# Topological Algebras 

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## Summer Semester 2018

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The primary sources for these notes are [18] and [3]. However, we also referred to [2] and [25]. The references to results from the theory of topological vector spaces appear in the following according to the enumeration used in [15] and [16].

## Introduction

The theory of topological algebras has its first roots in the famous works by Gelfand on "normed rings" of 1939 (see [6, 7, 8, 9]) followed by about fifteen years of successful activity on this subject which culminated in the publication of the book dealing with the commutative theory and its applications. From there the theory of normed and Banach algebras gained more and more importance (see [10] for a thorough account) until, with the development of the theories of topological rings and topological vector spaces, the investigation of general topological algebras became unavoidable. On the one hand, there was a great interest in better understanding which are the advantages of having in the same structure both the properties of topological rings and topological vector spaces. On the other hand it was desirable to understand how far one can go beyond normed and Banach algebras still retaining their distinguished features. The need for such an extension has been apparent since the early days of the theory of general topological algebras, more precisely with the introduction of locally multiplicative convex algebras by Arens in [1] and Michael in [21] (they introduced the notion independently). Moreover, it is worth noticing that the previous demand was due not only to a theoretical interest but also to concrete applications of this general theory to a variety of other disciplines (such as quantum filed theory and more in general theoretical physics). This double impact of the theory of topological algebras is probably the reason for which, after almost 80 years from its foundation, this is still an extremely active subject which is indeed recently enjoying very fast research developments.

## Chapter 1

## General Concepts

In this chapter we are going to consider vector spaces over the field $\mathbb{K}$ of real or complex numbers which is given the usual euclidean topology defined by means of the modulus.

### 1.1 Brief reminder about algebras over a field

Let us first recall the basic vocabulary needed to discuss about algebras.
Definition 1.1.1. $A \mathbb{K}$-algebra $A$ is a vector space over $\mathbb{K}$ equipped with an additional binary operation which is bilinear:

$$
\begin{array}{ccc}
A \times A & \rightarrow & A \\
(a, b) & \mapsto & a \cdot b
\end{array}
$$

called vector multiplication.
In other words, $(A,+, \cdot)$ is a ring such that the vector operations are both compatible with the multiplication by scalars in $\mathbb{K}$.

If a $\mathbb{K}$-algebra has an associative (resp. commutative) vector multiplication then it is said to be an associative (resp. commutative ) $\mathbb{K}$-algebra. Furthermore, if a $\mathbb{K}$-algebra $A$ has an identity element for the vector multiplication (called the unity of $A$ ), then $A$ is referred to as unital.

## Examples 1.1.2.

1. The real numbers form a unital associative commutative $\mathbb{R}$-algebra.
2. The complex numbers form a unital associative commutative $\mathbb{R}$-algebra.
3. Given $n \in \mathbb{N}$, the polynomial ring $\mathbb{R}\left[x_{1}, \ldots, x_{n}\right]$ (real coefficients and $n$ variables) equipped with pointwise addition and multiplication is a unital associative commutative $\mathbb{R}$-algebra.
4. The space $\mathcal{C}(X)$ of $\mathbb{K}$-valued continuous function on a topological space $X$ equipped with pointwise addition and multiplication is a unital associative commutative $\mathbb{K}$-algebra.
5. Given $n \in \mathbb{N}$, the ring $\mathbb{R}^{n \times n}$ of real square matrices of order $n$ equipped with the standard matrix addition and matrix multiplication is a unital associative $\mathbb{R}$-algebra but not commutative.
6. The set of quaternions $\mathbb{H}:=\{a+b i+c j+d k: a, b, c, d \in \mathbb{R}\}$ equipped with the componentwise addition and scalar multiplication is a real vector space with basis $\{1, i, j, k\}$. Let us equip $\mathbb{H}$ with the Hamilton product which is defined first on the basis elements by setting

$$
\begin{gathered}
i \cdot 1=1=1 \cdot i, \quad j \cdot 1=1=1 \cdot j, \quad k \cdot 1=1=1 \cdot k, \quad i^{2}=j^{2}=k^{2}=-1 \\
i j=k, \quad j i=-k, \quad j k=i, \quad k j=-i, \quad k i=j, \quad i k=-j,
\end{gathered}
$$

and then it is extended to all quaternions by using the distributive property and commutativity with real quaternions. Note that the multiplication formulas are equivalent to $i^{2}=j^{2}=k^{2}=i j k=-1$.
Then $\mathbb{H}$ is a unital, associative but non-commutative $\mathbb{R}$-algebra since e.g. $i j=k$ but $j i=-k$.
7. The three-dimensional Euclidean space $\mathbb{R}^{3}$ equipped with componentwise addition and scalar multiplication and with the vector cross product $\wedge$ as multiplication is a non-unital, non-associative, non-commutative $\mathbb{R}$-algebra. Non-associative since e.g. $(i \wedge j) \wedge j=k \wedge j=-i$ but $i \wedge(j \wedge j)=i \wedge 0=0$, non-commutative since e.g. $i \wedge j=k$ but $j \wedge i=-k$ and non-unital because if there was a unit element $u$ then for any $x \in \mathbb{R}^{3}$ we would have $u \wedge x=x=x \wedge u$, which is equivalent to say that $x$ is perpendicular to itself and so that $x=0$. (Here $i=(1,0,0)$, $j=(0,1,0)$ and $k=(0,0,1))$.
If we replace the vector cross product by the componentwise multiplication then $\mathbb{R}^{3}$ becomes a unital associative commutative $\mathbb{R}$-algebra with unity $(1,1,1)$.

Recall that:
Definition 1.1.3. Let $A$ be a $\mathbb{K}$-algebra. Then

1. $A$ subalgebra $B$ of $A$ is a linear subspace of $A$ closed under vector multiplication, i.e. $\forall b, b^{\prime} \in B, \quad b b^{\prime} \in B$.
2. A left ideal (resp. right ideal) $I$ of $A$ is a linear subspace of $A$ such that $\forall a \in A, \forall b \in I, a b \in I$ (resp. $\forall a \in A, \forall b \in I, b a \in I$. An ideal of $A$ is a linear subspace of $A$ which is simultaneously left and right ideal of $A$.
3. $A$ homomorphism between two $\mathbb{K}$-algebras $(A, \cdot)$ and $(B, *)$ is a linear map $\varphi: A \rightarrow B$ such that $\varphi(a \cdot b)=\varphi(a) * \varphi(b)$ for all $a, b \in A$. Its kernel $\operatorname{Ker}(\varphi)$ is an ideal of $A$ and its image $\varphi(A)$ is a subalgebra of $B$. A homomorphism between two unital $\mathbb{K}$-algebras has the additional property that $\varphi\left(1_{A}\right)=1_{B}$ where $1_{A}$ and $1_{B}$ are respectively the unit element in $A$ and the unit element in $B$.
4. The vector space $A_{1}=\mathbb{K} \times A$ equipped with the following operations:

$$
\begin{gathered}
(\lambda, a)+(\mu, b):=(\lambda+\mu, a+b), \forall \lambda, \mu \in \mathbb{K}, a, b \in A \\
\mu(\lambda, a):=(\mu \lambda, \mu a), \forall \lambda, \mu \in \mathbb{K}, a \in A \\
(\lambda, a) \cdot(\mu, b):=(\lambda \mu, \lambda b+\mu a+a b), \forall \lambda, \mu \in \mathbb{K}, a, b \in A
\end{gathered}
$$

is called the unitization of $A$.
Proposition 1.1.4. $A \mathbb{K}$-algebra $A$ can be always embedded in its unitization $A_{1}$ which is a unital algebra.

Proof. It is easy to check that $A_{1}$ fulfils the assumptions of $\mathbb{K}$-algebra and that the map

$$
\mathbb{E}: A \rightarrow A_{1}, a \mapsto(0, a)
$$

is an injective homomorphism, i.e. a monomorphism. The unit element of $A_{1}$ is given by $(1, o)$ as $(\lambda, a) \cdot(1, o)=(\lambda, a)=(1, o) \cdot(\lambda, a), \forall \lambda \in \mathbb{K}, a \in A$. Identifying $a$ and $\mathbb{e}(a)$ for any $a \in A$, we can see $A$ as a subalgebra of $A_{1}$.

### 1.2 Definition and main properties of a topological algebra

Definition 1.2.1. $A \mathbb{K}$-algebra $A$ is called a topological algebra (TA) if $A$ is endowed with a topology $\tau$ which makes the vector addition and the scalar multiplication both continuous and the vector multiplication separately continuous. (Here $\mathbb{K}$ is considered with the euclidean topology and, $A \times A$ and $\mathbb{K} \times A$ with the corresponding product topologies.)

If the vector multiplication in a TA is jointly continuous then we just speak of a TA with a continuous multiplication. Recall that jointly continuous implies separately continuous but the converse is false in general. In several books, the definition of TA is given by requiring a jointly continuous vector multiplication but we prefer here the more general definition according to [18].

An alternative definition of TA can be given in connection to TVS. Let us recall the definition:

Definition 1.2.2. A vector space $X$ over $\mathbb{K}$ is called a topological vector space (TVS) if $X$ is provided with a topology $\tau$ which is compatible with the vector space structure of $X$, i.e. $\tau$ makes the vector addition and the scalar multiplication both continuous. (Here $\mathbb{K}$ is considered with the euclidean topology and, $X \times X$ and $\mathbb{K} \times X$ with the corresponding product topologies.)

Then it is clear that
Definition 1.2.3. A topological algebra over $\mathbb{K}$ is a TVS over $\mathbb{K}$ equipped with a separately continuous vector multiplication.

Therefore, TAs inherit all the advantageous properties of TVS. In the following we will try to characterize topologies which make a $\mathbb{K}$-algebra into a TA. To do that we will make use of the results already available from the theory of TVS and see the further properties brought in by the additional structure of being a TA.

In this spirit, let us first recall that the topology of a TVS is always translation invariant that means, roughly speaking, that any TVS topologically looks about any point as it does about any other point. More precisely:

## Proposition 1.2.4.

The filter ${ }^{1} \mathcal{F}(x)$ of neighbourhoods of $x$ in a TVS $X$ coincides with the family of the sets $O+x$ for all $O \in \mathcal{F}(o)$, where $\mathcal{F}(o)$ is the filter of neighbourhoods of the origin o (i.e. neutral element of the vector addition).
(see [15, Corollary 2.1.9]]). This result easily implies that:
Proposition 1.2.5. Let $X, Y$ be two t.v.s. and $f: X \rightarrow Y$ linear. The map $f$ is continuous if and only if $f$ is continuous at the origin o.

Proof. (see [15, Corollary 2.1.15-3]]).
Thus, the topology of a TVS (and in particular the one of a TA) is completely determined by the filter of neighbourhoods of any of its points, in particular by the filter of neighbourhoods of the origin $o$ or, more frequently, by a base of neighbourhoods of the origin $o$. We would like to derive a criterion on a collection of subsets of a $\mathbb{K}$-algebra $A$ which ensures that it is a basis of neighbourhoods of the origin $o$ for some topology $\tau$ making $(A, \tau)$ a TA. To this aim let us recall the following result from TVS theory:

[^0]Theorem 1.2.6. A filter $\mathcal{F}$ of a vector space $X$ over $\mathbb{K}$ is the filter of neighbourhoods of the origin for some topology $\tau$ making $X$ into a TVS iff

1. The origin belongs to every set $U \in \mathcal{F}$
2. $\forall U \in \mathcal{F}, \exists V \in \mathcal{F}$ s.t. $V+V \subset U$
3. $\forall U \in \mathcal{F}, \forall \lambda \in \mathbb{K}$ with $\lambda \neq 0$ we have $\lambda U \in \mathcal{F}$
4. $\forall U \in \mathcal{F}, U$ is absorbing.
5. $\forall U \in \mathcal{F}, \exists V \in \mathcal{F}$ balanced s.t. $V \subset U$.

Proof. (see [15, Theorem 2.1.10]).

Recall that:
Definition 1.2.7. Let $U$ be a subset of a vector space $X$.

1. $U$ is absorbing (or radial) if $\forall x \in X \exists \rho>0$ s.t. $\forall \lambda \in \mathbb{K}$ with $|\lambda| \leq \rho$ we have $\lambda x \in U$.
2. $U$ is balanced (or circled) if $\forall x \in U, \forall \lambda \in \mathbb{K}$ with $|\lambda| \leq 1$ we have $\lambda x \in U$.
(see also [15, Examples 2.1.12, Proposition 2.1.13]).
A first interesting consequence of Theorem 1.2.6 for TA is that
Lemma 1.2.8. For $a T V S$ to be a TA with continuous multiplication it is necessary and sufficient that the vector multiplication is jointly continuous at the point $(o, o)$.

## Proof.

If $A$ is a TA with continuous multiplication, then clearly the multiplication is jointly continuous everywhere and so in particular at $(o, o)$. Conversely, let $A$ be a TVS with multiplication $M$ jointly continuous at the point $(o, o)$ and denote by $\mathcal{F}(o)$ the filter of neighbourhoods of the origin in $A$. Let $(o, o) \neq$ $(a, b) \in A \times A$ and $U \in \mathcal{F}(o)$. Then Theorem 1.2.6 guarantees that there exists $V \in \mathcal{F}(o)$ balanced and such that $V+V+V \subset U$. Moreover, the joint continuity of the multiplication at $(o, o)$ gives that there exists $U_{1}, U_{2} \in \mathcal{F}(0)$ such that $U_{1} U_{2} \subset V$. Taking $W:=U_{1} \cap U_{2}$ we have $W W \subseteq V$. Also, since $W$ is absorbing, there exists $\rho>0$ s.t. for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq \rho$ we have $\lambda a \in W, \lambda b \in W$. For $\theta:=\left\{\begin{array}{ll}\rho & \text { if } \rho \leq 1 \\ \frac{1}{\rho} & \text { if } \rho>1\end{array}\right.$, we have both $|\theta| \leq 1$ and $|\theta| \leq \rho$. Hence,

$$
\begin{aligned}
(a+\theta W)(b+\theta W) & \subseteq a b+a \theta W+W \theta b+\theta^{2} W W \subseteq a b+W W+W W+\theta^{2} V \\
& \subseteq a b+V+V+V \subseteq a b+U
\end{aligned}
$$

We showed that $\exists N \in \mathcal{F}(o)$ such that $M^{-1}(a b+U) \supseteq(a+N) \times(b+N)$ which proves that joint continuity of $M$ at the point $(a, b)$.

We are now ready to give a characterization for a basis ${ }^{2}$ of neighbourhoods of the origin in a TA (resp. TA with continuous multiplication).

Theorem 1.2.9. $A$ non-empty collection $\mathcal{B}$ of subsets of $a \mathbb{K}$-algebra $A$ is a basis of neighbourhoods of the origin for some topology making A into a TA if and only if
a) $\mathcal{B}$ is a basis of neighbourhoods of o for a topology making A into a TVS. b) $\forall U \in \mathcal{B}, \forall a \in A, \exists V, W \in \mathcal{B}$ s.t. $a V \subseteq U$ and $W a \subseteq U$.

## Proof.

Let $(A, \tau)$ be a TA and $\mathcal{B}$ be a basis of neighbourhoods of the origin of $A$. Then $(A, \tau)$ is in particular a TVS and so (a) holds. Also by definition of TA, the multiplication is separately continuous which means for any $a \in A$ the maps $L_{a}(y)=a y$ and $R_{a}(y)=y a$ are both continuous everywhere in $A$. Then by Proposition 1.2.5 they are continuous at o, i.e. $\forall U \in \mathcal{B}, \forall a \in A, \exists V, W \in \mathcal{B}$ s.t. $V \subset L_{a}^{-1}(U)$ and $W \subset R_{a}^{-1}(U)$, i.e. $a V \subseteq U$ and $W a \subseteq U$, that is (b).

Conversely, suppose that $\mathcal{B}$ is a collection of subsets of a $\mathbb{K}$-algebra $A$ fulfilling (a) and (b). Then (a) guarantees that there exists a topology $\tau$ having $\mathcal{B}$ as basis of neighbourhoods of $o$ and such that $(A, \tau)$ is a TVS. Hence, as we have already observed, (b) means that both $L_{a}$ and $R_{a}$ are continuous at $o$ and so by Proposition 1.2.5 continuous everywhere. This yields that the vector multiplication on $A$ is separately continuous and so that $(A, \tau)$ is a TA.

Theorem 1.2.10. A non-empty collection $\mathcal{B}$ of subsets of $a \mathbb{K}$-algebra $A$ is a basis of neighbourhoods of the origin for some topology making $A$ into a TA with continuous multiplication if and only if
a) $\mathcal{B}$ is a basis of neighbourhoods of o for a topology making A into a TVS. b') $\forall U \in \mathcal{B}, \exists V \in \mathcal{B}$ s.t. $V V \subseteq U$.

Proof. (Sheet 1).

## Examples 1.2.11.

1. Every $\mathbb{K}$-algebra $A$ endowed with the trivial topology $\tau$ (i.e. $\tau=\{\emptyset, A\}$ ) is a TA.

[^1]2. Let $S$ be a non-emptyset and $\mathbb{K}^{S}$ be the set of all functions from $S$ to $\mathbb{K}$ equipped with pointwise operations and the topology $\omega$ of pointwise convergence (or simple convergence), i.e. the topology generated by
$$
\mathcal{B}:=\left\{W_{\varepsilon}\left(x_{1}, \ldots, x_{n}\right): n \in \mathbb{N}, x_{1}, \ldots, x_{n} \in S, \varepsilon>0\right\},
$$
where $W_{\varepsilon}\left(x_{1}, \ldots, x_{n}\right):=\left\{f \in \mathbb{K}^{S}: f\left(x_{i}\right) \in B_{\varepsilon}(0), i=1, \ldots, n\right\}$ and $B_{\varepsilon}(0)=\{k \in \mathbb{K}:|k| \leq \varepsilon\}$. Then $\left(\mathbb{K}^{S}, \omega\right)$ is a TA with continuous multiplication. Indeed, for any $n \in \mathbb{N}, x_{1}, \ldots, x_{n} \in S, \varepsilon>0$ we have that
\[

$$
\begin{aligned}
W_{\sqrt{\varepsilon}}\left(x_{1}, \ldots, x_{n}\right) W_{\sqrt{\varepsilon}}\left(x_{1}, \ldots, x_{n}\right) & =\left\{f g: f\left(x_{i}\right), g\left(x_{i}\right) \in B_{\sqrt{\varepsilon}}(0), i=1, \ldots, n\right\} \\
& \subseteq\left\{h: h\left(x_{i}\right) \in B_{\varepsilon}(0), i=1, \ldots, n\right\} \\
& =W_{\varepsilon}\left(x_{1}, \ldots, x_{n}\right) .
\end{aligned}
$$
\]

As it is also easy to show that $\left(\mathbb{K}^{S}, \omega\right)$ is a TVS, the conclusion follows by Theorem 1.2.10.

