Universität Konstanz
Fachbereich Mathematik und Statistik
Dr. Maria Infusino
Patrick Michalski

TOPOLOGICAL ALGEBRAS–SS2018

Exercise Sheet 4

This exercise sheet aims to assess your progress and to explicitly work out more details of some of the results proposed in the previous lectures. Please, hand in your solutions in postbox 16 near F411 by Wednesday the 13th of June at 15:15. The solutions to this assignment will be discussed in the tutorial on Friday 15th of June (10:00–11:30 in F420).

1) *(5 points)* Prove the following lemma.

Lemma 1. *Let A be a unital commutative $\mathbb{K}$–algebra. Assume that there are $a, b_n \in A$ ($n \in \mathbb{N}$) such that*

$$b_n \neq 0 \quad \text{and} \quad ab_n = nb_n \quad \text{for all } n \in \mathbb{N},$$

then A does not admit a submutliplicative norm. If in addition the b_n can be chosen s.t. there are c_n satisfying $b_n c_n = 1$ for all $n \in \mathbb{N}$, then A does not admit a non-zero submultiplicative seminorm.

Use this lemma to show that the algebra $\mathcal{C}(X)$ of continuous $\mathbb{K}$–valued functions on a topological space X (with operations defined pointwise) does not admit a submultiplicative norm if $\mathcal{C}(X)$ contains an unbounded function.

2) *(5 points)* Let X be a $\mathbb{K}$–algebra and p be a seminorm on X. Show that (X, p) is an A-convex algebra if and only if p is absorbing. Give an example of a lc algebra which is not A-convex.

3) *(5 points)* Consider the $\mathbb{R}$–algebra $\mathcal{C}([0,1])$ of continuous real valued functions on $[0,1]$ (with operations defined pointwise) and

$$p(f) := \sup_{x \in [0,1]} |xf(x)| \quad \text{for all } f \in \mathcal{C}([0,1]).$$

Show that $(\mathcal{C}([0,1]), p)$ is an A-convex algebra which is not lmc.

4) *(5 points)* Let $\mathbb{R}[X]$ denote the polynomial ring in a single variable and set $A := \{p \in \mathbb{R}[X] : p(0) = 0\}$. Show that

a) The family $\mathcal{B} := \{U_\alpha : 0 < \alpha \leq 1\}$, where $U_\alpha := \mathrm{conv_b}(\{\alpha^n X^n : n \in \mathbb{N}\})$ is a basis of neighbourhoods of the origin for the finest lmc topology τ on A,

b) Define $B := \mathrm{conv_b}(\{\frac{1}{2^{n^2}} X^n : n \in \mathbb{N}\})$, then $\overline{B}$ is a barrel and $\overline{B} \notin \tau$.

Conclude that (A, τ) provides an example of an m-barrelled lmc algebra which is not barrelled.