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TOPOLOGICAL ALGEBRAS–SS2018

Exercise Sheet 2

This exercise sheet aims to assess your progress and to explicitly work out more details of some of the results proposed in the previous lectures. Please, hand in your solutions in postbox 16 near F411 by Wednesday the 16th of May at 15:15. The solutions to this assignment will be discussed in the tutorial on Friday 18th of May (10:00–11:30 in F420).

1) Let $(A, \|\cdot\|)$ be a commutative Banach algebra without unit and assume that $\|\cdot\|$ is regular, i.e.

$$\|a\| = \sup_{x\in A, \|x\|\leq 1} \|ax\| \quad \text{for all } a \in A.$$

Denote by A_1 the unitization of A. Prove the following statements and conclude that all unitization norms on A_1 are equivalent.

a) For $(\lambda, a) \in A_1$

$$\|(\lambda, a)\|_{\text{op}} := \sup\{\|\lambda x + ax\| : x \in A, \|x\| \leq 1\}$$

is a unitization norm.

b) $(A_1, \|\cdot\|_{\text{op}})$ is a Banach algebra.

Hint: Consider the algebra of operators $L_{(\lambda,a)} : x \mapsto \lambda x + ax$ endowed with the operator topology. Show that the linear map $T : L_{(\lambda,a)} \mapsto \lambda$ is continuous.

c) $\|\cdot\|_{\text{op}}$ is continuous w.r.t. any unitization norm on A_1.

Hint: Consider Lemma 1.2.16.

2) Prove the following proposition (Proposition 1.4.3 in the lecture notes).

Proposition 1. *Let (X, ω) be a TA over $\mathbb{K}$, Y a $\mathbb{K}$–algebra and $\varphi : X \to Y$ a surjective homomorphism. Denote by $\mathcal{B}_\omega$ a base of neighbourhoods of the origin in (X, ω). Then $\mathcal{B} := \{\varphi(U) : U \in \mathcal{B}_\omega\}$ is a base of neighbourhoods of the origin for a topology τ on Y such that (Y, τ) is a TA.*

3) Prove the following proposition (c.f. Proposition 1.4.9 in the lecture notes).

Proposition 2. *Let $(A, \|\cdot\|)$ be a Banach algebra and I a closed ideal of A. For $x \in A$ define*

$$q(\phi(x)) := \inf_{y\in I} \|x + y\|,$$

where $\phi : A \to A/I$ denotes the quotient map. Then $(A/I, q)$ is a Banach algebra and the topology induced by q on A/I coincides with the quotient topology.

4) Prove the following proposition (Proposition 2.1.3 in the lecture notes).

Proposition 3. *Let A be a $\mathbb{K}-$algebra and $U \subset A$ multiplicative, then*

a) The convex hull of U is an m-convex set in A.

b) The balanced hull of U is an m-balanced set in A.

c) The convex balanced hull of U is an absolutely m-convex set in A.

d) Any direct or inverse image via a homomorphism is a m-set.