Two fundamental classes of TA are the following ones:
Definition 1.2.12 (Normed Algebra). A normed algebra is a $\mathbb{K}$-algebra $A$ endowed with the topology induced by a submultiplicative norm $\|\cdot\|$, i.e. $\|x y\| \leq$ $\|x\|\|y\|, \forall x, y \in A$.

Definition 1.2.13 (Banach Algebra). A normed algebra whose underlying space is Banach (i.e. complete normed space) is said to be a Banach algebra.

Proposition 1.2.14. Any normed algebra is a TA with continuous multiplication.

Proof.
Let $(A,\|\cdot\|)$ be a normed algebra. It is easy to verify that the topology $\tau$ induced by the norm $\|\cdot\|$ (i.e. the topology generated by the collection $\mathcal{B}:=\left\{B_{\varepsilon}(o): \varepsilon>0\right\}$, where $\left.B_{\varepsilon}(o):=\{x \in A:\|x\| \leq \varepsilon\}\right)$ makes $A$ into a TVS. Moreover, the submultiplicativity of the norm $\|\cdot\|$ ensures that for any $\varepsilon>0$ we have: $B_{\sqrt{\varepsilon}}(o) B_{\sqrt{\varepsilon}}(o) \subseteq B_{\varepsilon}(o)$. Hence, $\mathcal{B}$ fulfills both a) and b') in Theorem 1.2.10 and so we get the desired conclusion.

## Examples 1.2.15.

1. Let $n \in \mathbb{N}$. $\mathbb{K}^{n}$ equipped with the componentwise operations of addition, scalar and vector multiplication, and endowed with the supremum norm $\|x\|:=\max _{i=1, \ldots, n}\left|x_{i}\right|$ for all $x:=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{K}^{n}$ is a Banach algebra.
2. Let $n \in \mathbb{N}$. The algebra $\mathbb{R}^{n \times n}$ of all real square matrices of order $n$ equipped with the following norm is a Banach algebra:

$$
\|A\|:=\sup _{x \in \mathbb{R}^{n} \backslash\{o\}} \frac{|A x|}{|x|}, \forall A \in \mathbb{R}^{n \times n}
$$

where $|\cdot|$ is the usual euclidean norm on $\mathbb{R}^{n}$. Indeed, from the previous example it is easy to see that $\left(\mathbb{R}^{n \times n},\|\cdot\|\right)$ is a Banach space. Also, for any $A, B \in \mathbb{R}^{n \times n}$ we have that:

$$
\|A B\|=\sup _{x \in \mathbb{R}^{n} \backslash\{o\}} \frac{|A(B x)|}{|x|} \leq\|A\| \sup _{x \in \mathbb{R}^{n} \backslash\{o\}} \frac{|B x|}{|x|}=\|A\|\|B\|
$$

3. Let $(X, \tau)$ be a topological space and $\mathcal{C}_{c}(X)$ the set of all $\mathbb{K}$-valued continuous functions with compact support. If we equip $\mathcal{C}_{c}(X)$ with the pointwise operations and the supremum norm $\|f\|:=\sup _{x \in X}|f(x)|$, then $\left(\mathcal{C}_{c}(X),\|\cdot\|\right)$ is a Banach algebra.
Before coming back to general TA, let us observe a further nice property of normed and so of Banach algebras, which will allow us to assume w.l.o.g. that in a unital normed algebra the unit has always unitary norm.

Proposition 1.2.16. If $(A, p)$ is a unital normed algebra with unit $1_{A}$, then there always exists a subultiplicative norm $q$ on $A$ equivalent to $p$ and such that $q\left(1_{A}\right)=1$.

Proof. Suppose that $p\left(1_{A}\right) \neq 1$ and define

$$
q(a):=\sup _{x \in A \backslash\{o\}} \frac{p(a x)}{p(x)}, \forall a \in A
$$

Immediately from the definition, we see that $q\left(1_{A}\right)=1$ and $p(a y) \leq q(a) p(y)$ for all $a, y \in A$. The latter implies at once that

$$
\begin{equation*}
p(a)=p\left(a 1_{A}\right) \leq q(a) p\left(1_{A}\right), \quad \forall a \in A \tag{1.1}
\end{equation*}
$$

and

$$
\begin{equation*}
q(a b)=\sup _{x \in A \backslash\{o\}} \frac{p(a b x)}{p(x)} \leq \sup _{x \in A \backslash\{o\}} \frac{q(a) p(b x)}{p(x)}=q(a) q(b), \forall a, b \in A \tag{1.2}
\end{equation*}
$$

Moreover, since $p$ is submultiplicative, we have that for all $a \in A$

$$
q(a) \leq \sup _{x \in A \backslash\{o\}} \frac{p(a) p(x)}{p(x)}=p(a)
$$

The latter together with (1.1) guarantees that $q$ is equivalent to $p$, while (1.2) its submultiplicativity.

So far we have seen only examples of TA with continuous multiplication. In the following example, we will introduce a TA whose multiplication is separately continuous but not jointly continuous.

## Example 1.2.17.

Let $(H,\langle\cdot, \cdot\rangle$,$) be an infinite dimensional separable Hilbert space over \mathbb{K}$. Denote by $\|\cdot\|_{H}$ the norm on $H$ defined as $\|x\|_{H}:=\sqrt{\langle x, x\rangle}$ for all $x \in H$, and by $L(H)$ the set of all linear and continuous maps from $H$ to $H$. The set $L(H)$ equipped with the pointwise addition a, the pointwise scalar multiplication m and the composition of maps $\circ$ as multiplication is a $\mathbb{K}$-algebra.
Let $\tau_{w}$ be the weak operator topology on $L(H)$, i.e. the coarsest topology on $L(H)$ such that all the maps $E_{x, y}: L(H) \rightarrow \mathbb{K}, T \mapsto\langle T x, y\rangle(x, y \in H)$ are continuous. A basis of neighbourhoods of the origin in $\left(L(H), \tau_{w}\right)$ is given by:

$$
\mathcal{B}_{w}:=\left\{V_{\varepsilon}\left(x_{i}, y_{i}, n\right): \varepsilon>0, n \in \mathbb{N}, x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n} \in H\right\}
$$

where $V_{\varepsilon}\left(x_{i}, y_{i}, n\right):=\left\{W \in L(H):\left|\left\langle W x_{i}, y_{i}\right\rangle\right|<\varepsilon, i=1, \ldots, n\right\}$.

- $\left(L(H), \tau_{w}\right)$ is a $T A$.

For any $\varepsilon>0, n \in \mathbb{N}, x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n} \in H$, using the bilinearity of the inner product we easily have:

$$
\begin{aligned}
& V_{\frac{\varepsilon}{2}}\left(x_{i}, y_{i}, n\right) \times V_{\frac{\varepsilon}{2}}\left(x_{i}, y_{i}, n\right)=\bigcap_{i=1}^{n}\left\{(T, S):\left|\left\langle T x_{i}, y_{i}\right\rangle\right|<\frac{\varepsilon}{2},\left|\left\langle S x_{i}, y_{i}\right\rangle\right|<\frac{\varepsilon}{2}\right\} \\
& \subseteq \bigcap_{i=1}^{n}\left\{(T, S):\left|\left\langle(T+S) x_{i}, y_{i}\right\rangle\right|<\varepsilon\right\} \\
&=\left\{(T, S):(T+S) \in V_{\varepsilon}\left(x_{i}, y_{i}, n\right)\right\} \\
&=\mathrm{a}^{-1}\left(V_{\varepsilon}\left(x_{i}, y_{i}, n\right)\right) \\
& B_{1}(0) \times V_{\varepsilon}\left(x_{i}, y_{i}, n\right)=\bigcap_{i=1}^{n}\left\{(\lambda, T) \in \mathbb{K} \times L(H):|\lambda|<1,\left|\left\langle T x_{i}, y_{i}\right\rangle\right|<\varepsilon\right\} \\
& \subseteq \bigcap_{i=1}^{n}\left\{(\lambda, T):\left|\left\langle(\lambda T) x_{i}, y_{i}\right\rangle\right|<\varepsilon\right\}=\mathbb{m}^{-1}\left(V_{\varepsilon}\left(x_{i}, y_{i}, n\right)\right)
\end{aligned}
$$

which prove that a and m are both continuous. Hence, $\left(L(H), \tau_{w}\right)$ is a TVS.
Furthermore, we can show that the multiplication in $\left(L(H), \tau_{w}\right)$ is separately continuous. For a fixed $T \in L(H)$ denote by $T^{*}$ the adjoint of $T$ and set $z_{i}:=T^{*} y_{i}$ for $i=1, \ldots, n$. Then

$$
\begin{aligned}
T \circ V_{\varepsilon}\left(x_{i}, z_{i}, n\right) & =\left\{T \circ S:\left|\left\langle S x_{i}, z_{i}\right\rangle\right|<\varepsilon, i=1, \ldots, n\right\} \\
& \subseteq\left\{W \in L(H):\left|\left\langle W x_{i}, y_{i}\right\rangle\right|<\varepsilon, i=1, \ldots, n\right\}=V_{\varepsilon}\left(x_{i}, y_{i}, n\right)
\end{aligned}
$$

where in the latter inequality we used that

$$
\left|\left\langle(T \circ S) x_{i}, y_{i}\right\rangle\right|=\left|\left\langle T\left(S x_{i}\right), y_{i}\right\rangle\right|=\left|\left\langle S x_{i}, T^{*} y_{i}\right\rangle\right|=\left|\left\langle S x_{i}, z_{i}\right\rangle\right|<\varepsilon .
$$

Similarly, we can show that $V_{\varepsilon}\left(x_{i}, z_{i}, n\right) \circ T \subseteq V_{\varepsilon}\left(x_{i}, y_{i}, n\right)$. Hence, $\mathcal{B}_{w}$ fulfills a) and b) in Theorem 1.2.9 and so we have that $\left(L(H), \tau_{w}\right)$ is a TA.

- the multiplication in $\left(L(H), \tau_{w}\right)$ is not jointly continuous.

Let us preliminarily observe that a sequence $\left(W_{j}\right)_{j \in \mathbb{N}}$ of elements in $L(H)$ converges to $W \in L(H)$ w.r.t. $\tau_{w}$, in symbols $W_{j} \xrightarrow{\tau_{w}} W$, if and only if for all $x, y \in H$ we have $\left\langle W_{j} x, y\right\rangle \rightarrow\langle W x, y\rangle^{3}$. As $H$ is separable, there exists a countable orthonormal basis $\left\{e_{k}\right\}_{k \in \mathbb{N}}$ for $H$. Define $S \in L(H)$ such that $S\left(e_{1}\right):=o$ and $S\left(e_{k}\right):=e_{k-1}$ for all $k \in \mathbb{N}$ with $k \geq 2$. Then the operator

$$
\begin{equation*}
T_{n}:=S^{n}=(\underbrace{S \circ \cdots \circ S}_{n \text { times }}), \quad n \in \mathbb{N} \tag{1.3}
\end{equation*}
$$

is s.t. $T_{n} \xrightarrow{\tau_{W}} o$ as $n \rightarrow \infty$. Indeed, $\forall x \in H, \exists!\lambda_{k} \in \mathbb{K}: x=\sum_{k=1}^{\infty} \lambda_{k} e_{k}{ }^{4}$ so

$$
\begin{aligned}
\left\|T_{n} x\right\| & =\left\|\sum_{k=1}^{\infty} \lambda_{k} T_{n}\left(e_{k}\right)\right\|=\left\|\sum_{k=n+1}^{\infty} \lambda_{k} T_{n}\left(e_{k}\right)\right\|=\left\|\sum_{k=n+1}^{\infty} \lambda_{k} e_{k-n}\right\| \\
& =\left\|\sum_{k=1}^{\infty} \lambda_{k+n} e_{k}\right\| \stackrel{4}{=} \sum_{k=1}^{\infty}\left|\lambda_{k+n}\right|^{2}=\sum_{k=n+1}^{\infty}\left|\lambda_{k}\right|^{2} \rightarrow 0 \text {, as } n \rightarrow \infty
\end{aligned}
$$

which implies that $\left\langle T_{n} x, y\right\rangle \rightarrow 0$ as $n \rightarrow \infty$ since $\left|\left\langle T_{n} x, y\right\rangle\right| \leq\left\|T_{n} x\right\|\|y\|$.
Moreover, the adjoint of $S$ is the continuous linear operator $S^{*}: H \rightarrow H$ such that $S^{*}\left(e_{k}\right)=e_{k+1}$ for all $k \in \mathbb{N}$. Hence, for any $n \in \mathbb{N}$ we have that $T_{n}^{*}=\left(S^{n}\right)^{*}=\left(S^{*}\right)^{n}$ and we can easily show that also $T_{n}^{*} \xrightarrow{\tau_{w}} o$. In fact, for any $x, y \in H$ we have that $\left|\left\langle T_{n}^{*} x, y\right\rangle\right|=\left|\left\langle x, T_{n} y\right\rangle\right| \leq\|x\|\left\|T_{n} y\right\| \rightarrow 0$ as $n \rightarrow \infty$. However, we have $S^{*} S=I$ where $I$ denotes the identity map on $H$, which gives in turn that $T_{n}^{*} \circ T_{n}=I$ for any $n \in \mathbb{N}$. Hence, for any $n \in \mathbb{N}$ and any $x, y \in H$ we have that $\left\langle\left(T_{n}^{*} \circ T_{n}\right) x, y\right\rangle=\langle x, y\rangle$ and so that $T_{n}^{*} \circ T_{n} \stackrel{\tau_{\mu}}{\nrightarrow} \circ$ as $n \rightarrow \infty$, which proves that $\circ$ is not jointly continuous.

```
\({ }^{3}\) Indeed, we have
    \(W_{j} \xrightarrow{\tau_{\psi}} W \quad \Longleftrightarrow \quad \forall \varepsilon>0, n \in \mathbb{N}, x_{i}, y_{i} \in H, \exists \bar{j} \in \mathbb{N}: \forall j \geq \bar{j}, W_{j}-W \in V_{\varepsilon}\left(x_{i}, y_{i}, n\right)\)
    \(\Longleftrightarrow \quad \forall \varepsilon>0, n \in \mathbb{N}, x_{i}, y_{i} \in H, \exists \bar{j} \in \mathbb{N}: \forall j \geq \bar{j},\left|\left\langle\left(W_{j}-W\right) x_{i}, y_{i}\right\rangle\right|<\varepsilon\)
    \(\Longleftrightarrow \quad \forall n \in \mathbb{N}, x_{i}, y_{i} \in H,\left\langle\left(W_{j}-W\right) x_{i}, y_{i}\right\rangle \rightarrow 0\), as \(j \rightarrow \infty\)
    \(\Longleftrightarrow \quad \forall x, y \in H,\left\langle\left(W_{j}-W\right) x, y\right\rangle \rightarrow 0\), as \(j \rightarrow \infty\).
```

${ }^{4}$ Recall that if $\left\{h_{i}\right\}_{i \in I}$ is an orthonormal basis of a Hilbert space $H$ then for each $y \in H$ $y=\sum_{i \in I}\left\langle y, h_{i}\right\rangle h_{i}$ and $\|y\|^{2}=\sum_{i \in I}\left|\left\langle y, h_{i}\right\rangle\right|^{2}$ (see e.g. [22, Theorem II.6] for a proof)

Let $\tau_{s}$ be the strong operator topology or topology of pointwise convergence on $L(H)$, i.e. the coarsest topology on $L(H)$ such that all the maps $E_{x}: L(H) \rightarrow H, T \mapsto T x(x \in H)$ are continuous. A basis of neighbourhoods of the origin in $\left(L(H), \tau_{s}\right)$ is given by:

$$
\mathcal{B}_{s}:=\left\{U_{\varepsilon}\left(x_{i}, n\right): \varepsilon>0, n \in \mathbb{N}, x_{1}, \ldots, x_{n} \in H\right\}
$$

where $U_{\varepsilon}\left(x_{i}, n\right):=\left\{T \in L(H):\left\|T x_{i}\right\|_{H}<\varepsilon, i=1, \ldots, n\right\}$.

- $\left(L(H), \tau_{s}\right)$ is a TA.

