$r_b^{-1}(V) := \{a \in A : ab \in V\}$ is closed. For any $k \in \mathbb{N}$, set $U_k := \bigcap_{b \in W_k} r_b^{-1}(V)$. Then each U_k is closed and[1] $A = \bigcup_{k \in \mathbb{N}} U_k$.

Since A is a Baire space by Proposition 3.1.13, we have that there exists $h \in \mathbb{N}$ such that $\mathring{U}_h \neq \emptyset$. Therefore, there exists $x \in \mathring{U}_h$, i.e. there exists $j \in \mathbb{N}$ such that $x + W_j \subseteq U_h$. This in turn provides that

$$U_h - U_h \supseteq x + W_j - x - W_j = W_j - W_j \supseteq W_j.$$

Since $\mathcal{B}$ is a basis for the filter of neighbourhoods of the origin, we can find $m \in \mathbb{N}$ such that $W_m \subseteq W_j \cap W_h$ and therefore

$$W_m W_m \subseteq W_j W_h \subseteq (U_h - U_h) W_h = U_h W_h - U_h W_h \subseteq V - V \subseteq W_n,$$

where in the last inclusion we have just used the definition of U_h. Hence, the multiplication in A is jointly continuous. $\square$

Example 3.1.17.

1) Let $\mathcal{C}^\infty([0,1])$ be the space of all real valued infinitely differentiable functions on $[0,1]$ equipped with pointwise operations. We endow the algebra $\mathcal{C}^\infty([0,1])$ with the topology $\tau_\mathcal{P}$ generated by $\mathcal{P} := \{p_n : n \in \mathbb{N}_0\}$ with $p_n(f) := \sup_{x \in [0,1]} |(D^{(n)} f)(x)|$ for any $f \in \mathcal{C}^\infty([0,1])$ (here $D^{(n)} f$ denotes the $n-$th derivative of f). $\mathcal{P}$ is a countable separating family of seminorms so $(\mathcal{C}^\infty([0,1]), \tau_\mathcal{P})$ is a metrizable lc algebra but the seminorms in $\mathcal{P}$ are not submultiplicative since if for example we take $f(t) := t$ then $p_1(f^2) = 2 > 1 = p_1(f)p_1(f)$. However, we are going to show that $\tau_\mathcal{P}$ can be in fact generated by a countable separating family of submultiplicative seminorms and so it is actually an lmc algebra. First, let us consider the family $\mathcal{R} := \{r_n := \max_{j=0,\dots,n} p_j : n \in \mathbb{N}_0\}$. As each $p_n \leq r_n$, we have that $\tau_\mathcal{P} = \tau_\mathcal{R}$ and also for all $n \in \mathbb{N}_0$, $f, g \in \mathcal{C}^\infty([0,1])$ the following holds:

$$\begin{aligned}
r_n(fg) &= \max_{j=0,\dots,n} p_j(fg) = \max_{j=0,\dots,n} \sup_{x \in [0,1]} |(D^{(n)} fg)(x)| \\
&\leq \max_{j=0,\dots,n} \sum_{k=0}^{j} \binom{j}{k} \sup_{x \in [0,1]} |(D^{(j-k)} f)(x)| \sup_{x \in [0,1]} |(D^{(k)} g)(x)| \\
&\leq \max_{j=0,\dots,n} \sum_{k=0}^{j} \binom{j}{k} p_{j-k}(f) p_k(g) \\
&\leq \left(\max_{j=0,\dots,n} \sum_{k=0}^{j} \binom{j}{k} \right) r_n(f) r_n(g) = 2^n r_n(f) r_n(g).
\end{aligned}$$

[1] Clearly, each $U_k \subset A$ and so $\bigcup_{k \in \mathbb{N}} U_k \subseteq A$. Conversely, if $x \in A$, then the continuity of the left multiplication implies that there exists $j \in \mathbb{N}$ such that $xW_j \subseteq V$ and so $x \in r_b^{-1}(V)$ for all $b \in W_j$, i.e. $x \in \bigcup_{k \in \mathbb{N}} U_k$.

