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TOPOLOGICAL ALGEBRAS–SS2018

Exercise Sheet 5

This exercise sheet aims to assess your progress and to explicitly work out more details of some of the results proposed in the previous lectures. Please, hand in your solutions in postbox 16 near F411 by Wednesday the 27th of June at 15:15. The solutions to this assignment will be discussed in the tutorial on Friday 29th of June (10:00–11:30 in F420).

1) *(5 points)* Prove the result of Gel'fand using Michael's Theorem (i.e. show Theorem 2.5.6 in the lecture notes).

Theorem 1. *If X is a $\mathbb{K}$–algebra endowed with a norm which makes it into a Banach space and a TA, then there exists an equivalent norm on X which makes it into a Banach algebra.*

2) *(5 points)* Consider the algebra $L^p([0,1])$ for $0 < p < 1$ endowed with pointwise operations and the topology induced by the quasi-norm $\|\cdot\|_p$ on $L^p([0,1])$ defined by

$$\|f\|_p := (\int_0^1 |f(x)|^p \,\mathrm{d}x)^{\frac{1}{p}}$$

for $f \in L^p([0,1])$ introduced in Example 3.2.7. Show that $(L^p([0,1]), \|\cdot\|_p)$ is lb but not lc.

Hint: Show that each non-empty convex neighbourhood of o equals $L^p([0,1])$.

3) *(5 points)* Let $0 < p < 1$ and consider the algebra $l_p := \{(x_i)_{i\in\mathbb{N}} \subseteq \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^p < \infty\}$ with operations defined pointwise. For $n \in \mathbb{N}$ define a map $q_n : l_p \to \mathbb{R}$ by

$$q_n(x) := \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \quad \text{for all } x \in l_p.$$

Further, define $\mathcal{B} := \{\frac{1}{n} U_{q_n} : n \in \mathbb{N}\}$, where $U_{q_n} := \{x \in l_p : q_n(x) < 1\}$. Show that:

a) $\mathcal{B}$ is a basis of neighbourhoods of the origin for a topology τ on l_p such that (l_p, τ) is a TA with continuous multiplication.

b) (l_p, τ) is not lb.

c) (l_p, τ) is metrizable but not complete.

4) *(5 points)* Show that a TVS (X, τ) is lb if and only τ is induced by an α–norm for some $0 < \alpha < \alpha_0$, where $0 < \alpha_0 \leq 1$ is such that $2^{\frac{1}{\alpha_0}} = C(X)$, where $C(X)$ denotes the module of concavity of X.

Hint: Show that if $U \subseteq X$ is bounded and balanced, then $2^{-\frac{k_1}{\alpha}} U + \cdots + 2^{-\frac{k_n}{\alpha}} U \subseteq U$ for all $k_1, \ldots, k_n \in \mathbb{N}$ that satisfy $\sum_{i=1}^{n} 2^{-k_i} \leq 1$.