For any $r>0$, denote by $B_{r}(o)\left(\right.$ resp. $\left.B_{r}(0)\right)$ the open unit ball centered at o in $H$ (resp. at 0 in $\mathbb{K}$ ). Then for any $\varepsilon>0, n \in \mathbb{N}, x_{1}, \ldots, x_{n} \in H$ we have:

$$
\begin{aligned}
& U_{\frac{\varepsilon}{2}}\left(x_{i}, n\right) \times U_{\frac{\varepsilon}{2}}\left(x_{i}, n\right)=\left\{(T, S): T x_{i}, S x_{i} \in B_{\frac{\varepsilon}{2}}(o), i=1, \ldots, n\right\} \\
& \subseteq\left\{(T, S):\left\|(T+S) x_{i}\right\|_{H}<\varepsilon, i=1, \ldots, n\right\} \\
&=\left\{(T, S):(T+S) \in U_{\varepsilon}\left(x_{i}, n\right)\right\}=\mathrm{a}^{-1}\left(U_{\varepsilon}\left(x_{i}, n\right)\right) \\
& \begin{aligned}
B_{1}(0) \times U_{\varepsilon}\left(x_{i}, n\right) & =\left\{(\lambda, T) \in \mathbb{K} \times L(H):|\lambda|<1,\left\|T x_{i}\right\|_{H}<\varepsilon, i=1, \ldots, n\right\} \\
\subseteq & \left.\subseteq(\lambda, T):\left\|(\lambda T) x_{i}\right\|_{H}<\varepsilon, i=1, \ldots, n\right\}=\mathrm{m}^{-1}\left(U_{\varepsilon}\left(x_{i}, n\right)\right)
\end{aligned}
\end{aligned}
$$

which prove that a and m are both continuous.
Furthermore, we can show that the multiplication in $\left(L(H), \tau_{s}\right)$ is separately continuous. Fixed $T \in L(H)$, its continuity implies that $T^{-1}\left(B_{\varepsilon}(o)\right)$ is a neighbourhood of o in $H$ and so that there exists $\eta>0$ such that $B_{\eta}(o) \subseteq$ $T^{-1}\left(B_{\varepsilon}(o)\right)$. Therefore, we get:

$$
\begin{aligned}
T \circ U_{\eta}\left(x_{i}, n\right) & =\left\{T \circ S: S \in L(H) \text { with } S x_{i} \in B_{\eta}(o), i=1, \ldots, n\right\} \\
& \subseteq\left\{W \in L(H): W x_{i} \in B_{\varepsilon}(o), i=1, \ldots, n\right\} \\
& =U_{\varepsilon}\left(x_{i}, n\right),
\end{aligned}
$$

where in the latter inequality we used that

$$
(T \circ S) x_{i}=T\left(S x_{i}\right) \in T\left(B_{\eta}(o)\right) \subseteq T\left(T^{-1}\left(B_{\varepsilon}(o)\right)\right) \subseteq B_{\varepsilon}(o) .
$$

Similarly, we can show that $U_{\eta}\left(x_{i}, n\right) \circ T \subseteq U_{\varepsilon}\left(x_{i}, n\right)$. Hence, $\mathcal{B}_{s}$ fulfills a) and b) in Theorem 1.2.9 and so we have that $\left(L(H), \tau_{s}\right)$ is a TA.

- the multiplication in $\left(L(H), \tau_{s}\right)$ is not jointly continuous

It is enough to show that there exists a neighbourhood of the origin in $\left(L(H), \tau_{s}\right)$ which does not contain the product of any other two such neighbourhoods. More precisely, we will show $\exists \varepsilon>0, \exists x_{0} \in H$ s.t. $\forall \varepsilon_{1}, \varepsilon_{2}>0, \forall p, q \in \mathbb{N}$,
$\forall x_{1}, \ldots, x_{p}, y_{1}, \ldots, y_{q} \in H$ we have $U_{\varepsilon_{1}}\left(x_{i}, p\right) \circ U_{\varepsilon_{2}}\left(y_{i}, q\right) \nsubseteq U_{\varepsilon}\left(x_{0}\right)$, i.e. there exist $A \in U_{\varepsilon_{1}}\left(x_{i}, p\right)$ and $B \in U_{\varepsilon_{2}}\left(y_{i}, q\right)$ with $B \circ A \notin U_{\varepsilon}\left(x_{0}\right)$.

Choose $0<\varepsilon<1$ and $x_{0} \in H$ s.t. $\left\|x_{0}\right\|=1$. For any $\varepsilon_{1}, \varepsilon_{2}>0, p, q \in \mathbb{N}$, $x_{1}, \ldots, x_{p}, y_{1}, \ldots, y_{q} \in H$, take

$$
\begin{equation*}
0<\delta<\frac{\varepsilon_{2}}{\max _{i=1, \ldots, q}\left\|y_{i}\right\|} \tag{1.4}
\end{equation*}
$$

and $n \in \mathbb{N}$ such that

$$
\begin{equation*}
\left\|T_{n}\left(x_{k}\right)\right\|<\delta \varepsilon_{1}, \text { for } k=1, \ldots, p \tag{1.5}
\end{equation*}
$$

where $T_{n}$ is defined as in (1.3). (Note that we can choose such an $n$ as we showed above that $\left\|T_{j} x\right\| \rightarrow 0$ as $\left.j \rightarrow \infty\right)$. Setting $A:=\frac{1}{\delta} T_{n}$ and $B:=\delta T_{n}^{*}$ we get that:

$$
\left\|A x_{k}\right\|=\frac{1}{\delta}\left\|T_{n} x_{k}\right\| \stackrel{(1.5)}{<} \varepsilon_{1}, \text { for } k=1, \ldots, p
$$

and

$$
\left\|B y_{i}\right\|=\delta\left\|T_{n}^{*} y_{i}\right\| \stackrel{(4)}{=} \delta\left\|y_{i}\right\| \stackrel{(1.4)}{<} \varepsilon_{2}, \text { for } i=1, \ldots, q
$$

Hence, $A \in U_{\varepsilon_{1}}\left(x_{i}, p\right)$ and $B \in U_{\varepsilon_{2}}\left(y_{i}, q\right)$ but $B \circ A \notin U_{\varepsilon}\left(x_{0}\right)$ because

$$
\left\|(B \circ A) x_{0}\right\|=\left\|\left(T_{n}^{*} T_{n}\right) x_{0}\right\|=\left\|x_{0}\right\|=1>\varepsilon
$$

Note that $L(H)$ endowed with the operator norm $\|\cdot\|$ is instead a normed algebra and so has jointly continuous multiplication. Recall that the operator norm is defined by $\|T\|:=\sup _{x \in H \backslash\{o\}} \frac{\|T x\|_{H}}{\|x\|_{H}}, \forall T \in L(H)$.

### 1.3 Hausdorffness and unitizations of a TA

Topological algebras are in particular topological spaces so their Hausdorfness can be established just by verifying the usual definition of Hausdorff topological space.

Definition 1.3.1. A topological space $X$ is said to be Hausdorff or (T2) if any two distinct points of $X$ have neighbourhoods without common points; or equivalently if two distinct points always lie in disjoint open sets.

However, a TA is more than a mere topological space but it is also a TVS. This provides TAs with the following characterization of their Hausdorfness which holds in general for any TVS.

Proposition 1.3.2. For a TVS $X$ the following are equivalent:
a) $X$ is Hausdorff.
b) $\{o\}$ is closed in $X$.
c) The intersection of all neighbourhoods of the origin o is just $\{o\}$.
d) $\forall o \neq x \in X, \exists U \in \mathcal{F}(o)$ s.t. $x \notin U$.

Since the topology of a TVS is translation invariant, property (d) means that the TVS is a (T1) ${ }^{5}$ topological space. Recall for general topological spaces (T2) always implies (T1), but the converse does not always hold (c.f. Example 1.1.41-4 in [15]). However, Proposition 1.3.2 ensures that for TVS and so for TAs the two properties are equivalent.

## Proof.

Let us just show that (d) implies (a) (for a complete proof see [15, Proposition 2.2.3, Corollary 2.2.4] or even better try it yourself!).

Suppose that (d) holds and let $x, y \in X$ with $x \neq y$, i.e. $x-y \neq o$. Then there exists $U \in \mathcal{F}(o)$ s.t. $x-y \notin U$. By (2) and (5) of Theorem 1.2.6, there exists $V \in \mathcal{F}(o)$ balanced and s.t. $V+V \subset U$. Since $V$ is balanced $V=-V$ then we have $V-V \subset U$. Suppose now that $(V+x) \cap(V+y) \neq \emptyset$, then there exists $z \in(V+x) \cap(V+y)$, i.e. $z=v+x=w+y$ for some $v, w \in V$. Then $x-y=w-v \in V-V \subset U$ and so $x-y \in U$ which is a contradiction. Hence, $(V+x) \cap(V+y)=\emptyset$ and by Proposition 1.2.4 we know that $V+x \in \mathcal{F}(x)$ and $V+y \in \mathcal{F}(y)$. Hence, $X$ is Hausdorff.

We have already seen that a $\mathbb{K}$-algebra can be always embedded in a unital one, called unitization see Definition 1.1.3-4). In the rest of this section, we will discuss about which topologies on the unitization of a $\mathbb{K}$-algebra makes it into a TA. To start with, let us look at normed algebras.

Proposition 1.3.3. If $A$ is a normed algebra, then there always exists a norm on its unitization $A_{1}$ making both $A_{1}$ into a normed algebra and the canonical embedding an isometry. Such a norm is called a unitization norm.

Proof.
Let $(A,\|\cdot\|)$ be a normed algebra and $A_{1}=\mathbb{K} \times A$ its unitization. Define

$$
\|(k, a)\|_{1}:=|k|+\|a\|, \forall k \in \mathbb{K}, a \in A
$$

[^2]Then $\|(1, o)\|_{1}=1$ and it is straightforward that $\|\cdot\|_{1}$ is a norm on $A_{1}$ since $|\cdot|$ is a norm on $\mathbb{K}$ and $\|\cdot\|$ is a norm on $A$. Also, for any $\lambda, k \in \mathbb{K}, a, b \in A$ we have:

$$
\begin{aligned}
\|(k, a)(\lambda, b)\|_{1} & =\|(k \lambda, k a+\lambda b+a b)\|_{1}=|k \lambda|+\|k a+\lambda b+a b\| \\
& \leq|k||\lambda|+k\|a\|+\lambda\|b\|+\|a\|\|b\|=|k|(|\lambda|+\|b\|)+\|a\|(|\lambda|+\|b\|) \\
& =(|k|+\|a\|)(|\lambda|+\|b\|)=\|(k, a)\|_{1}\|(\lambda, b)\|_{1}
\end{aligned}
$$

This proves that $\left(A_{1},\|\cdot\|_{1}\right)$ is a unital normed algebra. Moreover, the canonical embedding $\mathbb{e}: A \rightarrow A_{1}, a \mapsto(0, a)$ is an isometry because $\|\mathbb{C}(a)\|_{1}=|0|+\|a\|=$ $\|a\|$ for all $a \in A$. This in turn gives that $\mathbb{e}$ is continuous and so a topological embedding.

Remark 1.3.4. Note that $\|\cdot\|_{1}$ induces the product topology on $A_{1}$ given by $(\mathbb{K},|\cdot|)$ and $(A,\|\cdot\|)$ but there might exist other unitization norms on $A_{1}$ not necessarily equivalent to $\|\cdot\|_{1}$ (see Sheet 1, Exercise 3).

The latter remark suggests the following generalization of Proposition 1.3.3 to any TA.

Proposition 1.3.5. Let $A$ be a TA. Its unitization $A_{1}$ equipped with the corresponding product topology is a TA and $A$ is topologically embedded in $A_{1}$. Note that $A_{1}$ is Hausdorff if and only if $A$ is Hausdorff.

Proof. Suppose $(A, \tau)$ is a TA. By Proposition 1.1.4, we know that the unitization $A_{1}$ of $A$ is a $\mathbb{K}$-algebra. Moreover, since $(\mathbb{K},|\cdot|)$ and $(A, \tau)$ are both TVS, we have that $A_{1}:=\mathbb{K} \times A$ endowed with the corresponding product topology $\tau_{\text {prod }}$ is also a TVS. Then the definition of multiplication in $A_{1}$ together with the fact that the multiplication in $A$ is separately continuous imply that the multiplication in $A_{1}$ is separately continuous, too. Hence, $\left(A_{1}, \tau_{\text {prod }}\right)$ is a TA.

The canonical embedding $\mathbb{e}$ of $A$ in $A_{1}$ is then a continuous monomorphism, since for any $U$ neighbourhood of $(0, o)$ in $\left(A_{1}, \tau_{\text {prod }}\right)$ there exist $\varepsilon>0$ and a neighbourhood $V$ of $o$ in $(A, \tau)$ such that $B_{\varepsilon}(0) \times V \subseteq U$ and so $V=\mathbb{E}^{-1}\left(B_{\varepsilon}(0) \times V\right) \subseteq \mathbb{e}^{-1}(U)$. Hence, $(A, \tau)$ is topologically embedded in $\left(A_{1}, \tau_{\text {prod }}\right)$.

Finally, recall that the cartesian product of topological spaces endowed with the corresponding product topology is Hausdorff iff each of them is Hausdorff. Then, as $(\mathbb{K},|\cdot|)$ is Hausdorff, it is clear that $\left(A_{1}, \tau_{\text {prod }}\right)$ is Hausdorff iff $(A, \tau)$ is Hausdorff. ${ }^{6}$

[^3]If $A$ is a TA with continuous multiplication, then $A_{1}$ endowed with the corresponding product topology is also a TA with continuous multiplication. Moreover, from Remark 1.3.4, it is clear that the product topology is not the unique one making the unitization of a TA into a TA itself.

### 1.4 Subalgebras and quotients of a TA

In this section we are going to see some methods which allow us to construct new TAs from a given one. In particular, we will see under which conditions the TA structure is preserved under taking subalgebras and quotients.

Let us start with an immediate application of Theorem 1.2.9.
Proposition 1.4.1. Let $X$ be a $\mathbb{K}$-algebra, $(Y, \omega)$ a TA (resp. TA with continuous multiplication) over $\mathbb{K}$ and $\varphi: X \rightarrow Y$ a homomorphism. Denote by $\mathcal{B}_{\omega}$ a basis of neighbourhoods of the origin in $(Y, \omega)$. Then the collection $\mathcal{B}:=\left\{\varphi^{-1}(U): U \in \mathcal{B}_{\omega}\right\}$ is a basis of neighbourhoods of the origin for a topology $\tau$ on $X$ such that $(X, \tau)$ is a TA (resp. TA with continuous multiplication).

The topology $\tau$ constructed in the previous proposition is usually called initial topology or inverse image topology induced by $\varphi$.

## Proof.

We first show that $\mathcal{B}$ is a basis for a filter in $X$.
For any $B_{1}, B_{2} \in \mathcal{B}$, we have $B_{1}=\varphi^{-1}\left(U_{1}\right)$ and $B_{2}=\varphi^{-1}\left(U_{2}\right)$ for some $U_{1}, U_{2} \in \mathcal{B}_{\omega}$. Since $\mathcal{B}_{\omega}$ is a basis of the filter of neighbourhoods of the origin in $(Y, \omega)$, there exists $U_{3} \in \mathcal{B}_{\omega}$ such that $U_{3} \subseteq U_{1} \cap U_{2}$ and so $B_{3}:=\varphi^{-1}\left(U_{3}\right) \subseteq$ $\varphi^{-1}\left(U_{1}\right) \cap \varphi^{-1}\left(U_{2}\right)=B_{1} \cap B_{2}$ and clearly $B_{3} \in \mathcal{B}$.

Now consider the filter $\mathcal{F}$ generated by $\mathcal{B}$. For any $M \in \mathcal{F}$, there exists $U \in \mathcal{B}_{\omega}$ such that $\varphi^{-1}(U) \subseteq M$ and so we have the following:

1. $o_{Y} \in U$ and so $o_{X} \in \varphi^{-1}\left(o_{Y}\right) \in \varphi^{-1}(U)=M$.
2. by Theorem 1.2.6-2 applied to the TVS $(Y, \omega)$, we have that there exists $V \in \mathcal{B}_{\omega}$ such that $V+V \subseteq U$. Hence, setting $N:=\varphi^{-1}(V) \in \mathcal{F}$ we have $N+N \subseteq \varphi^{-1}(V+V) \subseteq \varphi^{-1}(U)=M$.
3. by Theorem 1.2.6-3 applied to the TVS $(Y, \omega)$, we have that for any $\lambda \in \mathbb{K} \backslash\{0\}$ there exists $V \in \mathcal{B}_{\omega}$ such that $V \subseteq \lambda U$. Therefore, setting $N:=\varphi^{-1}(V) \in \mathcal{B}$ we have $N \subseteq \varphi^{-1}(\lambda U)=\lambda \varphi^{-1}(U) \subseteq \lambda M$, and so $\lambda M \in \mathcal{F}$.
4. For any $x \in X$ there exists $y \in Y$ such that $x=\varphi^{-1}(y)$. As $U$ is absorbing (by Theorem 1.2.6-4 applied to the TVS $(Y, \omega)$ ), we have that there exists $\rho>0$ such that $\lambda y \in U$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq \rho$.

This yields $\lambda x=\lambda \varphi^{-1}(y)=\varphi^{-1}(\lambda y) \in \varphi^{-1}(U)=M$ and hence, $M$ is absorbing in $X$.
5. by Theorem 1.2.6-5 applied to the TVS $(Y, \omega)$, we have that there exists $V \in \mathcal{B}_{\omega}$ balanced such that $V \subseteq U$. By the linearity of $\varphi$ also $\varphi^{-1}(V)$ is balanced and so, setting $N:=\varphi^{-1}(V)$ we have $N \subseteq \varphi^{-1}(U)=M$.
Therefore, we have showed that $\mathcal{F}$ fulfills itself all the 5 properties of Theorem 1.2.6 and so it is a filter of neighbourhoods of the origin for a topology $\tau$ making $(X, \tau)$ a TVS.