Therefore, setting $q_n(f) := 2^n r_n(f)$ for any $n \in \mathbb{N}_0$ and $f \in \mathcal{C}^\infty([0,1])$, we have that the family $\mathcal{Q} := \{q_n : n \in \mathbb{N}_0\}$ is a countable family of submultiplicative seminorms such that $\tau_{\mathcal{Q}} = \tau_{\mathcal{R}} = \tau_{\mathcal{P}}$. Hence, $(\mathcal{C}^\infty([0,1]), \tau_{\mathcal{P}})$ is a metrizable lmc algebra. Actually, it is also complete and so a Fréchet lmc algebra. Indeed, as it is metrizable we said it is enough to show sequentially complete. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathcal{C}^\infty([0,1]), \tau_{\mathcal{P}})$, i.e. $\forall U \in \tau_{\mathcal{P}} \exists N \in \mathbb{N}$ s.t. $f_n - f_m \in U \forall n, m \geq N$. Thus,

$$\forall \varepsilon > 0 \forall k \in \mathbb{N}_0 \exists N \in \mathbb{N} : p_k(f_n - f_m) \leq \varepsilon \quad \forall n, m \geq N \tag{3.3}$$

which yields

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : \sup_{x\in[0,1]} |f_n(x) - f_m(x)| \leq \varepsilon \quad \forall n, m \geq N \tag{3.4}$$

so that $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ for all $x \in [0,1]$. Since $\mathbb{R}$ is complete for all $x \in [0,1]$ there exists $y_x \in \mathbb{R}$ s.t. $f_n(x) \to y_x$ as $n \to \infty$. Set $f(x) := y_x$ for all $x \in [0,1]$, then $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f. The latter combined with (3.4) yields that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f which implies that $f \in \mathcal{C}([0,1])$ by [11, Lemma 1.2.2]. By (3.4) for $k = 1$, we get $((D^{(1)} f_n)(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ for all $x \in [0,1]$ and reasoning as above $(D^{(1)} f_n)_{n\in\mathbb{N}}$ uniformly converges to some g on $[0,1]$. By [11, Lemma 1.2.3], $g = D^{(1)} f$ and so $f \in \mathcal{C}^1([0,1])$. Proceeding by induction, we can show that $(D^{(j)} f_n)_{n\in\mathbb{N}}$ converges uniformly to $D^{(j)} f$ on $[0,1]$ and $f \in \mathcal{C}^j([0,1])$ for all $j \in \mathbb{N}_0$, i.e.

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : \sup_{x\in[0,1]} \left| D^{(j)} f_n(x) - D^{(j)} f(x) \right| \leq \varepsilon \quad \forall n, m \geq N.$$

Therefore, $(f_n)_{n\in\mathbb{N}}$ converges to $f \in \mathcal{C}^\infty([0,1])$ in $\tau_{\mathcal{P}}$. Hence, completeness is proven.

Note that we could have first proved completeness and then used Corollary 3.1.15 to show that $(\mathcal{C}^\infty([0,1]), \tau_{\mathcal{P}})$ is m-barrelled. Then, observing that $\tau_{\mathcal{P}} = \tau_{\mathcal{R}}$ and that the seminorms in $\mathcal{R}$ are all absorbing, we could have applied Theorem 2.5.5 and concluded that $(\mathcal{C}^\infty([0,1]), \tau_{\mathcal{P}})$ is an lmc algebra.

