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TOPOLOGICAL ALGEBRAS–SS2018

Exercise Sheet 3

This exercise sheet aims to assess your progress and to explicitly work out more details of some of the results proposed in the previous lectures. Please, hand in your solutions in postbox 16 near F411 by Wednesday the 30th of May at 15:15. The solutions to this assignment will be discussed in the tutorial on Friday 1st of June (10:00–11:30 in F420).

1) *(5 points)* Prove the following proposition (c.f. Proposition 2.2.8 in the lecture notes).

Proposition 1. *Let X be a $\mathbb{K}-$vector space. The following hold:*

c) If p and q are seminorms on X, then $p \leq q$ if and only if $\mathring{U}_q \subseteq \mathring{U}_p$.

d) If $n \in \mathbb{N}$ and $p_1, \ldots, p_n$ are seminorms on X, then their maximum *p defined as*

$$p(x) := \max_{i=1,\ldots,n} p_i(x) \quad \text{for all } x \in X$$

is also seminorm on X and $\mathring{U}_p = \bigcap_{i=1}^n \mathring{U}_{p_i}$. In particular, if X is a $\mathbb{K}-$algebra and all p_i's are submultiplicative seminorms then p is also submultiplicative.

2) *(10 points)* Consider the algebra $L^\omega([0,1]) := \bigcap_{p \geq 1} L^p([0,1])$ endowed with pointwise operations and the topology induced by the family of norms $\mathcal{P} := \{\|\cdot\|_p : p \geq 1\}$, where for each $p \geq 1$

$$\|f\|_p := \left(\int_0^1 |f(t)|^p dt\right)^{\frac{1}{p}} \qquad \text{for all } f \in L^\omega([0,1]),$$

introduced in Examples 2.2.13–4. Show that if $U \subseteq L^\omega([0,1])$ is a non-empty open and m-convex set, then $U = L^\omega([0,1])$ using the following steps:

a) There exist $p \geq 1$ and $r > 0$ s.t. $\|f\|_p < r \Rightarrow f \in U$.
Hint: Show that $\mathcal{P}$ is directed.

b) Let $M \subseteq [0,1]$ be measurable and $\rho > 0$. If $\lambda(M) < (\frac{r}{\rho})^p$ then $\alpha\chi_M \in U$ for all $|\alpha| \leq \rho$, where λ is the Lebesgue measure on $[0,1]$ and χ_M denotes the characteristic function of M.

c) Let $N \subseteq [0,1]$ be measurable and $\beta \in \mathbb{R}$, then $\beta\chi_N \in U$.
Hint: Decompose $\beta\chi_N$ as a suitable convex combination of elements in U by covering N with sets of sufficiently small measure and using b).

d) All simple functions belong to U.

e) Use the fact that the simple functions are dense in $L^\omega([0,1])$ to show that $U = L^\omega([0,1])$.

Now, conclude that the TA $(L^\omega([0,1]), \tau_{\mathcal{P}})$ is not lmc.

3) *(5 points)* Let A be an lmc algebra whose topology τ is induced by a directed family $\mathcal{Q}$ of submutliplicative seminorms on A, i.e. $\tau = \tau_{\mathcal{Q}}$. Show that

$$\mathcal{B}_{\mathrm{d}} = \{rU_q : q \in \mathcal{Q}, 0 < r \leq 1\}$$

is a basis of neighbourhoods of the origin for $\tau_{\mathcal{Q}}$.