Furthermore, for any $x \in X$ and any $B \in \mathcal{B}$ we have that there exist $y \in Y$ and $U \in \mathcal{B}_{\omega}$ such that $x=\varphi^{-1}(y)$ and $B=\varphi^{-1}(U)$. Then, as $(Y, \omega)$ is a TA, Theorem 1.2.9 guarantees that there exist $V_{1}, V_{2} \in \mathcal{B}_{\omega}$ such that $y V_{1} \subseteq U$ and $V_{2} y \subseteq U$. Setting $N_{1}:=\varphi^{-1}\left(V_{1}\right)$ and $N_{2}:=\varphi^{-1}\left(V_{2}\right)$, we obtain that $N_{1}, N_{2} \in \mathcal{B}$ and $x N_{1}=\varphi^{-1}(y) \varphi^{-1}\left(V_{1}\right)=\varphi^{-1}\left(y V_{1}\right) \subseteq \varphi^{-1}(U)=B$ and $x N_{2}=\varphi^{-1}(y) \varphi^{-1}\left(V_{2}\right)=\varphi^{-1}\left(y V_{2}\right) \subseteq \varphi^{-1}(U)=B$. (Similarly, if $(Y, \omega)$ is a TA with continuous multiplication, then one can show that for any $B \in \mathcal{B}$ there exists $N \in \mathcal{B}$ such that $N N \subseteq B$.)

Hence, by Theorem 1.2.9 (resp. Theorem 1.2.10), $(X, \tau)$ is a TA (resp. TA with continuous multiplication).

Corollary 1.4.2. Let $(A, \omega)$ be a $T A$ (resp. TA with continuous multiplication) and $M$ a subalgebra of $A$. If we endow $M$ with the relative topology $\tau_{M}$ induced by $A$, then $\left(M, \tau_{M}\right)$ is a TA (resp. TA with continuous multiplication).

Proof.
Consider the identity map $i d: M \rightarrow A$ and let $\mathcal{B}_{\omega}$ a basis of neighbourhoods of the origin in $(A, \omega)$ Clearly, $i d$ is a homomorphism and the initial topology induced by $i d$ on $M$ is nothing but the relative topology $\tau_{M}$ induced by $A$ since

$$
\left\{i d^{-1}(U): U \in \mathcal{B}_{\omega}\right\}=\left\{U \cap M: U \in \mathcal{B}_{\omega}\right\}=\tau_{M} .
$$

Hence, Proposition 1.4.1 ensures that $\left(M, \tau_{M}\right)$ is a TA (resp. TA with continuous multiplication).

With similar techniques to the ones used in Proposition 1.4.1 one can show:
Proposition 1.4.3. Let $(X, \omega)$ be a $T A$ (resp. TA with continuous multiplication) over $\mathbb{K}, Y a \mathbb{K}$-algebra and $\varphi: X \rightarrow Y$ a surjective homomorphism. Denote by $\mathcal{B}_{\omega}$ a basis of neighbourhoods of the origin in $(X, \omega)$. Then $\mathcal{B}:=\left\{\varphi(U): U \in \mathcal{B}_{\omega}\right\}$ is a basis of neighbourhoods of the origin for a topology $\tau$ on $Y$ such that $(Y, \tau)$ is a $T A$ (resp. TA with continuous multiplication).

Proof. (Sheet 2)

Using the latter result one can show that the quotient of a TA over an ideal endowed with the quotient topology is a TA (Sheet 2). However, in the following we are going to give a direct proof of this fact without making use of bases. Before doing that, let us briefly recall the notion of quotient topology.

Given a topological space $(X, \omega)$ and an equivalence relation $\sim$ on $X$. The quotient set $X / \sim$ is defined to be the set of all equivalence classes w.r.t. to $\sim$. The map $\phi: X \rightarrow X / \sim$ which assigns to each $x \in X$ its equivalence class $\phi(x)$ w.r.t. $\sim$ is called the canonical map or quotient map. Note that $\phi$ is surjective. Thequotient topology on $X / \sim$ is the collection of all subsets $U$ of $X / \sim$ such that $\phi^{-1}(U) \in \omega$. Hence, the quotient map $\phi$ is continuous and actually the quotient topology on $X / \sim$ is the finest topology on $X / \sim$ such that $\phi$ is continuous.

Note that the quotient map $\phi$ is not necessarily open or closed.
Example 1.4.4. Consider $\mathbb{R}$ with the standard topology given by the modulus and define the following equivalence relation on $\mathbb{R}$ :

$$
x \sim y \Leftrightarrow(x=y \vee\{x, y\} \subset \mathbb{Z}) .
$$

Let $\mathbb{R} / \sim$ be the quotient set w.r.t $\sim$ and $\phi: \mathbb{R} \rightarrow \mathbb{R} / \sim$ the correspondent quotient map. Let us consider the quotient topology on $\mathbb{R} / \sim$. Then $\phi$ is not an open map. In fact, if $U$ is an open proper subset of $\mathbb{R}$ containing an integer, then $\phi^{-1}(\phi(U))=U \cup \mathbb{Z}$ which is not open in $\mathbb{R}$ with the standard topology. Hence, $\phi(U)$ is not open in $\mathbb{R} / \sim$ with the quotient topology.

For an example of not closed quotient map see e.g. [15, Example 2.3.3].
Let us consider now a $\mathbb{K}$-algebra $A$ and an ideal $I$ of $A$. We denote by $A / I$ the quotient set $A / \sim_{I}$, where $\sim_{I}$ is the equivalence relation on $A$ defined by $x \sim_{I} y$ iff $x-y \in I$. The canonical (or quotient) map $\phi: A \rightarrow A / I$ which assigns to each $x \in A$ its equivalence class $\phi(x)$ w.r.t. the relation $\sim_{I}$ is clearly surjective.

Using the fact that $I$ is an ideal of the algebra $A$ (see Definition 1.1.3-2), it is easy to check that:

1. if $x \sim_{I} y$, then $\forall \lambda \in \mathbb{K}$ we have $\lambda x \sim_{I} \lambda y$.
2. if $x \sim_{I} y$, then $\forall z \in A$ we have $x+z \sim_{I} y+z$.
3. if $x \sim_{I} y$, then $\forall z \in A$ we have $x z \sim_{I} y z$ and $z x \sim_{I} z y$.

These three properties guarantee that the following operations are well-defined on $A / I$ :

- vector addition: $\forall \phi(x), \phi(y) \in A / I, \phi(x)+\phi(y):=\phi(x+y)$
- scalar multiplication: $\forall \lambda \in \mathbb{K}, \forall \phi(x) \in A / I, \lambda \phi(x):=\phi(\lambda x)$
- vector multiplication: $\forall \phi(x), \phi(y) \in A / I, \phi(x) \cdot \phi(y):=\phi(x y)$
$A / I$ equipped with the three operations defined above is a $\mathbb{K}$-algebra which is often called quotient algebra. Then the quotient map $\phi$ is clearly a homomorphism. Moreover, if $A$ is unital and $I$ proper then also the quotient algebra $A / I$ is unital. Indeed, as $I$ is a proper ideal of $A$, the unit $1_{A}$ does not belong to $I$ and so we have $\phi\left(1_{A}\right) \neq o$ and for all $x \in A$ we get $\phi(x) \phi\left(1_{A}\right)=\phi\left(x \cdot 1_{A}\right)=\phi(x)=\phi\left(1_{A} \cdot x\right)=\phi\left(1_{A}\right) \phi(x)$.

Suppose now that $(A, \omega)$ is a TA and $I$ an ideal of $A$. Since $A$ is in particular a topological space, we can endow it with the quotient topology w.r.t. the equivalence relation $\sim_{I}$. We already know that in this setting $\phi$ is a continuous homomorphism but actually the structure of TA on $A$ guarantees also that it is open. Indeed, the following holds for any TVS and so for any TA:

Proposition 1.4.5. For a linear subspace $M$ of a t.v.s. $X$, the quotient mapping $\phi: X \rightarrow X / M$ is open (i.e. carries open sets in $X$ to open sets in $X / M$ ) when $X / M$ is endowed with the quotient topology.

Proof.
Let $V$ be open in $X$. Then we have

$$
\phi^{-1}(\phi(V))=V+M=\bigcup_{m \in M}(V+m) .
$$

Since $X$ is a t.v.s, its topology is translation invariant and so $V+m$ is open for any $m \in M$. Hence, $\phi^{-1}(\phi(V))$ is open in $X$ as union of open sets. By definition, this means that $\phi(V)$ is open in $X / M$ endowed with the quotient topology.

Theorem 1.4.6. Let $(A, \omega)$ be a TA (resp. TA with continuous multiplication) and $I$ an ideal of $A$. Then the quotient algebra $A / I$ endowed with the quotient topology is a TA (resp. TA with continuous multiplication).

Proof.
For convenience, in this proof we denote by a (resp. m) the vector addition (resp. vector multiplication) in $A / I$ and just by + (resp. •) the vector addition (resp. vector multiplication) in $A$. Let $W$ be a neighbourhood of the origin $o$ in $A / I$ endowed with the quotient topology $\tau_{Q}$. We first aim to prove that $\mathrm{a}^{-1}(W)$ is a neighbourhood of $(o, o)$ in $A / I \times A / I$.

By definition of $\tau_{Q}, \phi^{-1}(W)$ is a neighbourhood of the origin in $(A, \omega)$ and so, by Theorem 1.2.6-2 (we can apply the theorem because $(A, \omega)$ is a TA and so a TVS), there exists $V$ neighbourhood of the origin in $(A, \omega)$ s.t. $V+V \subseteq \phi^{-1}(W)$. Hence, by the linearity of $\phi$, we get
$\mathrm{a}(\phi(V) \times \phi(V))=\phi(V+V) \subseteq \phi\left(\phi^{-1}(W)\right) \subseteq W$, i.e. $\phi(V) \times \phi(V) \subseteq \mathrm{a}^{-1}(W)$.

Since $\phi$ is also an open map, $\phi(V)$ is a neighbourhood of the origin $o$ in $\left(A / I, \tau_{Q}\right)$ and so $\mathrm{a}^{-1}(W)$ is a neighbourhood of $(o, o)$ in $A / I \times A / I$ endowed with the product topology given by $\tau_{Q}$. A similar argument gives the continuity of the scalar multiplication. Hence, $A / I$ endowed with the quotient topology is a TVS.

Furthermore, for any $\tilde{x} \in A / I$ and any $W$ neighbourhood of the origin in $\left(A / I, \tau_{Q}\right)$, we know that $\tilde{x}=\phi(x)$ for some $x \in A$ and $\phi^{-1}(W)$ is a neighbourhood of the origin in $(A, \omega)$. Since $(A, \omega)$ is a TA, the multiplication - in $A$ is separately continuous so there exist $V_{1}, V_{2}$ neighbourhoods of the origin in $(A, \omega)$ such that $x \cdot V_{1} \subseteq \phi^{-1}(W)$ and $V_{2} \cdot x \subseteq \phi^{-1}(W)$. Setting $N_{1}:=\phi\left(V_{1}\right)$ and $N_{2}:=\phi\left(V_{2}\right)$, we get $\mathfrak{m}\left(\tilde{x} \times N_{1}\right)=\mathfrak{m}\left(\phi(x) \times \phi\left(V_{1}\right)\right)=$ $\phi\left(x \cdot V_{1}\right) \subseteq \phi\left(\phi^{-1}(W)\right) \subseteq W$ and similarly $\mathrm{m}\left(N_{2} \times \tilde{x}\right) \subseteq \phi\left(\phi^{-1}(W)\right) \subseteq W$. This yields that m is separately continuous as the quotient map is open and so $N_{1}, N_{2}$ are both neighbourhoods of the origin in $\left(A / I, \tau_{Q}\right)$.

Proposition 1.4.7. Let $A$ be a $T A$ and $I$ an ideal of $A$. Consider $A / I$ endowed with the quotient topology. Then the two following properties are equivalent:
a) $I$ is closed
b) $A / I$ is Hausdorff

Proof.
In view of Proposition 1.3.2, (b) is equivalent to say that the complement of the origin in $A / I$ is open w.r.t. the quotient topology. But the complement of the origin in $A / I$ is exactly the image under the canonical map $\phi$ of the complement of $I$ in $A$. Since $\phi$ is an open continuous map, the image under $\phi$ of the complement of $I$ in $X$ is open in $A / I$ iff the complement of $I$ in $A$ is open, i.e. (a) holds.

Corollary 1.4.8. If $A$ is a $T A$, then $A / \overline{\{o\}}$ endowed with the quotient topology is a Hausdorff TA. $A / \overline{\{o\}}$ is said to be the Hausdorff TA associated with $A$. When $A$ is a Hausdorff TA, $A$ and $A / \overline{\{o\}}$ are topologically isomorphic.

Proof.
First of all, let us observe that $\overline{\{o\}}$ is a closed ideal of $A$. Indeed, since $A$ is a TA, the multiplication is separately continuous and so for all $x, y \in A$ we have $x \overline{\{o\}} \subseteq \overline{\{x \cdot o\}}=\overline{\{o\}}$ and $\overline{\{o\}} y \subseteq \overline{\{o \cdot y\}}=\overline{\{o\}}$. Then, by Theorem 1.4.6 and Proposition 1.4.7, $A / \overline{\{o\}}$ is a Hausdorff TA. If in addition $A$ is also Hausdorff, then Proposition 1.3.2 guarantees that $\overline{\{o\}}=\{o\}$ in $A$. Therefore, the quotient map $\phi: A \rightarrow A / \overline{\{o\}}$ is also injective because in this case $\operatorname{Ker}(\phi)=\{o\}$.

Hence, $\phi$ is a topological isomorphism (i.e. bijective, continuous, open, linear) between $A$ and $A / \overline{\{o\}}$ which is indeed $A /\{o\}$.

Let us finally focus on quotients of normed algebra. If $(A,\|\cdot\|)$ is a normed (resp. Banach) algebra and $I$ a closed ideal of $A$, then Theorem 1.4.6 guarantees that $A / I$ endowed with the quotient topology is a TA with continuous multiplication but, actually, the latter is also a normed (resp. Banach) algebra. Indeed, one can easily show that the quotient topology is generated by the so-called quotient norm defined by

$$
q(\phi(x)):=\inf _{y \in I}\|x+y\|, \quad \forall x \in A
$$

which has the nice property to be submultiplicative and so the following holds.
Proposition 1.4.9. If $(A,\|\cdot\|)$ is a normed (resp. Banach) algebra and $I$ a closed ideal of $A$, then $A / I$ equipped with the quotient norm is a normed (resp. Banach) algebra.

Proof. (Sheet 2)

## Chapter 2

## Locally multiplicative convex algebras

### 2.1 Neighbourhood definition of Imc algebras

In the study of locally multiplicative convex algebras a particular role will be played by multiplicative sets. Therefore, before starting the study of this class of topological algebras we are going to have a closer look to this concept.

Definition 2.1.1. A subset $U$ of $a \mathbb{K}$-algebra $A$ is said to be a multiplicative set or m-set if $U \cdot U \subseteq U$. We call m-convex (resp. m-balanced) $a$ multiplicative convex (resp. balanced) subset of $A$ and absolutely m-convex $a$ multiplicative subset of $A$ which is both balanced and convex.

The notions defined above are totally algebraic and so independent from the topological structure with which the algebra is endowed.

## Example 2.1.2.

- Any ideal of an algebra is an m-set.
- Fixed an element $a \neq o$ of an algebra, the set $\left\{a^{n}: n \in \mathbb{N}\right\}$ is an m-set.
- Given a normed algebra $(A,\|\cdot\|)$ and an integer $n \in \mathbb{N}$, the open and the closed ball centered at origin with radius $\frac{1}{n}$ are both examples of absolutely m-convex sets in $A$.

The following proposition illustrates some operations under which the multiplicativity of a subset of an algebra is preserved.

Proposition 2.1.3. Let $A$ be a $\mathbb{K}$-algebra and $U \subset A$ multiplicative, then
a) The convex hull of $U$ is an m-convex set in $A$.
b) The balanced hull of $U$ is an $m$-balanced set in $A$.
c) The convex balanced hull of $U$ is an absolutely m-convex set in $A$.
d) Any direct or inverse image via a homomorphism is a m-set.

Proof. (Sheet 2)

Recall that
Definition 2.1.4. Let $S$ be any subset of a vector space $X$ over $\mathbb{K}$. The convex (resp. balanced) hull of $S$, denoted by $\operatorname{conv}(S)$ (resp. bal( $S$ )) is the smallest convex (resp. balanced) subset of $X$ containing $S$, i.e. the intersection of all convex (resp. balanced) subsets of $X$ containing S. Equivalently,

$$
\operatorname{conv}(S):=\left\{\sum_{i=1}^{n} \lambda_{i} x_{i}: x_{i} \in S, \lambda_{i} \in[0,1], \sum_{i=1}^{n} \lambda_{i}=1, n \in \mathbb{N}\right\}
$$

and the balanced hull of $S$, denoted by $\operatorname{bal}(S)$ as

$$
\operatorname{bal}(S):=\bigcup_{\lambda \in \mathbb{K},|\lambda| \leq 1} \lambda S .
$$

The convex balanced hull of $S$, denoted by $\operatorname{conv}_{b}(S)$, is defined as the smallest convex and balanced subset of $X$ containing $S$ and it can be easily proved that $\operatorname{conv}_{b}(S)=\operatorname{conv}(\operatorname{bal}(S))$.

Let us come back now to topological algebras.
Proposition 2.1.5. In any topological algebra, the operation of closure preserves the multiplicativity of a subset as well as its m-convexity and absolute $m$-convexity.