2) Let $\mathbb{K}^{\mathbb{N}} = \{\underline{a} = (a_n)_{n\in\mathbb{N}} : a_n \in \mathbb{K}\}$ be the space of all $\mathbb{K}$–valued sequences endowed $\mathbb{K}^{\mathbb{N}}$ with the topology $\tau_{\mathcal{P}}$ generated by $\mathcal{P} := \{p_n : n \in \mathbb{N}\}$ with $p_n(\underline{a}) := \max_{k\leq n} |a_k|$ for any $\underline{a} \in \mathbb{K}^{\mathbb{N}}$ ($n \in \mathbb{N}$). Since $\mathcal{P}$ is an increasing family of submultiplicative seminorms and separating, $(\mathbb{K}^{\mathbb{N}}, \tau_{\mathcal{P}})$ is a *metrizable lmc algebra by Theorem 3.1.3. Indeed, if $\underline{a}, \underline{b} \in \mathbb{K}^{\mathbb{N}}$, then*

$$p_n(\underline{ab}) = \max_{k \leq n} |a_k b_k| \leq \max_{k \leq n} |a_k| \max_{k \leq n} |b_k| = p_n(\underline{a}) p_n(\underline{b})$$

for all $n \in \mathbb{N}$. Further, if $p_n(\underline{a}) = 0$ for all $n \in \mathbb{N}$, then

$$\max_{k \leq n} |a_k| = 0, \forall n \in \mathbb{N} \Rightarrow |a_k| = 0, \forall k \in \mathbb{N} \Rightarrow a \equiv 0.$$

Moreover, $(\mathbb{K}^{\mathbb{N}}, \tau_{\mathcal{P}})$ is sequentially complete and so complete (prove it yourself). Hence, it is a Fréchet lmc algebra.

3) *The Arens–algebra $L^{\omega}([0,1]) := \bigcap_{p \geq 1} L^p([0,1])$ endowed with the topology $\tau_{\mathcal{P}}$ generated by $\mathcal{P} := \{\| \cdot \|_p : p \geq \mathbb{N}\}$ is a Fréchet lc algebra which is not lmc. We have already showed that it is an lc algebra but not lmc. Metrizability comes from the fact that the family of seminorms is countable and increasing (Hölder–inequality). Proving completeness is more complicated which we will maybe see it later on.*

3.2 Locally bounded algebras

The TAs we are going to study in this section were first introduced by W. Zelazko in the 1960's and provide non-trivial examples of TAs whose underlying space is not necessarily locally convex (so they are neither necessarily lc algebras nor lmc algebras) but they still share several nice properties of Banach and/or lmc algebras.

Definition 3.2.1. *A TA is* locally bounded *(lb) if there exists a neighbourhood of the origin which is bounded. Equivalently, a locally bounded algebra is a TA which is in particular a locally bounded TVS (i.e. the space has a bounded neighbourhood of the origin).*

Recall that:

Definition 3.2.2. *A subset B of a TVS X is* bounded *if $\forall U \in \mathcal{F}(o) \exists \lambda > 0$ s.t. $B \subseteq \lambda U$ (i.e. B can be swallowed by any neighbourhood of the origin).*

This generalizes the concept of boundedness we are used to in the theory of normed and metric spaces, where a subset is bounded whenever we can find a ball large enough to contain it.

Example 3.2.3. *The subset $Q := [0,1]^2$ is bounded in $(\mathbb{R}^2, \| \cdot \|)$ as for any $\varepsilon > 0$ there exists $\lambda > 0$ s.t. $Q \subseteq \lambda B_{\varepsilon}(o)$ namely, if $\varepsilon \geq \sqrt{2}$ take $\lambda = 1$, otherwise take $\lambda = \frac{\sqrt{2}}{\varepsilon}$.*

Proposition 3.2.4. *Every Hausdorff locally bounded algebra is metrizable.*

Proof.
Let (A, τ) be a Hausdorff locally bounded algebra. Then there exists $U \in \mathcal{F}(o)$ bounded. W.l.o.g. we can assume that U is balanced. Indeed, if this is not the case, then we can replace it by some $V \in \mathcal{F}(o)$ balanced s.t. $V \subseteq U$. Then the boundedness of U provides that $\forall N \in \mathcal{F}(o)\ \exists \lambda > 0$ s.t. $U \subseteq \lambda N$ and so $V \subseteq \lambda N$, i.e. V is bounded and balanced.

