V}) \neq \emptyset$. This implies that there exists $x \in (k\mathring{V})$, i.e. there exists a neighbourhood N of the origin in (X,τ) such that $x + N \subseteq V$. As (X,τ) is in particular an lc TVS, we can assume that N is absolutely convex. Then we get

$$N = \frac{1}{2}N - \frac{1}{2}N = \frac{1}{2}(x+N) + \frac{1}{2}(-x-N) \subseteq \frac{1}{2}V + \frac{1}{2}(-V) = V,$$

where in the last equality we used that V is a barrel and so absolutely convex. Hence, we can conclude that V is a neighbourhood of the origin and so (X,τ) is barrelled.

If (X,τ) is a Fréchet algebra, then it is in particular a Fréchet TVS and so the previous part of the proof guarantees that every m-barrelled subset of X is a neighbourhood of the origin, i.e. (X,τ) is an m-barrelled algebra. □

This result together with Theorem 2.5.1 (resp. Theorem 2.5.5) clearly provides that every Fréchet algebra having a basis of neighbourhoods of the origin which satifies (2.10) (resp. every A-convex Fréchet algebra) is lmc. Proposition 3.1.13 plays also a fundamental role in proving the following general property of complete metrizable algebras and so of Fréchet algebras.

Proposition 3.1.16. *Every complete metrizable algebra is a TA with continuous multiplication.*

Proof.
Let A be a complete metrizable algebra. The metrizability provides the existence of a countable basis $\mathcal{B} := \{W_n : n \in \mathbb{N}\}$ of neighbourhoods of the origin. We aim to show that for any $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $W_m W_m \subseteq W_n$.

Fixed $n \in \mathbb{N}$, as we are in a TVS, there always exists a closed neighbourhood V of the origin such that $V - V \subseteq W_n$. As for any $b \in A$ the right multiplication $r_b : A \to A, a \mapsto ab$ is continuous we have that

$r_b^{-1}(V) := \{a \in A : ab \in V\}$ is closed. For any $k \in \mathbb{N}$, set $U_k := \bigcap_{b \in W_k} r_b^{-1}(V)$. Then each U_k is closed and[1] $A = \bigcup_{k \in \mathbb{N}} U_k$.

Since A is a Baire space by Proposition 3.1.13, we have that there exists $h \in \mathbb{N}$ such that $\mathring{U}_h \neq \emptyset$. Therefore, there exists $x \in \mathring{U}_h$, i.e. there exists $j \in \mathbb{N}$ such that $x + W_j \subseteq U_h$. This in turn provides that

$$U_h - U_h \supseteq x + W_j - x - W_j = W_j - W_j \supseteq W_j.$$

Since $\mathcal{B}$ is a basis for the filter of neighbourhoods of the origin, we can find $m \in \mathbb{N}$ such that $W_m \subseteq W_j \cap W_h$ and therefore

$$W_m W_m \subseteq W_j W_h \subseteq (U_h - U_h)W_h = U_h W_h - U_h W_h \subseteq V - V \subseteq W_n,$$

where in the last inclusion we have just used the definition of U_h. Hence, the multiplication in A is jointly continuous. $\square$

Example 3.1.17.

1) *Let $\mathcal{C}^\infty([0,1])$ be the space of all real valued infinitely differentiable functions on $[0,1]$ equipped with pointwise operations. We endow the algebra $\mathcal{C}^\infty([0,1])$ with the topology $\tau_\mathcal{P}$ generated by $\mathcal{P} := \{p_n : n \in \mathbb{N}_0\}$ with $p_n(f) := \sup_{x \in [0,1]} |(D^{(n)} f)(x)|$ for any $f \in \mathcal{C}^\infty([0,1])$ (here $D^{(n)} f$ denotes the $n-$th derivative of f). $\mathcal{P}$ is a countable separating family of seminorms so $(\mathcal{C}^\infty([0,1]), \tau_\mathcal{P})$ is a metrizable lc algebra but the seminorms in $\mathcal{P}$ are not submultiplicative since if for example we take $f(t) := t$ then $p_1(f^2) = 2 > 1 = p_1(f)p_1(f)$. However, we are going to show that $\tau_\mathcal{P}$ can be in fact generated by a countable separating family of submultiplicative seminorms and so it is actually an lmc algebra. First, let us consider the family $\mathcal{R} := \{r_n := \max_{j=0,\ldots,n} p_j : n \in \mathbb{N}_0\}$. As each $p_n \leq r_n$, we have that $\tau_\mathcal{P} = \tau_\mathcal{R}$ and also for all $n \in \mathbb{N}_0$, $f, g \in \mathcal{C}^\infty([0,1])$ the following holds:*

$$\begin{aligned} r_n(fg) &= \max_{j=0,\ldots,n} p_j(fg) = \max_{j=0,\ldots,n} \sup_{x \in [0,1]} |(D^{(n)} fg)(x)| \\ &\leq \max_{j=0,\ldots,n} \sum_{k=0}^{j} \binom{j}{k} \sup_{x \in [0,1]} |(D^{(j-k)} f)(x)| \sup_{x \in [0,1]} |(D^{(k)} g)(x)| \\ &\leq \max_{j=0,\ldots,n} \sum_{k=0}^{j} \binom{j}{k} p_{j-k}(f) p_k(g) \\ &\leq \left(\max_{j=0,\ldots,n} \sum_{k=0}^{j} \binom{j}{k} \right) r_n(f) r_n(g) = 2^n r_n(f) r_n(g). \end{aligned}$$

[1] Clearly, each $U_k \subset A$ and so $\bigcup_{k \in \mathbb{N}} U_k \subseteq A$. Conversely, if $x \in A$, then the continuity of the left multiplication implies that there exists $j \in \mathbb{N}$ such that $xW_j \subseteq V$ and so $x \in r_b^{-1}(V)$ for all $b \in W_j$, i.e. $x \in \bigcup_{k \in \mathbb{N}} U_k$.