## Proof.

First of all let us show that the following property holds in any TA $(A, \tau)$ :

$$
\begin{equation*}
\forall V, W \subseteq A, \bar{V} \cdot \bar{W} \subseteq \overline{V W} \tag{2.1}
\end{equation*}
$$

where the closure in $A$ is here clearly intended w.r.t. the topology $\tau$. Let $x \in \bar{V}, y \in \bar{W}$ and $O \in \mathcal{F}(o)$ where $\mathcal{F}(o)$ denotes the filter of neighbourhoods of the origin in $A$. As $A$ is in particular a TVS, Theorem 1.2.6-2 ensures that there exists $N \in \mathcal{F}(o)$ s.t. $N+N \subseteq O$. Then for each $a \in A$, by Theorem 1.2.9, there exist $N_{1}, N_{2} \in \mathcal{F}(o)$ such that $N_{1} a \subseteq N$ and $a N_{2} \subseteq N$. Moreover, since $x \in \bar{V}$ and $y \in \bar{W}$, there exist $v \in V$ and $w \in W$ s.t. $v \in x+N_{1}$ and $w \in y+N_{2}$. Putting all together, we have that

$$
\begin{aligned}
v w \in\left(x+N_{1}\right) w & =x w+N_{1} w \subseteq x w+N \subseteq x\left(y+N_{2}\right)+N \\
& =x y+x N_{2}+N \subseteq x y+N+N \subseteq x y+O .
\end{aligned}
$$

Hence, $(x y+O) \cap V W \neq \emptyset$, which proves that $x y \in \overline{V W}$. Therefore, if $U$ is an m-set in $A$ then by (2.1) we get $\bar{U} \cdot \bar{U} \subseteq \overline{U \cdot U} \subseteq \bar{U}$, which proves that $\bar{U}$ is an m-set.

Suppose now that $U$ is m-convex. The first part of the proof guarantees that $\bar{U}$ is an m-set. Moreover, using that $A$ is in particular a TVS, we have that for any $\lambda \in[0,1]$ the mapping

$$
\varphi_{\lambda}: \begin{array}{ll}
A \times A & \rightarrow A \\
(x, y) & \mapsto \lambda x+(1-\lambda) y
\end{array}
$$

is continuous and so $\varphi_{\lambda}(\overline{U \times U}) \subseteq \overline{\varphi_{\lambda}(U \times U)}$. Since $U$ is also convex, for any $\lambda \in[0,1]$ we have that $\varphi_{\lambda}(U \times U) \subseteq U$ and so $\overline{\varphi_{\lambda}(U \times U)} \subseteq \bar{U}$. Putting all together, we can conclude that $\varphi_{\lambda}(\bar{U} \times \bar{U})=\varphi_{\lambda}(\overline{U \times U}) \subseteq \bar{U}$, i.e. $\bar{U}$ is convex. Hence, $\bar{U}$ is an m-convex set.

Finally, assume that $U$ is absolutely m-convex. As $U$ is in particular mconvex, by the previous part of the proof, we can conclude immediately that $\bar{U}$ is an m-convex set. Furthermore, since $U$ is balanced and $A$ has the TVS structure, we can conclude that $\bar{U}$ is also balanced. Indeed, in any TVS the closure of a balanced set is still balanced because the multiplication by scalar is continuous and so for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$ we have $\lambda \bar{U} \subseteq \overline{\lambda U} \subseteq U$.

Definition 2.1.6. A closed absorbing absolutely convex multiplicative subset of a TA is called a m-barrel.

Proposition 2.1.7. Every multiplicative neighbourhood of the origin in a TA is contained in a neighbourhood of the origin which is an m-barrel.
Proof.
Let $U$ be a multiplicative neighbourhood of the origin and define $T(U):=$ $\operatorname{conv}_{b}(U)$. Clearly, $U \subseteq T(U)$. Therefore, $T(U)$ is a neighbourhood of the origin and so it is absorbing by Theorem 1.2.6-4). By Proposition 2.1.3-c), $\operatorname{conv}_{b}(U)$ is an absolutely m-convex set as $U$ is an m-set. Hence, Proposition 2.1.5 ensures that $T(U)$ is closed and absolutely m-convex, i.e. an m-barrel.

Note that the converse inclusion in Proposition 2.1.7 does not hold in general. Indeed, in any TA not every neighbourhood of the origin (not even every multiplicative one) contains another one which is a m-barrel. This means that not every TA has a basis of neighbourhoods consisting of m-barrels. However, this is true for any lmc TA.

Definition 2.1.8. A $T A$ is said to be locally multiplicative convex (lmc) if it has a basis of neighbourhoods of the origin consisting of m-convex sets.

It is then easy to show that

Proposition 2.1.9. A locally multiplicative convex algebra is a TA with continuous multiplication.
Proof.
Let $(A, \tau)$ be an lmc algebra and let $\mathcal{B}$ denote a basis of neighbourhoods of the origin in $(A, \tau)$ consisting of m-convex sets. Then $(A, \tau)$ is in particular a TVS and for any $U \in \mathcal{B}$ we have $U \cdot U \subset U$. Hence, both conditions of Theorem 1.2 .10 are fulfilled by $\mathcal{B}$, which proves that $(A, \tau)$ is a TA with continuous multiplication.

Note that any lmc algebra is in particular a locally convex TVS, i.e. a TVS having a basis of neighbourhoods of the origin consisting of convex sets. Hence, in the study of this class of TAs we can make use of all the powerful results about locally convex TVS. To this aim let us recall that the class of locally convex TVS can be characterized in terms of absorbing absolutely convex neighbourhoods of the origin.

Theorem 2.1.10. If $X$ is a lc TVS then there exists a basis $\mathcal{B}$ of neighbourhoods of the origin consisting of absorbing absolutely convex subsets s.t.
a) $\forall U, V \in \mathcal{B}, \exists W \in \mathcal{B}$ s.t. $W \subseteq U \cap V$
b) $\forall U \in \mathcal{B}, \forall \rho>0, \exists W \in \mathcal{B}$ s.t. $W \subseteq \rho U$

Conversely, if $\mathcal{B}$ is a collection of absorbing absolutely convex subsets of a vector space $X$ s.t. a) and b) hold, then there exists a unique topology compatible with the linear structure of $X$ s.t. $\mathcal{B}$ is a basis of neighbourhoods of the origin in $X$ for this topology (which is necessarily locally convex).

## Proof.

Let $N$ be a neighbourhood of the origin in the lc TVS $(X, \tau)$. The local convexity ensures that there exists $W$ convex neighbourhood of the origin in $(X, \tau)$ s.t. $W \subseteq N$. Moreover, by Theorem 1.2.6-5), there exists $U$ balanced neighbourhood of the origin in $X$ s.t. $U \subseteq W$. Then, using that $W$ is a convex set containing $U$, we get $\operatorname{conv}(U) \subseteq W \subseteq N$. Now $\operatorname{conv}(U)$ is convex by definition, balanced because $U$ is balanced and it is also a neighbourhood of the origin (and so an absorbing set) since $U \subseteq \operatorname{conv}(U)$. Hence, the collection $\mathcal{B}:=$ $\left\{\operatorname{conv}(U): U \in \mathcal{B}_{b}\right\}$ is a basis of absorbing absolutely convex neighbourhoods of the origin in $(X, \tau)$; here $\mathcal{B}_{b}$ denotes a basis of balanced neighbourhoods of the origin in $(X, \tau)$. Observing that for any $U, W \in \mathcal{B}_{b}$ and any $\rho>0$ we have $\operatorname{conv}(U \cap W) \subseteq \operatorname{conv}(U) \cap \operatorname{conv}(W)$ and $\operatorname{conv}(\rho U) \subseteq \rho \operatorname{conv}(U)$, we see that $\mathcal{B}$ fulfills both a) and b).

The converse direction is left as an exercise for the reader.
This theorem will be a handful tool in the proof of the following characterization of lmc algebras in terms of neighbourhood basis.

Theorem 2.1.11. Let $A$ be $a \mathbb{K}$-algebra. Then the following are equivalent:
a) $A$ is an lmc algebra
b) A is a TVS having a basis of neighbourhoods consisting of m-barrels.
c) There exists a basis for a filter on A consisting of absorbing absolutely m-convex subsets.

## Proof.

$a) \Rightarrow b$ ) If $A$ is an lmc algebra, then we have already observed that it is a lc TVS. Let $\mathcal{F}(o)$ be the filter of neighbourhoods of the origin in $A$ and let $N \in \mathcal{F}(o)$. The TVS structure ensures that there exists $V \in \mathcal{F}(o)$ closed s.t. $V \subseteq N^{1}$ and the local convexity allows to apply Theorem 2.1.10 which guarantees that we can always find $M \in \mathcal{F}(o)$ absolutely convex s.t. $M \subseteq V$. Finally, since $A$ is an lmc algebra, we know that there exists $C \in \mathcal{F}(o)$ mconvex s.t. $C \subseteq M$. Using the previous inclusions we have that

$$
T(C):=\overline{\operatorname{conv}_{b}(C)} \subseteq \bar{M} \subseteq \bar{V}=V \subseteq N .
$$

(Note that the first inclusion follows from the fact that $M$ is a convex and balanced subset containing $C$.) Hence, the conclusion holds because $T(C)$ is an m-barrel set as $C$ is a multiplicative neighbourhood of the origin (see last part of proof of Proposition 2.1.7).
$b) \Rightarrow c)$ This is clear because every m-barrelled neighbourhood of the origin is an absorbing absolutely m-convex subsets of $A$.
$c) \Rightarrow a)$ Suppose that $\mathcal{M}$ is a basis for a filter on $A$ consisting of absorbing absolutely convex m-subsets. Then it is easy to verify that the collection $\widetilde{\mathcal{M}}:=\{\lambda U: U \in \mathcal{M}, 0<\lambda \leq 1\}$ also consists of absorbing absolutely m-convex subsets of $A$. Moreover, for any $U, V \in \mathcal{M}$ we know that there exists $W \in \mathcal{M}$ s.t. $W \subseteq U \cap W$ and so for any $0<\lambda, \mu \leq 1$ we have that $\delta W \subseteq \delta(U \cap V)=\delta U \cap \delta V \subseteq \lambda U \cap \mu V$ where $\delta:=\min \{\lambda, \mu\}$. As $\delta W \in \widetilde{\mathcal{M}}$ we have that a) of Theorem 2.1.10. Also b) of this same theorem is satisfied because for any $\rho>0,0<\lambda \leq 1$ and $U \in \mathcal{M}$ we easily get that there exists $M \in \widetilde{\mathcal{M}}$ s.t. $M \subseteq \rho(\lambda U)$ by choosing $M=\rho(\lambda U)$ when $0<\rho \leq 1$ and $M=\lambda U$ when $\rho>1$. Hence, $\widetilde{M}$ fulfills all the assumptions of the second part of Theorem 2.1.10 and so it is a basis of neighbourhoods of the origin for a uniquely defined topology $\tau$ on $A$ making $(A, \tau)$ a lc TVS. As every set in $\widetilde{\mathcal{M}}$ is m-convex, $(A, \tau)$ is in fact a lmc algebra.

[^4]From the last part of the proof we can immediately see that
Corollary 2.1.12. If $\mathcal{M}$ is a basis for a filter on a $\mathbb{K}$-algebra $A$ consisting of absorbing absolutely convex $m$-subsets, then there exists a unique topology $\tau$ on A both having $\widetilde{\mathcal{M}}:=\{\lambda U: U \in \mathcal{M}, 0<\lambda \leq 1\}$ as a basis of neighbourhoods of the origin and making $(A, \tau)$ an lmc algebra.

Theorem 2.1.11 shows that in an lmc algebra every neighbourhood of the origin contains an m-barrel set. However, it is important to remark that not every m-barrel subset of a topological algebra, not even of an lmc algebra, is a neighbourhood of the origin (see Examples 2.2.19)! Topological algebras having this property are called $m$-barrelled algebras.

### 2.2 Seminorm characterization of Imc algebras

In this section we will investigate the intrinsic and very useful connection between lmc algebras and seminorms. Therefore, let us briefly recall this concept and focus in particular on submultiplicative seminorms.
Definition 2.2.1. Let $X$ be a $\mathbb{K}$-vector space. A function $p: X \rightarrow \mathbb{R}$ is called $a$ seminorm if it satisfies the following conditions:

1. $p$ is subadditive: $\forall x, y \in X, p(x+y) \leq p(x)+p(y)$.
2. $p$ is positively homogeneous: $\forall x \in X, \forall \lambda \in \mathbb{K}, p(\lambda x)=|\lambda| p(x)$.

A seminorm on a $\mathbb{K}$-algebra $X$ is called submultiplicative if

$$
\forall x, y \in X, p(x y) \leq p(x) p(y)
$$

Definition 2.2.2. A seminorm $p$ on a vector space $X$ is a norm if $p(x)=0$ implies $x=o$ (i.e. if $p^{-1}(\{0\})=\{o\}$ ).

The following properties are an easy consequence of Definition 2.2.1.
Proposition 2.2.3. Let $p$ be a seminorm on a vector space $X$. Then:

- $p$ is symmetric, i.e. $p(x)=p(-x), \forall x \in X$.
- $p(o)=0$.
- $|p(x)-p(y)| \leq p(x-y), \forall x, y \in X$.
- $p(x) \geq 0, \forall x \in X$.
- $\operatorname{ker}(p)$ is a linear subspace of $X$.


## Examples 2.2.4.

a) Suppose $X=\mathbb{R}^{n}$ is equipped with the componentwise operations of addition, scalar and vector multiplication. Let $M$ be a linear subspace of $X$. For any $x \in X$, set

$$
q_{M}(x):=\inf _{m \in M}\|x-m\|,
$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^{n}$, i.e. $q_{M}(x)$ is the distance from the point $x$ to $M$ in the usual sense. If $\operatorname{dim}(M) \geq 1$ then $q_{M}$ is a submultiplicative seminorm but not a norm ( $M$ is exactly the kernel of $q_{M}$ ). When $M=\{o\}, p_{M}(\cdot)$ and $\|\cdot\|$ coincide.
b) Let $\mathcal{C}(\mathbb{R})$ be the vector space of all real valued continuous functions on the real line equipped with the pointwise operations of addition, multiplication and scalar multiplication. For any $a \in \mathbb{R}^{+}$, we define

$$
p_{a}(f):=\sup _{-a \leq t \leq a}|f(t)|, \forall f \in \mathcal{C}(\mathbb{R}) .
$$

Then $p_{a}$ is a submultiplicative seminorm but is never a norm because it might be that $f(t)=0$ for all $t \in[-a, a]$ (and so that $p_{a}(f)=0$ ) but $f \not \equiv 0$.
c) Let $n \geq 2$ be an integer and consider the algebra $\mathbb{R}^{n \times n}$ of real square matrices of order $n$. Then

$$
q(A):=\max _{i, j=1, \ldots, n}\left|A_{i j}\right|, \quad \forall A=\left(A_{i j}\right)_{i, j=1}^{n} \in \mathbb{R}^{n \times n}
$$

is a norm (so in particular a seminorm) but it is not submultiplicative because for example if $A$ is the matrix with all entries equal to 1 then it is easy to check that $\left\|A^{2}\right\|>\|A\|$.

Seminorms on vector spaces are strongly related to a special kind of functionals, i.e. Minkowski functionals. Let us investigate more in details such a relation. Note that we are still in the realm of vector spaces with no topology!
Definition 2.2.5. Let $X$ be a vector space and $V$ a non-empty subset of $X$. We define the Minkowski functional (or gauge) of $V$ to be the mapping:

$$
\begin{aligned}
p_{V}: & X
\end{aligned} \rightarrow \mathbb{R},=\inf \{\lambda>0: x \in \lambda V\}
$$

(where $p_{V}(x)=\infty$ if the set $\{\lambda>0: x \in \lambda V\}$ is empty).
It is then natural to ask whether there exists a class of subsets for which the associated Minkowski functionals are actually seminorms, and in particular submultiplicative seminorms in the context of algebras. The answer is positive in both cases as established in the following lemma.

Notation 2.2.6. Let $X$ be a vector space and $p$ a seminorm on $X$. The sets

$$
\stackrel{\circ}{U}_{p}=\{x \in X: p(x)<1\} \text { and } U_{p}=\{x \in X: p(x) \leq 1\} .
$$

are said to be, respectively, the open and the closed unit semiball of $p$.

Lemma 2.2.7. Let $X$ be $a \mathbb{K}$-vector space (resp. $\mathbb{K}$-algebra).
a) If $V$ is a non-empty subset of $X$ which is absorbing and absolutely convex (resp. absolutely m-convex), then the associated Minkowski functional $p_{V}$ is a seminorm (resp. submultiplicative seminorm) and ${\stackrel{\circ}{U_{p_{V}}}}^{\text {( }}$ ( $\subseteq U_{p_{V}}$.
b) If $q$ is a seminorm (resp. submultiplicative seminorm) on $X$ then both $U_{q}$ and $U_{q}$ are absorbing absolutely convex sets [resp. absolutely m-convex] and for any absorbing absolutely convex (resp. absolutely $m$-convex) $V$ such that $\stackrel{\circ}{U}_{q} \subseteq V \subseteq U_{q}$ we have $q=p_{V}$.

## Proof.

a) Let $V$ be a non-empty subset of $X$ which is absorbing and absolutely convex and denote by $p_{V}$ the associated Minkowski functional. We want to show that $p_{V}$ is a seminorm.