The collection $\{\frac{1}{n}U : n \in \mathbb{N}\}$ is a countable basis of neighbourhoods of the origin o. In fact, for any $N \in \mathcal{F}(o)$ there exists $\lambda > 0$ s.t. $U \subseteq \lambda N$, i.e. $\frac{1}{\lambda}U \subseteq N$, and so $\frac{1}{n}U \subseteq \frac{1}{\lambda}U$ for all $n \geq \lambda$ as U is balanced. Hence, we obtain that for any $N \in \mathcal{F}(o)$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n}U \subseteq N$. Then we can apply Theorem 3.1.2 which gives that (A, τ) is a metrizable algebra. □

The converse is not true in general as for example the countable product of 1–dimensional metrizable TVS is metrizable but not locally bounded.

Corollary 3.2.5. *Every complete Hausdorff lb algebra has continuous multiplication.*

Proof. Since local boundedness and Hausdorfness imply metrizability, Proposition 3.1.16, ensures that the multiplication is continuous. □

The concept of lb TVS and so of lb TA can be characterized through extensions of the notion of norm, which will allow us to see how some results can be extended from Banach algebras to complete lb algebras.

Definition 3.2.6. *Let X be a $\mathbb{K}$–vector space. A map $\|\cdot\| : X \to \mathbb{R}^+$ is said to be a* quasi-norm *if*
1. $\forall x \in X : \|x\| = 0 \iff x = 0$,
2. $\forall x \in X \forall \lambda \in \mathbb{K} : \|\lambda x\| = |\lambda|\, \|x\|$,
3. $\exists k \geq 1 : \|x + y\| \leq k(\|x\| + \|y\|), \forall x, y \in X$.
If $k = 1$ this coincides with the notion of norm.

Example 3.2.7. *Let $0 < p < 1$ and consider the space $L^p([0,1])$ together with the map $\|\cdot\|_p : L^p([0,1]) \to \mathbb{R}^+$ defined by $\|f\|_p := (\int_0^1 |f(x)|^p\, dx)^{\frac{1}{p}}$ for all $f \in L^p([0,1])$. Then the Minkowski inequality does not hold but we still have that $\|f + g\|_p \leq 2^{\frac{1-p}{p}}(\|f\| + \|g\|)$ for all $f, g \in L^p([0,1])$ and so that $\|\cdot\|_p$ is a quasi-norm.*

Proposition 3.2.8. *Let (X,τ) be a Hausdorff TVS. Then (X,τ) is lb if and only if τ is induced by a quasi-norm on X.*

Proof.
Assume (X,τ) is lb. Then there exists balanced and bounded $U \in \mathcal{F}(o)$ and $\mathcal{B} := \{\alpha U : \alpha > 0\}$ is a basis of neighbourhoods of the origin in (X,τ) because $\forall N \in \mathcal{F}(o) \exists \lambda > 0$ s.t. $U \subseteq \lambda N \Rightarrow \mathcal{B} \ni \frac{1}{\lambda} U \subseteq N$. Consider the Minkowski functional $p_U(x) := \inf\{\alpha > 0 : x \in \alpha U\}$. In the proof of Lemma 2.2.7 we have already seen that if U is absorbing and balanced, then $0 \leq p_U(x) < \infty$ and $p_U(\lambda x) = |\lambda|\, p_U(x)$ for all $x \in X$ and all $\lambda \in \mathbb{K}$. If $p_U(x) = 0$, then $x \in \alpha U$ for all $\alpha > 0$ and so $x \in \bigcap_{\alpha>0} \alpha U = \{o\}$, i.e. $x = o$. Since X is a TVS, $\exists\ V \in \mathcal{F}(o)$ s.t. $V + V \subseteq U$ and also $\exists\ \alpha > 0$ s.t. $\alpha U \subseteq V$ as $\mathcal{B}$ is a basis of neighbourhoods. Therefore, $\alpha U + \alpha U \subseteq V + V \subseteq U$ and taking $k \geq \max\{1, \frac{1}{\alpha}\}$, we obtain $U + U \subseteq \frac{1}{\alpha} U \subseteq kU$ as U is balanced.

