Universität Konstanz
Fachbereich Mathematik und Statistik
Dr. Maria Infusino
Patrick Michalski

TOPOLOGICAL ALGEBRAS–SS2018

Exercise Sheet 1

This exercise sheet aims to assess your progress and to explicitly work out more details of some of the results proposed in the previous lectures. Please, hand in your solutions in postbox 16 near F411 by Wednesday the 2nd of May at 15:15. The solutions to this assignment will be discussed in the tutorial on Friday, May the 4th (10:00–11:30 in F420).

1) Prove the following characterization of TA with continuous multiplication (i.e. Theorem 1.2.10 in the lecture notes).

Theorem 1. *A non-empty collection $\mathcal{B}$ of subsets of a $\mathbb{K}$–algebra A is a basis of neighbourhoods of the origin for some topology making A into a TA with continuous multiplication if and only if*

a) $\mathcal{B}$ is a basis of neighbourhoods of o for a topology making A into a TVS.

b) $\forall U \in \mathcal{B}, \exists V \in \mathcal{B}$ s.t. $VV \subseteq U$.

2) Let μ be the Lebesgue measure on $[0, 1]$ and consider the Lebesgue space $L^0(\mu)$ of measurable functions which becomes an $\mathbb{R}$–algebra with pointwise defined addition, multiplication and scalar multiplication. For $k, \varepsilon > 0$ set

$$B(k, \varepsilon) := \{f \in L^0(\mu) : \mu(\{|f| \geq k\}) < \varepsilon\}$$

and define $\mathcal{B} := \{B(k, \varepsilon) : k, \varepsilon > 0\}$. Show that $\mathcal{B}$ is a base of neighbourhoods of the origin for a topology τ on $L^0(\mu)$ such that $(L^0(\mu), \tau)$ is a Hausdorff TA with continuous multiplication.

3) Let $A := \{p \in \mathbb{R}[t] : p(0) = 0\}$ and identify it with its image under the embedding $\phi : A \to \mathcal{C}([1, 2]), p \mapsto p|_{[1,2]}$, where $\mathcal{C}([1, 2])$ denotes the $\mathbb{R}$–algebra of continuous real valued functions on $[1, 2]$. For $f \in \mathcal{C}([1, 2])$ set $\|f\| := \sup_{t\in[1,2]} |f(t)|$ and show that:

a) $(A, \|\cdot\|)$ is a normed algebra.

b) $(\mathcal{C}([1, 2]), \|\cdot\|)$ is a unital normed algebra.

Denote by A_1 the unitization of A and for $(\lambda, a) \in A_1$ define $\|(\lambda, a)\|_1 := |\lambda| + \|a\|$ as well as $\|(\lambda, a)\|_2 := \|\lambda + a\|$. Show that:

c) Both $\|\cdot\|_1$ and $\|\cdot\|_2$ are unitization norms but they are not equivalent.

4) Show that on a Banach algebra A all continuous complete norms are equivalent. Deduce that any complete unitization norm is equivalent to the one given in the lecture (i.e. $\|(k, a)\|_1 := |k| + \|a\|, \ \forall k \in \mathbb{K}, a \in A$).