- First of all, note that $p_{V}(x)<\infty$ for all $x \in X$ because $V$ is absorbing. Indeed, for any $x \in X$ there exists $\rho_{x}>0$ s.t. for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq \rho_{x}$ we have $\lambda x \in V$ and so the set $\{\lambda>0: x \in \lambda V\}$ is never empty, i.e. $p_{V}$ has only finite nonnegative values. Moreover, since $o \in V$, we also have that $o \in \lambda V$ for any $\lambda \in \mathbb{K}$ and so $p_{V}(o)=\inf \{\lambda>0: o \in \lambda V\}=0$.
- The balancedness of $V$ implies that $p_{V}$ is positively homogeneous. Since we have already showed that $p_{V}(o)=0$ it remains to prove the positive homogeneity of $p_{V}$ for non-zero scalars. Since $V$ is balanced we have that for any $x \in X$ and for any $\xi, \lambda \in \mathbb{K}$ with $\xi \neq 0$ the following holds:

$$
\begin{equation*}
\xi x \in \lambda V \text { if and only if } x \in \frac{\lambda}{|\xi|} V . \tag{2.2}
\end{equation*}
$$

Indeed, $V$ balanced guarantees that $\xi V=|\xi| V$ and so $x \in \frac{\lambda}{|\xi|} V$ is equivalent to $\xi x \in \lambda \frac{\xi}{|\xi|} V=\lambda V$. Using (2.2), we get that for any $x \in X$ and for any $\xi \in \mathbb{K}$ with $\xi \neq 0$ :

$$
\begin{aligned}
p_{V}(\xi x) & =\inf \{\lambda>0: \xi x \in \lambda V\} \\
& =\inf \left\{\lambda>0: x \in \frac{\lambda}{|\xi|} V\right\} \\
& =\inf \left\{|\xi| \frac{\lambda}{|\xi|}>0: x \in \frac{\lambda}{|\xi|} V\right\} \\
& =|\xi| \inf \{\mu>0: x \in \mu V\}=|\xi| p_{V}(x)
\end{aligned}
$$

- The convexity of $V$ ensures the subadditivity of $p_{V}$. Take $x, y \in X$. By the definition of Minkowski functional, for every $\varepsilon>0$ there exist $\lambda, \mu>0$ s.t.

$$
\lambda<p_{V}(x)+\frac{\varepsilon}{2} \text { and } x \in \lambda V
$$

and

$$
\mu<p_{V}(y)+\frac{\varepsilon}{2} \text { and } y \in \mu V .
$$

Then, by the convexity of $V$, we obtain that $\frac{\lambda}{\lambda+\mu} V+\frac{\mu}{\lambda+\mu} V \subseteq V$, i.e. $\lambda V+\mu V \subseteq(\lambda+\mu) V$, and therefore $x+y \in(\lambda+\mu) V$. Hence:

$$
p_{V}(x+y)=\inf \{\delta>0: x+y \in \delta V\} \leq \lambda+\mu<p_{V}(x)+p_{V}(y)+\varepsilon
$$

which proves the subadditivity of $p_{V}$ since $\varepsilon$ is arbitrary.
We can then conclude that $p_{V}$ is a seminorm. Furthermore, we have the following inclusions:

$$
{\stackrel{\circ}{U_{p_{V}}} \subseteq V \subseteq U_{p_{V}} .}
$$

In fact, if $x \in{\stackrel{\circ}{U^{\prime}}}_{p_{V}}$ then $p_{V}(x)<1$ and so there exists $0<\lambda<1$ s.t. $x \in \lambda V$. Since $V$ is balanced, for such $\lambda$ we have $\lambda V \subseteq V$ and therefore $x \in V$. On the other hand, if $x \in V$ then clearly $1 \in\{\lambda>0: x \in \lambda V\}$ which gives $p_{V}(x) \leq 1$ and so $x \in U_{p_{V}}$.

If $X$ is a $\mathbb{K}$-algebra and $V$ an absorbing absolutely m-convex subset of $X$, then the previous part of the proof guarantees that $p_{V}$ is a seminorm and $\stackrel{\circ}{U}_{p_{V}} \subseteq V \subseteq U_{p_{V}}$. Moreover, for any $a, b \in X$, the multiplicativity of $V$ implies that $\{\lambda>0: a \in \lambda V\}\{\mu>0: b \in \mu V\} \subseteq\{\delta>0: a b \in \delta V\}$ and so

$$
\begin{aligned}
p_{V}(a) p_{V}(b) & =\inf (\{\lambda>0: a \in \lambda V\}\{\mu>0: b \in \mu V\}) \\
& \geq \inf \{\delta>0: a b \in \delta V\}=p_{V}(a b) .
\end{aligned}
$$

Hence, $p_{V}$ is a submultiplicative seminorm.
b) Let us take any seminorm $q$ on $X$. Let us first show that $\dot{U}_{q}$ is absorbing and absolutely convex.

- $\stackrel{\circ}{U}_{q}$ is absorbing.

Let $x$ be any point in $X$. If $q(x)=0$ then clearly $x \in \stackrel{\circ}{U}_{q}$. If $q(x)>0$, we can take $0<\rho<\frac{1}{q(x)}$ and then for any $\lambda \in \mathbb{K}$ s.t. $|\lambda| \leq \rho$ the positive homogeneity of $q$ implies that $q(\lambda x)=|\lambda| q(x) \leq \rho q(x)<1$, i.e. $\lambda x \in \stackrel{\circ}{U}_{q}$.

- $\stackrel{\circ}{U}_{q}$ is balanced.

For any $x \in \stackrel{\circ}{U}_{q}$ and for any $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$, again by the positive homogeneity of $q$, we get: $q(\lambda x)=|\lambda| q(x) \leq q(x)<1$ i.e. $\lambda x \in \stackrel{\circ}{U}_{q}$.

- $\stackrel{\circ}{U}_{q}$ is convex.

For any $x, y \in \stackrel{\circ}{U}_{q}$ and any $t \in[0,1]$, by both the properties of seminorm, we have that $q(t x+(1-t) y) \leq t q(x)+(1-t) q(y)<t+1-t=1$ i.e. $t x+(1-t) y \in \stackrel{\circ}{U}_{q}$.

The proof above easily adapts to show that $U_{q}$ is absorbing and absolutely convex. Also, it is easy to check that

$$
\begin{equation*}
p_{\dot{U}_{q}}(x)=q(x)=p_{U_{q}}(x), \forall x \in X . \tag{2.3}
\end{equation*}
$$

Since for any absorbing absolutely convex subset $V$ of $X$ s.t. $\stackrel{\circ}{U}_{q} \subseteq V \subseteq U_{q}$ and for any $x \in X$ we have that

$$
p_{U_{q}}(x) \leq p_{V}(x) \leq p_{\dot{U}_{q}}(x),
$$

by (2.3) we can conclude that $p_{V}(x)=q(x)$.
If $X$ is a $\mathbb{K}$-algebra and $q$ is submultiplicative, then the previous part of the proof of b) applies but in addition we get that both $\stackrel{\circ}{U}_{q}$ and $U_{q}$ are multiplicative sets. Indeed, for any $a, b \in \dot{U}_{q}$ we have $q(a b) \leq q(a) q(b)<1$, i.e. $a b \in \dot{U}_{q}$ and similarly for $U_{q}$.

In a nutshell this lemma says that: a real-valued functional on $a \mathbb{K}$-vector space $X$ (resp. a $\mathbb{K}$-algebra) is a seminorm (resp-submultiplicative seminorm) if and only if it is the Minkowski functional of an absorbing absolutely convex (resp. absolutely m-convex) non-empty subset of $X$.

Let us collect some interesting properties of semiballs in a vector space, which we will repeatedly use in the following.

Proposition 2.2.8. Let $X$ be a $\mathbb{K}$-vector space and $p$ a seminorm on $X$. Then:
a) $\forall r>0, r \stackrel{\circ}{U}_{p}=\{x \in X: p(x)<r\}=\stackrel{\circ}{U}_{\frac{1}{r} p}$.
b) $\forall x \in X, x+\stackrel{\circ}{U}_{p}=\{y \in X: p(y-x)<1\}$.
c) If $q$ is also a seminorm on $X$, then $p \leq q$ if and only if $\dot{U}_{q} \subseteq \dot{U}_{p}$.
d) If $n \in \mathbb{N}$ and $s_{1}, \ldots, s_{n}$ are seminorms on $X$, then their maximum $s$ defined as $s(x):=\max _{i=1, \ldots, n} s_{i}(x), \forall x \in X$ is also seminorm on $X$ and $\stackrel{\circ}{U}_{s}=\bigcap_{i=1}^{n} \stackrel{\circ}{U}_{s_{i}}$.
In particular, if $X$ is a $\mathbb{K}$-algebra and all $s_{i}$ 's are submultiplicative seminorms, then $s(x)$ is also submultiplicative.
All the previous properties also hold for closed semballs.
Proof. (Sheet 3)
Let us start to put some topological structure on our space and so to consider continuous seminorms on it. The following result holds in any TVS and so in particular in any TA.

Proposition 2.2.9. Let $X$ be a TVS and $p$ a seminorm on $X$. Then the following conditions are equivalent:
a) The open unit semiball $\stackrel{\circ}{U}_{p}$ of $p$ is an open neighbourhood of the origin and coincides with the interior of $U_{p}$.
b) $p$ is continuous at the origin.
c) The closed unit semiball $U_{p}$ of $p$ is a closed neighbourhood of the origin and coincides with the closure of $\dot{U}_{p}$.
d) $p$ is continuous at every point.

Proof.
$a) \Rightarrow b)$ Suppose that $\stackrel{\circ}{U}_{p}$ is open in the topology on $X$. Then for any $\varepsilon>0$ we have that $p^{-1}(]-\varepsilon, \varepsilon[)=\{x \in X: p(x)<\varepsilon\}=\varepsilon \dot{U}_{p}$ is an open neighbourhood of the origin in $X$. This is enough to conclude that $p: X \rightarrow \mathbb{R}^{+}$ is continuous at the origin.
$b) \Rightarrow c)$ Suppose that $p$ is continuous at the origin, then $U_{p}=p^{-1}([0,1])$ is a closed neighbourhood of the origin. Also, by definition $U_{p} \subseteq U_{p}$ and so $\overline{\overleftarrow{U}_{p}} \subseteq \overline{U_{p}}=U_{p}$. To show the converse inclusion, we consider $x \in X$ s.t. $p(x)=1$ and take $\left(\lambda_{n}\right)_{n \in \mathbb{N}} \subset(0,1)$ s.t. $\lim _{n \rightarrow \infty} \lambda_{n}=1$. Then $\lambda_{n} x \in \stackrel{\circ}{U}_{p}$ and $\lim _{n \rightarrow \infty} \lambda_{n} x=x$ since the scalar multiplication is continuous. Hence, $x \in \overline{\dot{U}_{p}}$ which completes the proof of c ).
$c) \Rightarrow d)$ Assume that c) holds and fix $x \in X$. Using Proposition 2.2.8 and Proposition 2.2.3, we get that for any $\varepsilon>0: p^{-1}([-\varepsilon+p(x), p(x)+\varepsilon])=\{y \in$ $X:|p(y)-p(x)| \leq \varepsilon\} \supseteq\{y \in X: p(y-x) \leq \varepsilon\}=x+\varepsilon U_{p}$, which is a closed neighbourhood of $x$ since $X$ is a TVS and by the assumption $c$ ). Hence, $p$ is continuous at $x$.
$d) \Rightarrow a)$ If $p$ is continuous on $X$ then a) holds because $\stackrel{\circ}{U}_{p}=p^{-1}(]-1,1[)$ and the preimage of an open set under a continuous function is open. Also, by definition $\stackrel{\circ}{U}_{p} \subseteq U_{p}$ and so $\stackrel{\circ}{U}_{p}=\operatorname{int}\left(\stackrel{\circ}{U}_{p}\right) \subseteq \operatorname{int}\left(U_{p}\right)$. To show the converse inclusion, we consider $x \in \operatorname{int}\left(U_{p}\right)$. Then $p(x) \leq 1$ but, since $p(x)=p_{U_{p}}(x)$, we also have that for any $\varepsilon>0$ there exists $\lambda>0$ s.t. $x \in \lambda \dot{U}_{\circ}$ and $\lambda<p(x)+\varepsilon$. This gives that $p(x)<\lambda<1+\varepsilon$ and so $p(x)<1$, i.e. $x \in \dot{U}_{p}$ which completes the proof of a).

Definition 2.2.10. Let $X$ be a vector space and $\mathcal{P}:=\left\{p_{i}\right\}_{i \in I}$ a family of seminorms on $X$. The coarsest topology $\tau_{\mathcal{P}}$ on $X$ s.t. each $p_{i}$ is continuous is said to be the topology induced or generated by the family of seminorms $\mathcal{P}$.

We are now ready to see the connection between submultiplicative seminorms and locally convex multiplicative algebras.

Theorem 2.2.11. Let $X$ be a $\mathbb{K}$-algebra and $\mathcal{P}:=\left\{p_{i}\right\}_{i \in I}$ a family of submultiplicative seminorms. Then the topology induced by the family $\mathcal{P}$ is the unique topology both making $X$ into an lmc algebra and having as a basis of neighbourhoods of the origin the following collection:
$\mathcal{B}:=\left\{\left\{x \in X: p_{i_{1}}(x) \leq \varepsilon, \ldots, p_{i_{n}}(x) \leq \varepsilon\right\}: i_{1}, \ldots, i_{n} \in I, n \in \mathbb{N}, 0<\varepsilon \leq 1\right\}$.
Viceversa, the topology of an arbitrary lmc algebra is always induced by a family of submultiplicative seminorms (often called generating).

## Proof.

Let us first observe that

$$
\mathcal{B}=\left\{\bigcap_{j=1}^{n} \varepsilon U_{p_{i_{j}}}: n \in \mathbb{N}, i_{1}, \ldots, i_{n} \in I, 0<\varepsilon \leq 1\right\}
$$

and is a basis for a filter on $X$ as it is closed under finite intersections. Moreover, by Proposition $2.2 .8-\mathrm{a}$ ) and Lemma 2.2.7-b), we have that for any $i \in I$ the semiball $\varepsilon U_{p_{i}}$ is absorbing and absolutely m-convex. Therefore, any element in $\mathcal{B}$ is an absorbing absolutely m-convex subset of $X$ as finite intersection of sets having such properties. Hence, Corollary 2.1.12 guarantees that there exists a unique topology $\tau$ having $\mathcal{B}$ as a basis of neighbourhoods of the origin and s.t. $(X, \tau)$ is an lmc algebra.

Since for any $i \in I$ we have $U_{p_{i}} \in \mathcal{B}, U_{p_{i}}$ is a neighbourhood of the origin in $(X, \tau)$, then by Proposition 2.2.9, the seminorm $p_{i}$ is $\tau$-continuous. Therefore, the topology $\tau_{\mathcal{P}}$ induced by the family $\mathcal{P}$ is by definition coarser than $\tau$. On the other hand, each $p_{i}$ is also $\tau_{\mathcal{P}}$-continuous and so $U_{p_{i}}$ is a closed neighbourhood of the origin in $\left(X, \tau_{\mathcal{P}}\right)$. Then $\mathcal{B}$ consists of neighbourhoods of the origin in $\left(X, \tau_{\mathcal{P}}\right)$ which implies that $\tau$ is coarser than $\tau_{\mathcal{P}}$. Hence, $\tau \equiv \tau_{\mathcal{P}}$.

Viceversa, let us assume that $(X, \tau)$ is an lmc algebra. Then by Theorem 2.1.11 there exists a basis $\mathcal{N}$ of neighbourhoods of the origin in $(X, \tau)$ consisting of m-barrels. Consider now the family $\mathcal{S}:=\left\{p_{N}: N \in \mathcal{N}\right\}$. By Lemma 2.2.7-a), we know that each $p_{N}$ is a submultiplicative seminorm and that $\stackrel{\circ}{U}_{p_{N}} \subseteq N \subseteq U_{p_{N}}$. Now each $p_{N}$ is $\tau$-continuous because $U_{p_{N}} \supseteq N \in \mathcal{N}$ and hence, $\tau_{\mathcal{S}} \subseteq \tau$. Moreover, each $p_{N}$ is clearly $\tau_{\mathcal{S}}$-continuous and so, by Proposition 2.2.9, $\stackrel{\circ}{U}_{p_{N}}$ is open in $\left(X, \tau_{\mathcal{S}}\right)$. Since $\stackrel{\circ}{U}_{p_{N}} \subseteq N$, we have that $\mathcal{N}$ consists of neighbourhoods of the origin in $\left(X, \tau_{\mathcal{S}}\right)$, which implies $\tau \subseteq \tau_{\mathcal{S}}$.

Historically the following more general result holds for locally convex tvs and the previous theorem could be also derived as a corollary of:

Theorem 2.2.12. Let $X$ be a vector space and $\mathcal{P}:=\left\{p_{i}\right\}_{i \in I}$ a family of seminorms. Then the topology induced by the family $\mathcal{P}$ is the unique topology both making $X$ into a locally convex TVS and having as a basis of neighbourhoods of the origin the following collection:

$$
\mathcal{B}:=\left\{\left\{x \in X: p_{i_{1}}(x) \leq \varepsilon, \ldots, p_{i_{n}}(x) \leq \varepsilon\right\}: i_{1}, \ldots, i_{n} \in I, n \in \mathbb{N}, 0<\varepsilon \leq 1\right\} .
$$

Viceversa, the topology of an arbitrary locally convex TVS is always induced by a family of seminorms (often called generating).

Coming back to lmc algebras, Theorem 2.2.11 allows us to give another characterization of such a class, namely: ATA is lmc if and only if its topology is induced by a family of submultiplicative seminorms. This is very helpful in establishing whether a given topological algebras is lmc or not as we will see from the following examples.

## Examples 2.2.13.

1. Normed algebras are clearly lmc algebras.
2. A seminormed algebra, i.e. a $\mathbb{K}$-algebra endowed with the topology generated by a submultiplicative seminorm, is lmc.
3. The weak and the strong operator topologies on the space $L(H)$ introduced in Example 1.2.17 both make $L(H)$ into a locally convex algebra which is not lmc. Indeed, the weak operator topology $\tau_{w}$ is generated by the family of seminorms $\left\{p_{x, y}: x, y \in H\right\}$ where $p_{x, y}(T):=|\langle T x, y\rangle|$, while the strong operator topology $\tau_{s}$ is generated by the family of seminorms $\left\{p_{x}: x \in H\right\}$ where $p_{x}(T):=\|T x\|$. If $\left(L(H), \tau_{w}\right)$ and $\left(L(H), \tau_{s}\right)$ were lmc algebras, then by Proposition 2.1.9 the multiplication should have been jointly continuous in both of them but this is not the case as we have already showed in Example 1.2.17.
4. Consider $L^{\omega}([0,1]):=\bigcap_{p \geq 1} L^{p}([0,1])$, where for each $p \geq 1$ we define $L^{p}([0,1])$ to be the space of all equivalence classes of functions $f:[0,1] \rightarrow \mathbb{R}$ such that $\|f\|_{p}:=\left(\int_{0}^{1}|f(t)|^{p} d t\right)^{\frac{1}{p}}<\infty$ which agree almost everywhere. The set $L^{\omega}([0,1])$ endowed with the pointwise operations is a real algebra since for any $q, r \geq 1$ such that $\frac{1}{p}=\frac{1}{q}+\frac{1}{r}$ we have

$$
\|f g\|_{p} \leq\|f\|_{q}\|g\|_{r}, \forall f, g \in L^{\omega}([0,1]) .
$$

The algebra $L^{\omega}([0,1])$ endowed with the topology induced by the family $\mathcal{P}:=\left\{\|\cdot\|_{p}: p \geq 1\right\}$ of seminorms is a locally convex algebra. However, $\left(L^{\omega}([0,1]), \tau_{\mathcal{P}}\right)$ is not an lmc algebra because any $m$-convex subset $U$ is open in $\left(L^{\omega}([0,1]), \tau_{\mathcal{P}}\right)$ if and only if $U=L^{\omega}([0,1])$ (Sheet 3).