Let $x, y \in X$ and $\rho > p_U(x), \delta > p_U(y)$, then $x \in \rho U, y \in \delta U$ since U is balanced, and so $\frac{x}{\rho}, \frac{y}{\delta} \in U$. Thus,

$$\frac{x+y}{\rho+\delta} = \frac{\rho}{\rho+\delta}\frac{x}{\rho} + \frac{\delta}{\rho+\delta}\frac{y}{\delta} \in U + U \subseteq kU.$$

and we obtain $x + y \in k(\rho+\delta)U$ which implies $p_U(x+y) \leq k(\rho+\delta)$. As $\rho > p_U(x)$ and $\delta > p_U(y)$ were chosen arbitrarily, we conclude $p_U(x+y) \leq k(p_U(x) + p_U(y))$. Hence, p_U is a quasi-norm.

Let $B_1^{p_U} := \{x \in X : p_U(x) \leq 1\}$. Then we have $U \subseteq B_1^{p_U} \subseteq (1+\varepsilon)U$ for all $\varepsilon > 0$. Indeed, if $x \in U$, then $p_U(x) \leq 1$ and so $x \in B_1^{p_U}$. If $x \in B_1^{p_U}$, then $p_U(x) \leq 1$ and so $\forall \varepsilon > 0\ \exists\ \alpha$ with $\alpha \leq 1 + \varepsilon$ s.t. $x \in \alpha U$. This gives that $x \in (1+\varepsilon)U$ as U is balanced and so $\alpha U \subseteq (1+\varepsilon)U$. Since $\{\varepsilon B_1^{p_U} : \varepsilon > 0\}$ is a basis of τ_{p_U}, this implies $\tau = \tau_{p_U}$.

Conversely, assume that $\tau = \tau_q$ for a quasi-norm q on X. The collection $\mathcal{B} := \{\varepsilon B_1^q : \varepsilon > 0\}$ is a basis of neighbourhoods of the originin X (by Theorem 1.2.6). Let us just show that $\forall N \in \mathcal{F}^q(o)\ \exists V \in \mathcal{F}^q(o)$ s.t. $V + V \subseteq N$. Indeed, $\frac{1}{2k} B_1^q + \frac{1}{2k} B_1^q \subseteq B_1^q$ because if $x, y \in B_1^q$, then

$$q\left(\frac{x+y}{2k}\right) = \frac{1}{2k} q(x+y) \leq \frac{k(q(x)+q(y))}{2k} \leq \frac{2k}{2k} = 1$$

and so $\frac{x+y}{2k} \in B_1^q$. Then for all $N \in \mathcal{F}^q(o)$ there is some $\varepsilon > 0$ s.t. $\varepsilon B_1^q \subseteq N$ and so $\frac{\varepsilon}{2k} B_1^q + \frac{\varepsilon}{2k} B_1^q \subseteq \varepsilon B_1^q \subseteq N$. Since $\mathcal{B}$ is a basis for τ_q, clearly B_1^q is bounded: $\forall N \in \mathcal{F}^q(o) \exists \varepsilon > 0$ s.t. $\varepsilon B_1^q \subseteq N$ which implies $B_1^q \subseteq \frac{1}{\varepsilon} N$. Therefore, τ_q is lb. $\square$

Using the previous proposition and equipping the space in Example 3.2.7 with pointwise multiplication, we get an example of lb but not lc algebra (see Sheet 5).