Let us conclude this section with a further very useful property of lmc algebras.

Proposition 2.2.14. The topology of an lme algebra can be always induced by a directed family of submultiplicative seminorms.

Definition 2.2.15. A family $\mathcal{Q}:=\left\{q_{j}\right\}_{j \in J}$ of seminorms on a vector space $X$ is said to be directed (or fundamental or saturated) if
$\forall n \in \mathbb{N}, j_{1}, \ldots, j_{n} \in J, \exists j \in J, C>0$ s.t. $C q_{j}(x) \geq \max _{k=1, \ldots, n} q_{j_{k}}(x), \forall x \in X$.

To prove Proposition 2.2 .14 we need to recall an important criterion to compare topologies induced by families of seminorms.

Theorem 2.2.16.
Let $\mathcal{P}=\left\{p_{i}\right\}_{i \in I}$ and $\mathcal{Q}=\left\{q_{j}\right\}_{j \in J}$ be two families of seminorms on a $\mathbb{K}$-vector space $X$ inducing respectively the topologies $\tau_{\mathcal{P}}$ and $\tau_{\mathcal{Q}}$. Then $\tau_{\mathcal{P}}$ is finer than $\tau_{\mathcal{Q}}$ (i.e. $\tau_{\mathcal{Q}} \subseteq \tau_{\mathcal{P}}$ ) iff

$$
\begin{equation*}
\forall q \in \mathcal{Q} \exists n \in \mathbb{N}, i_{1}, \ldots, i_{n} \in I, C>0 \text { s.t. } C q(x) \leq \max _{k=1, \ldots, n} p_{i_{k}}(x), \forall x \in X . \tag{2.5}
\end{equation*}
$$

Proof.
Let us first recall that, by Theorem 2.2.12, we have that

$$
\mathcal{B}_{\mathcal{P}}:=\left\{\bigcap_{k=1}^{n} \varepsilon{\stackrel{\circ}{U_{i_{i_{k}}}}}: i_{1}, \ldots, i_{n} \in I, n \in \mathbb{N}, 0<\varepsilon \leq 1\right\}
$$

and

$$
\mathcal{B}_{\mathcal{Q}}:=\left\{\bigcap_{k=1}^{n} \varepsilon \dot{U}_{q_{j_{k}}}: j_{1}, \ldots, j_{n} \in J, n \in \mathbb{N}, 0<\varepsilon \leq 1\right\} .
$$

are respectively bases of neighbourhoods of the origin for $\tau_{\mathcal{P}}$ and $\tau_{\mathcal{Q}}$.
By using Proposition 2.2.8, the condition (2.5) can be rewritten as

$$
\forall q \in \mathcal{Q}, \exists n \in \mathbb{N}, i_{1}, \ldots, i_{n} \in I, C>0 \text { s.t. } C \bigcap_{k=1}^{n}{\stackrel{\circ}{U_{i_{k}}}}^{\overbrace{}^{\circ}} \stackrel{\circ}{q} \text {. }
$$

which means that

$$
\begin{equation*}
\forall q \in \mathcal{Q}, \exists B_{q} \in \mathcal{B}_{\mathcal{P}} \text { s.t. } B_{q} \subseteq \stackrel{\circ}{U}_{q} . \tag{2.6}
\end{equation*}
$$

since $C \bigcap_{k=1}^{n}{\stackrel{\circ}{U_{p_{k}}}} \in \mathcal{B}_{\mathcal{P}}$.
Condition (2.6) means that for any $q \in \mathcal{Q}$ the set $\stackrel{\circ}{U}_{q}$ is a neighbourhood of the origin in $\left(X, \tau_{\mathcal{P}}\right)$, which by Proposition 2.2.9 is equivalent to say that $q$ is continuous w.r.t. $\tau_{\mathcal{P}}$. By definition of $\tau_{\mathcal{Q}}$, this gives that $\tau_{\mathcal{Q}} \subseteq \tau_{\mathcal{P}}$.

This theorem allows us to easily see that the topology induced by a family of seminorms on a vector space does not change if we close the family under taking the maximum of finitely many of its elements. Indeed, the following result holds.

Proposition 2.2.17. Let $\mathcal{P}:=\left\{p_{i}\right\}_{i \in I}$ be a family of seminorms on a $\mathbb{K}$-vector space (resp. submultiplicative seminorms on a $\mathbb{K}$-algebra) $X$. Then we have that $\mathcal{Q}:=\left\{\max _{i \in B} p_{i}: \emptyset \neq B \subseteq I\right.$ with $B$ finite $\}$ is a directed family of seminorms (resp. submultiplicative seminorms) and $\tau_{\mathcal{P}}=\tau_{\mathcal{Q}}$, where $\tau_{\mathcal{P}}$ and $\tau_{\mathcal{Q}}$ denote the topology induced on $X$ by $\mathcal{P}$ and $\mathcal{Q}$, respectively.

## Proof.

First of all let us note that, by Proposition 2.2.8-d), $\mathcal{Q}$ is a family of seminorms. On the one hand, since $\mathcal{P} \subseteq \mathcal{Q}$, by definition of induced topology we have $\tau_{\mathcal{P}} \subseteq \tau_{\mathcal{Q}}$. On the other hand, for any $q \in \mathcal{Q}$ we have $q=\max _{i \in B} p_{i}$ for some $\emptyset \neq B \subseteq I$ finite. Then (2.5) is fulfilled for $n=|B|$ (where $|B|$ denotes the cardinality of the finite set $B), i_{1}, \ldots, i_{n}$ being the $n$ elements of $B$ and for any $0<C \leq 1$. Hence, by Theorem 2.2.16, $\tau_{\mathcal{Q}} \subseteq \tau_{\mathcal{P}}$. If $X$ is a $\mathbb{K}$-algebra and $\mathcal{P}$ consists of submultiplicative seminorms, then $\mathcal{Q}$ consists of submultiplicative seminorms by the second part of Proposition 2.2.8-d).

We claim that $\mathcal{Q}$ is directed.
Let $n \in \mathbb{N}$ and $q_{1}, \ldots, q_{n} \in \mathcal{Q}$. Then for each $j \in\{1, \ldots, n\}$ we have $q_{j}=$ $\max _{i \in B_{j}} p_{i}$ for some non-empty finite subset $B_{j}$ of $I$. Let us define $B:=\bigcup_{j=1}^{n} B_{j}$ and $q:=\max _{i \in B} p_{i}$. Then $q \in \mathcal{Q}$ and for any $C \geq 1$ we have that (2.4) is satisfied, because we get that for any $x \in X$

$$
C q(x) \geq \max _{i \in B} p_{i}(x)=\max _{j=1, \ldots, n}\left(\max _{i \in B_{j}} p_{i}(x)\right)=\max _{j=1, \ldots, n} q_{j}(x)
$$

Hence, $\mathcal{Q}$ is directed.

We are ready now to show Proposition 2.2.14.
Proof. of Proposition 2.2.14
Let $(X, \tau)$ be an lmc algebra. By Theorem 2.2.11, we have that there exists a family of submultiplicative seminorms $\mathcal{P}:=\left\{p_{i}\right\}_{i \in I}$ on $X$ s.t. $\tau=\tau_{\mathcal{P}}$. Let us define $\mathcal{Q}$ as the collection obtained by forming the maximum of finitely many elements of $\mathcal{P}$, i.e. $\mathcal{Q}:=\left\{\max _{i \in B} p_{i}: \emptyset \neq B \subseteq I\right.$ with $B$ finite $\}$. By

Proposition 2.2.17, $\mathcal{Q}$ is a directed family of submultiplicative seminorms and we have that $\tau_{\mathcal{P}}=\tau_{\mathcal{Q}}$.

It is possible to show (Sheet 3) that a basis of neighbourhoods of the origin for the lmc topology $\tau_{\mathcal{Q}}$ induced by a directed family of submultiplicative seminorms $\mathcal{Q}$ is given by:

$$
\begin{equation*}
\mathcal{B}_{d}:=\left\{r U_{q}: q \in \mathcal{Q}, 0<r \leq 1\right\} . \tag{2.7}
\end{equation*}
$$

Remark 2.2.18. The proof of Proposition 2.2.14 can be easily adapted to show that the topology of a lc tvs can be always induced by a directed family of seminorms $\tau_{\mathcal{Q}}$ and that the corresponding (2.7) is basis of neighbourhoods of the origin for $\tau_{\mathcal{Q}}$.

Example 2.2.19. $\operatorname{Let} \mathcal{C}_{b}(\mathbb{R})$ the set of all real-valued bounded continuous functions on the real line endowed with the pointwise operations of addition, multiplication and scalar multiplication and endowed with the topology $\tau_{\mathcal{Q}}$ induced by the family $\mathcal{Q}:=\left\{p_{a}: a>0\right\}$, where $p_{a}(f):=\sup _{-a \leq t \leq a}|f(t)|, \forall f \in \mathcal{C}_{b}(\mathbb{R})$. Since each $p_{a}$ is a submultiplicative seminorm (see Example 2.2.4-d)), the algebra $\left(\mathcal{C}_{b}(\mathbb{R}), \tau_{\mathcal{Q}}\right)$ is lmc.

Note that $\mathcal{Q}$ is directed since for any $n \in \mathbb{N}$ and any positive real numbers $a_{1}, \ldots, a_{n}$ we have that $\max _{i=1, \ldots, n} p_{a_{i}}(f)=\sup _{t \in[-b, b]}|f(t)|=p_{b}(f)$, where $b:=\max _{i=1, \ldots, n} a_{i}$, and so (2.4) is fulfilled. Hence, $\mathcal{B}_{d}$ as in (2.7) is a basis of neighbourhoods of the origin for the lmc topology $\tau_{\mathcal{Q}}$.

The algebra $\left(\mathcal{C}_{b}(\mathbb{R}), \tau_{\mathcal{Q}}\right)$ is not m-barrelled, because for instance the set $M:=\left\{f \in \mathcal{C}_{b}(\mathbb{R}): \sup _{t \in \mathbb{R}}|f(t)| \leq 1\right\}$ is an m-barrel but not a neighbourhood of the origin in $\left(\mathcal{C}_{b}(\mathbb{R}), \tau_{\mathcal{Q}}\right)$. Indeed, no elements of the basis $\mathcal{B}_{d}$ of neighbourhoods of the origin is entirely contained in $M$, because for any $a>0$ and any $0<r \leq 1$ the set $r U_{p_{a}}$ also contains continuous functions bounded by $r$ on $[-a, a]$ but bounded by $C>1$ on the whole $\mathbb{R}$ and so not belonging to $M$.

### 2.3 Hausdorff Imc algebras

In Section 1.3, we gave some characterization of Hausdorff TVS which can of course be applied to establish whether an lmc algebra is Hausdorff or not. However, in this section we aim to provide necessary and sufficient conditions bearing only on the family of seminorms generating an lmc topology for being a Hausdorff topology.

## Definition 2.3.1.

A family of seminorms $\mathcal{P}:=\left\{p_{i}\right\}_{i \in I}$ on a vector space $X$ is said to be separating if

$$
\begin{equation*}
\forall x \in X \backslash\{o\}, \exists i \in I \text { s.t. } p_{i}(x) \neq 0 \tag{2.8}
\end{equation*}
$$

Note that the separation condition (2.8) is equivalent to

$$
p_{i}(x)=0, \forall i \in I \Rightarrow x=o
$$

which by using Proposition 2.2 .8 can be rewritten as

$$
\begin{equation*}
\bigcap_{i \in I, c>0} c \dot{U}_{p_{i}}=\{o\}, \tag{2.9}
\end{equation*}
$$

since $p_{i}(x)=0$ is equivalent to say that $p_{i}(x)<c$, for all $c>0$.
It is clear that if any of the elements in a family of seminorms is actually a norm, then the the family is separating.

Lemma 2.3.2. Let $\tau_{\mathcal{P}}$ be the topology induced by a separating family of seminorms $\mathcal{P}:=\left(p_{i}\right)_{i \in I}$ on a vector space $X$. Then $\tau_{\mathcal{P}}$ is a Hausdorff topology.

Proof. ${ }^{2}$
Let $x, y \in X$ be such that $x \neq y$. Since $\mathcal{P}$ is separating, we have that $\exists i \in I$ with $p_{i}(x-y) \neq 0$. Then $\exists \varepsilon>0$ s.t. $p_{i}(x-y)=2 \varepsilon$. Take $V_{x}:=x+\varepsilon \dot{U}_{p_{i}}$ and $V_{y}:=y+\varepsilon U_{p_{i}}$. Since Theorem 2.2.12 guarantees that ( $X, \tau_{\mathcal{P}}$ ) is a TVS where the set $\varepsilon \dot{U}_{p_{i}}$ is a neighbourhood of the origin, $V_{x}$ and $V_{y}$ are neighbourhoods of $x$ and $y$, respectively. They are clearly disjoint. Indeed, if there would exist $u \in V_{x} \cap V_{y}$ then $p_{i}(x-y)=p_{i}(x-u+u-y) \leq p_{i}(x-u)+p_{i}(u-y)<2 \varepsilon$, which is a contradiction.

## Proposition 2.3.3.

a) A locally convex TVS is Hausdorff if and only if its topology can be induced by a separating family of seminorms.
b) An lmc algebra is Hausdorff if and only if its topology can be induced by a separating family of submultiplicative seminorms.

[^5]Proof.
a) Let $(X, \tau)$ be a locally convex TVS. Then we know that $\tau$ is induced by a directed family $\mathcal{P}$ of seminorms on $X$ and that $\mathcal{B}_{d}:=\left\{r U_{p}: p \in \mathcal{Q}, 0<r \leq 1\right\}$ (see Remark 2.2.18).

Suppose that $(X, \tau)$ is also Hausdorff. Then Proposition 1.3.2 ensures that for any $x \in X$ with $x \neq o$ there exists a neighbourhood $V$ of the origin in $X$ s.t. $x \notin V$. This implies that there exists at least $B \in \mathcal{B}_{d}$ s.t. $x \notin B,{ }^{3}$ i.e. there exist $p \in \mathcal{P}$ and $0<r \leq 1$ s.t. $x \notin r U_{p}$. Hence, $p(x)>r>0$ and so $p(x) \neq 0$, i.e. $\mathcal{P}$ is separating.

Conversely, if $\tau$ is induced by a separating family of seminorms $\mathcal{P}$, i.e. $\tau=\tau_{\mathcal{P}}$, then Lemma 2.3.2 ensures that $X$ is Hausdorff.
b) A Hausdorff lmc algebra $(X, \tau)$ is in particular a Hausdorff lc tvs, so by a) there exists a separating family $\mathcal{P}$ of seminorms s.t. $\tau=\tau_{\mathcal{P}}$. Since $(X, \tau)$ is an lmc algebra, Theorem 2.2.11 ensures that there exists $\mathcal{Q}$ family of submultiplicative seminorms s.t. $\tau=\tau_{\mathcal{Q}}$. Hence, we have got $\tau_{\mathcal{P}}=\tau_{\mathcal{Q}}$ which gives in turn that for any $p \in \mathcal{P}$ there exist $q_{1}, q_{2} \in \mathcal{Q}$ and $C_{1}, C_{2}>0$ s.t. $C_{1} q_{1}(x) \leq p(x) \leq C_{2} q_{2}(x), \forall x \in X$. This gives in turn that if $q(x)=0$ for all $q \in \mathcal{Q}$ then we have $p(x)=0$ for all $p \in \mathcal{P}$ which implies $x=0$ because $\mathcal{P}$ is separating. This shows that $\mathcal{Q}$ is a separating family of submultiplicative seminorms. Conversely, if $\tau$ is induced by a separating family of submultiplicative seminorms $\mathcal{P}$, i.e. $\tau=\tau_{\mathcal{P}}$, then Lemma 2.3.2 ensures that $X$ is Hausdorff and Theorem 2.2.11 that it is an lmc algebra.

## Examples 2.3.4.

1. Every normed algebra is a Hausdorff lmc algebra, since every submultiplicative norm is a submultiplicative seminorm satisfying the separation property. Therefore, every Banach algebra is a complete Hausdorff lmc algebra.
2. Every family of submultiplicative seminorms on a vector space containing a submultiplicative norm induces a Hausdorff llmc topology.
3. Given an open subset $\Omega$ of $\mathbb{R}^{d}$ with the euclidean topology, the space $\mathcal{C}(\Omega)$ of real valued continuous functions on $\Omega$ with the so-called topology of uniform convergence on compact sets is a lmc algebra. This topology is defined by the family $\mathcal{P}$ of all the submultiplicative seminorms on $\mathcal{C}(\Omega)$ given by

$$
p_{K}(f):=\max _{x \in K}|f(x)|, \forall K \subset \Omega \text { compact. }
$$

[^6]Moreover, $\left(\mathcal{C}(\Omega), \tau_{\mathcal{P}}\right)$ is Hausdorff, because the family $\mathcal{P}$ is clearly separating. In fact, if $p_{K}(f)=0, \forall K$ compact subsets of $\Omega$ then in particular $p_{\{x\}}(f)=|f(x)|=0, \forall x \in \Omega$, which implies $f \equiv 0$ on $\Omega$.