Definition 3.2.9. *Let X be a $\mathbb{K}$–vector space and $0 < \alpha \leq 1$. A map $q : X \to \mathbb{R}^+$ is a an α–*norm *if*
1. $\forall x \in X : q(x) = 0 \iff x = 0$*,*
2. $\forall x \in X \forall \lambda \in \mathbb{K} : q(\lambda x) = |\lambda|^\alpha q(x)$*,*
3. $\forall x, y \in X : q(x + y) \leq q(x) + q(y)$*.*
If $\alpha = 1$, this coincides with the notion of norm.

Definition 3.2.10. *A TVS (X, τ) is α–*normable *if τ can be induced by an α–norm for some $0 < \alpha \leq 1$.*

In order to understand how $\alpha-$norms relates to lb spaces we need to introduce a generalization of the concept of convexity.

Definition 3.2.11. *Let $0 < \alpha \leq 1$ and X a $\mathbb{K}-$vector space.*

- *A subset V of X is* α-convex *if for any $x, y \in V$ we have $tx + sy \in V$ for all $t, s > 0$ such that $t^\alpha + s^\alpha = 1$.*
- *A subset V of X is* absolutely α-convex *if for any $x, y \in V$ we have $tx + sy \in V$ for all $t, s \in \mathbb{K}$ such that $|t|^\alpha + |s|^\alpha \leq 1$.*

Proposition 3.2.12. *Let (X, τ) be a TVS and $0 < \alpha \leq 1$. Then (X, τ) is α–normable if and only if there exists an α–convex, bounded neighbourhood of the origin.*

Proof. (next lecture!) □

Corollary 3.2.13. *Every α–normable TVS is lb.*

The converse also holds and in proving it the following notion turns out to be very useful.

Definition 3.2.14. *If (X, τ) is lb TVS, then for any balanced, bounded, neighbourhood U of the origin in X we define*

$$C(U) := \inf\{\lambda : U + U \subseteq \lambda U\}.$$

Then the concavity module of X *is defined as follows*

$$C(X) := \inf\{C(U) : U \textit{ balanced, bounded, neighbourhood of } o \textit{ in } X\}.$$

Theorem 3.2.15. *Let (X, τ) be a TVS. Then (X, τ) is lb if and only if τ is induced by some α–norm for some $0 < \alpha \leq 1$.*

Proof. The sufficiency is given by the previous corollary. As for the necessity, it is possible to show that if (X,τ) is lb then there exists a bounded $\alpha-$convex neighbourhood of the origin for all $0<\alpha<\alpha_0$ where $\alpha_0 := \frac{\log 2}{\log C(X)}$ (see Sheet 5). Hence, the conclusion follows by Proposition 3.2.12. $\square$

In the context of lb algebras, it might happen that the $\alpha-$norm defining the topology is actually submultiplicative. This is actually the case if the considered algebra is complete.

Definition 3.2.16. *An* $\alpha-$normed algebra *is a* $\mathbb{K}-$*algebra endowed with the topology induced by a submultiplicative* $\alpha-$*norm.*

Theorem 3.2.17. *Any lb complete algebra can be made into an* $\alpha-$*normed algebra for some* $0<\alpha\leq 1$.

Proof. Sketch
Let (X,τ) be a complete lb algebra. For convenience let us assume that X is unital but the proof can be adapted also to the non-unital case.

As (X,τ) is lb, Theorem 3.2.15 ensures that the exists $0<\alpha\leq 1$ such that τ is induced by an $\alpha-$norm q. Consider the space $L(X)$ of all linear continuous operators on X equipped with pointwise addition and scaler multiplication and with the composition as multiplication. Then the operator norm on $L(X)$ defined by $\|\ell\| := \sup_{x\in X\setminus\{o\}} \frac{q(\ell(x))}{q(x)}$ for all $\ell \in L(X)$ is a submultiplicative $\alpha-$norm. Since (X,q) is complete, it is possible to show that it is topologically isomorphic to $(L(X),\|\cdot\|)$. If we denote by φ such an isomorphism, we then get that (X,p) with $p(x) := \|\varphi(x)\|$for all $x\in X$ is an $\alpha-$normed algebra. $\square$