### 2.4 The finest Imc topology

In the previous sections we have seen how to generate topologies on an algebra which makes it into an lmc algebra. Among all of them, there is the finest one (i.e. the one having the largest number of open sets).

Proposition 2.4.1. The finest lmc topology on an algebra $X$ is the topology induced by the family of all submultiplicative seminorms on $X$.

Proof.
Let us denote by $\mathcal{S}$ the family of all submultiplicative seminorms on the vector space $X$. By Theorem 2.2.11, we know that the topology $\tau_{\mathcal{S}}$ induced by $\mathcal{S}$ makes $X$ into an lmc algebra. We claim that $\tau_{\mathcal{S}}$ is the finest lmc topology. In fact, if there was a finer $\operatorname{lmc}$ topology $\tau$ (i.e. $\tau_{\mathcal{S}} \subseteq \tau$ with $(X, \tau) \mathrm{lmc}$ algebra) then Theorem 2.2 .11 would give that $\tau$ is also induced by a family $\mathcal{P}$ of submultiplicative seminorms. But then $\mathcal{P} \subseteq \mathcal{S}$ and so $\tau=\tau_{\mathcal{P}} \subseteq \tau_{\mathcal{S}}$ by definition of induced topology. Hence, $\tau=\tau_{\mathcal{S}}$.

An alternative way of describing the finest lmc topology on an algebra without using the seminorms is the following:

Proposition 2.4.2. The collection of all absorbing absolutely m-convex sets of an algebra $X$ is a basis of neighbourhoods of the origin for the finest lmc topology on $X$.

Proof.
Let $\tau_{\text {max }}$ be the finest lmc topology on $X$ and $\mathcal{M}$ the collection of all absorbing absolutely m-convex sets of $X$. Since $\mathcal{M}$ fulfills all the properties required in Corollary 2.1.12, there exists a unique topology $\tau$ which makes $X$ into an lmc algebra having as basis of neighbourhoods of the origin $\mathcal{M}$. Hence, by definition of finest lmc topology, $\tau \subseteq \tau_{\max }$. On the other hand, $\left(X, \tau_{\max }\right)$ is itself an lmc algebra and so Theorem 2.2.11 ensures that has a basis $\mathcal{B}_{\text {max }}$ of neighbourhoods of the origin consisting of absorbing absolutely m-convex subsets of $X$. Then clearly $\mathcal{B}_{\text {max }}$ is contained in $\mathcal{M}$ and, hence, $\tau_{\text {max }} \subseteq \tau$.

This result can be proved also using Proposition 2.4.1 and the correspondence between Minkowski functionals and absorbing absolutely convex subsets of $X$ introduced in the Section 2.2.

Corollary 2.4.3. Every $\mathbb{K}$-algebra endowed with the finest lme topology is an m-barrelled algebra.

Proof.
Let $U$ be an m-barrel of $\left(X, \tau_{\max }\right)$. Then $U$ is closed absolutely m-convex and so it is a neighbourhood of the origin by the previous proposition.

Using basically the same proofs, we could show the analogous results for the finest lc topology, namely:

The finest lc topology on a $\mathbb{K}$-vector space $X$ is the topology induced by the family of all seminorms on $X$ or equivalently the topology having the collection of all absorbing absolutely convex sets of $X$ as a basis of neighbourhoods of the origin. Hence, every vector space endowed with the finest lc topology is a barrelled space.

Recall that
Definition 2.4.4. A closed absorbing absolutely convex subset of a TVS is called a barrel. A TVS in which every barrel is a neighbourhood of the origin is called barrelled space.

It is also important to remark that while the finest lc topology on a $\mathbb{K}$-vector space (and in particular on a $\mathbb{K}$-algebra) is always Hausdorff, the finest lmc topology on a $\mathbb{K}$-algebra does not have necessarily this property.

Proposition 2.4.5. Any $\mathbb{K}$-vector space endowed with the finest lc topology is a Hausdorff TVS.
Proof.
Let $X$ be any non-empty $\mathbb{K}$-vector space and $\mathcal{S}$ the family of all seminorms on $X$. By Proposition 2.3.3-a), it is enough to show that $\mathcal{S}$ is separating. We will do that, by proving that there always exists a non-zero norm on $X$. In fact, let $\mathcal{B}=\left(b_{i}\right)_{i \in I}$ be an algebraic basis of $X$ then for any $x \in X$ there exist a finite subset $J$ of $I$ and $\lambda_{j} \in \mathbb{K}$ for all $j \in J$ s.t. $x=\sum_{j \in J} \lambda_{j} b_{j}$ and so we can define $\|x\|:=\max _{j \in J}\left|\lambda_{j}\right|$. Then it is easy to check that $\|\cdot\|$ is a norm on $X$ and so $\|\cdot\| \in \mathcal{S}$.

Note that if $X$ is a $\mathbb{K}$-algebra, then the previous proof does not guaranteed the existence of a non-zero norm on $X$ because, depending on the multiplication in $X$, the norm $\|\cdot\|$ might be or not submultiplicative. In fact, there exist algebras on which no submultiplicative norm can be defined. For instance, if the algebra $\mathcal{C}(Y)$ of all complex valued continuous functions on a topological space $Y$ contains an unbounded function then it does not admit a
submultiplicative norm (see Sheet 4). Actually, there exist algebras on which no non-zero submultiplicative seminorms can be defined, e.g. the algebra of all linear operator on an infinite dimensional complex vector space (see [20, Theorem 3] ). The finest lmc topology on such algebras is the trivial topology which is obviously not Hausdorff.

We conclude this section with a nice further property of the finest lmc topology involving characters of an algebra.

Definition 2.4.6. Let $A$ be a $\mathbb{K}$-algebra. $A$ character of $A$ is a non-zero homomorpism of $A$ into $\mathbb{K}$. The set of all characters is denoted by $\mathcal{X}(A)$.

Proposition 2.4.7. Every character on a $\mathbb{K}$-algebra $A$ is continuous w.r.t. the finest lmc topology on $A$.

Proof. Let $\alpha: A \rightarrow \mathbb{K}$ be a character on $A$. For any $\varepsilon>0$, we denote by $B_{\varepsilon}(0)$ the open ball in $\mathbb{K}$ of radius $\varepsilon$ and center $0 \in \mathbb{K}$, i.e. $B_{\varepsilon}(0):=\{k \in \mathbb{K}:|k|<\varepsilon\}$. Set $p(a):=|\alpha(a)|$ for all $a \in A$. Then $p$ is a submultiplicative seminorm on $A$ since for any $a, b \in A$ and $\lambda \in \mathbb{K} \backslash\{0\}$ we have that:

- $p(a+b)=|\alpha(a+b)|=|\alpha(a)+\alpha(b)| \leq|\alpha(a)|+|\alpha(b)|=p(a)+p(b)$
- $p(\lambda a)=|\alpha(\lambda a)|=|\lambda \alpha(a)|=|\lambda||\alpha(a)|=|\lambda| p(a)$
- $p(a b)=|\alpha(a b)|=|\alpha(a) \alpha(b)|=|\alpha(a)||\alpha(b)|=p(a) p(b)$.

Then $\alpha^{-1}\left(B_{\varepsilon}(0)\right)=\{a \in A:|\alpha(a)|<\varepsilon\}=\varepsilon \dot{U}_{p}$, which is an absorbing absolutely m-convex subset of $X$ and so, by Proposition 2.4.2, it is a neighbourhood of the origin in the finest lmc topology on $X$. Hence $\alpha$ is continuous at the origin and so continuous everywhere in $A$.

With a proof similar to the previous one, we can deduce that
Proposition 2.4.8. Every linear functional on $a \mathbb{K}$-vector space $X$ is continuous w.r.t. the finest lc topology on $X$.

### 2.5 Topological algebras admitting Imc topologies

In this section we will look for sufficient conditions on a TA to be an lmc algebra. More precisely, we would like to find out under which conditions a locally convex algebra (i.e. a TA which is a locally convex TVS) is in fact an lmc algebra. The main result in this direction was proved by Michael in 1952 (see [21, Proposition 4.3]) and it is actually a generalization of a well-known theorem by Gel'fand within the theory of Banach algebras (see [9]).
Theorem 2.5.1 (Michael's Theorem). Let $A$ be a lc algebra. If
a) $A$ is m-barrelled, and
b) there exists a basis $\mathcal{M}$ of neighbourhoods of the origin in $A$ such that

$$
\begin{equation*}
\forall a \in A, \forall U \in \mathcal{M}, \exists \lambda>0: a U \subseteq \lambda U, \tag{2.10}
\end{equation*}
$$

then $A$ is an lmc algebra.
Proof.
Let us first give the main proof structure and then proceed to show the more technical details.

Claim 1 W.l.o.g. we can assume that $\mathcal{M}$ consists of barrels.
Consider the unitization $A_{1}$ of $A$ equipped with the product topology (see Definition 1.1.3 and Section 1.3). Denote by $\cdot$ the multiplication in $A_{1}$ and by $\bar{B}_{1}(0):=\{k \in \mathbb{K}:|k| \leq 1\}$. Then the family $\left\{\bar{B}_{1}(0) \times U: U \in \mathcal{M}\right\}$ is a basis of neighbourhoods of the origin $(0, o)$ in $A_{1}$ and the following holds.

Claim 2 For any $U \in \mathcal{M}, V(U):=\left\{x \in A:(0, x) \cdot\left(\bar{B}_{1}(0) \times U\right) \subseteq\right.$ $\left.\left(\bar{B}_{1}(0) \times U\right)\right\}$ is an m-barrel subset of $A$.
Then the assumption a) ensures that each $V(U)$ is a neighbourhood of the origin in $A$. Moreover, for any $U \in \mathcal{M},(1, o) \in\left(\bar{B}_{1}(0) \times U\right)$ and so

$$
\forall x \in V(U),(0, x)=(0, x) \cdot(1, o) \in\left(\bar{B}_{1}(0) \times U\right)
$$

which provides that $V(U) \subseteq U$. Hence, $\{V(U): U \in \mathcal{M}\}$ is a basis of neighbourhoods of the origin in $A$ consisting of m-barrels and so, by Theorem 2.1.11, $A$ is an lmc algebra.

Proof. Claim 1
If $\mathcal{M}$ is not already consisting of all barrels, then we can always replace it by $\widetilde{M}:=\left\{\overline{\operatorname{conv}_{\mathrm{b}}(U)}: U \in \mathcal{M}\right\}$, because $\widetilde{M}$ is a basis of neighbourhoods of the origin in $A$ fulfilling (2.10).

In fact, since $A$ is a lc TVS, then there exists a basis $\mathcal{N}$ of neighbourhoods of the origin in $A$ consisting of barrels. Then, since also $\mathcal{M}$ is a basis of neighbourhoods of the origin in $A$, we have that:

$$
\forall V \in \mathcal{N}, \exists U \in \mathcal{M}: U \subseteq V
$$

As $\overline{\operatorname{conv}_{\mathrm{b}}(U)}$ is the smallest closed convex balanced subset of $A$ containing $U$ and $V$ has all such properties, we get that $\overline{\operatorname{conv}_{\mathrm{b}}(U)} \subseteq V$. Hence, $\widetilde{M}$ is a basis of neighbourhoods of the origin in $A$.

Moreover, let $a \in A$ and $U \in \mathcal{M}$. By assumption b), we know that there exists $\lambda>0$ such that $a U \subseteq \lambda U$. Now recalling that $\operatorname{conv}_{\mathrm{b}}(U)=\operatorname{conv}(\operatorname{bal}(U))$, we can write any $x \in \operatorname{conv}_{\mathrm{b}}(U)$ as $x=\sum_{i=1}^{n} \mu_{i} \delta_{i} u_{i}$ for some $n \in \mathbb{N}, u_{i} \in U$,
$\mu_{i} \in[0,1]$ with $\sum_{i=1}^{n} \mu_{i}=1$, and $\delta_{i} \in \mathbb{K}$ with $\left|\delta_{i}\right| \leq 1$. Then for each $i \in\{1, \ldots, n\}$ there exist $\tilde{u}_{i} \in U$ such that:

$$
a x=\sum_{i=1}^{n} \mu_{i} \delta_{i} a u_{i}=\sum_{i=1}^{n} \mu_{i} \delta_{i} \lambda \tilde{u}_{i}=\lambda \sum_{i=1}^{n} \mu_{i}\left(\delta_{i} \tilde{u}_{i}\right)
$$

and so $a x \in \lambda \cdot \operatorname{conv}_{\mathrm{b}}(U)$. Hence, $a \cdot \operatorname{conv}_{\mathrm{b}}(U) \subseteq \lambda \cdot \operatorname{conv}_{\mathrm{b}}(U)$. This together with the separate continuity of the multiplication in $A$ and the fact that the scalar multiplication is a homeomorphism imply that

$$
a \cdot \overline{\operatorname{conv}_{\mathrm{b}}(U)} \subseteq \overline{a \cdot \operatorname{conv}_{\mathrm{b}}(U)} \subseteq \overline{\lambda \cdot \operatorname{conv}_{\mathrm{b}}(U)}=\lambda \cdot \overline{\operatorname{conv}_{\mathrm{b}}(U)}
$$

This shows that $\widetilde{M}$ fulfills (2.10).
Proof. Claim 2
Let $U \in \mathcal{M}$ and $V(U):=\left\{x \in A:(0, x) \cdot\left(\bar{B}_{1}(0) \times U\right) \subseteq\left(\bar{B}_{1}(0) \times U\right)\right\}$. Then:

- $V(U)$ is multiplicative.

For any $a, b \in V(U)$ we have
$(0, a b) \cdot\left(\bar{B}_{1}(0) \times U\right)=(0, a) \cdot(0, b) \cdot\left(\bar{B}_{1}(0) \times U\right) \subseteq(0, a) \cdot\left(\bar{B}_{1}(0) \times U\right) \subseteq\left(\bar{B}_{1}(0) \times U\right)$, i.e. $a b \in V(U)$.

- $V(U)$ is closed.

Let us show that $A \backslash V(U)$ is open, i.e. that for any $x \in A \backslash V(U)$ there exists $N \in \mathcal{M}$ such that $x+N \subseteq A \backslash V(U)$. If $x \in A \backslash V(U)$, then there exist $t \in \bar{B}_{1}(0)$ and $u \in U$ such that $(0, x) \cdot(t, u) \notin\left(\bar{B}_{1}(0) \times U\right)$, i.e. $t x+u x \in A \backslash U$. As $U$ is closed, $A \backslash U$ is open and so there exists $W \in \mathcal{M}$ s.t.

$$
\begin{equation*}
t x+u x+W \subseteq A \backslash U \tag{2.11}
\end{equation*}
$$

Take $N \in \mathcal{M}$ s.t. $u N \subseteq \frac{1}{2} W$ and $N \subseteq \frac{1}{2} W$ (this exists because left multiplication is continuous and $\mathcal{M}$ is basis of neighbourhoods of the origin). Then $x+N \subseteq A \backslash U$, because otherwise there would exists $n \in N$ such that $x+n \in$ $V(U)$ and so $(0, n+x) \cdot(t, u) \in\left(\bar{B}_{1}(0) \times U\right)$ that is $n t+n u+x t+x u \in U$, which in turns implies $x t+x u \in U-t N-u N \subseteq U-N-\frac{1}{2} W \subseteq U-\frac{1}{2} W-\frac{1}{2} W \subseteq U-W$, i.e. $x t+x u+W \subseteq U$ which contradicts (2.11).

- $V(U)$ is absorbing.

Let $a \in A$. Then (2.10) ensures that there exists $\lambda>0$ s.t. $a U \subseteq \lambda U$. Also, since $U$ is absorbing, there exists $\mu>0$ such that $a \in \mu U$. Take $\rho:=\frac{1}{\lambda+\mu}$. Then for all $k \in \mathbb{K}$ with $|k| \leq \rho$ and for any $(t, u) \in\left(\bar{B}_{1}(0) \times U\right)$ we get that $k t a+k a u \in k t \mu U+k \lambda U \subseteq k \mu U+k \lambda U=k(\mu+\lambda) U \subseteq U$ where in both inclusions we have used that $U$ is balanced together with $|t| \leq 1$ in the first and $|k(\mu+\lambda)| \leq \rho|\mu+\lambda|=1$ in the second. Hence, we have obtained that $(0, k a) \cdot(t, u)=(0, k t a+k a u) \in\left(\bar{B}_{1}(0) \times U\right)$ which gives that $k a \in V(U)$.

- $V(U)$ is balanced.

Let $a \in V(U)$ and $k \in \mathbb{K}$ with $|k| \leq 1$. Then
$(0, k a) \cdot\left(\bar{B}_{1}(0) \times U\right)=k(0, a) \cdot\left(\bar{B}_{1}(0) \times U\right) \subseteq\left(k \bar{B}_{1}(0) \times k U\right) \subseteq\left(\bar{B}_{1}(0) \times U\right)$
where in the last inclusion we used that both $\bar{B}_{1}(0)$ and $U$ are balanced.

- $V(U)$ is convex.

Let $a, b \in V(U)$ and $\mu \in[0,1]$. Then for any $(t, u) \in\left(\bar{B}_{1}(0) \times U\right)$ we know that $(0, a) \cdot(t, u) \in\left(\bar{B}_{1}(0) \times U\right)$ and $(0, b) \cdot(t, u) \in\left(\bar{B}_{1}(0) \times U\right)$, which give in turn that $a t+a u \in U$ and $b t+b u \in U$. Therefore, the convexity of $U$ implies that $\mu(a t+a u)+(1-\mu)(b t+b u) \in U$ and so we obtain $(0, \mu a+(1-\mu) b) \cdot(t, u)=(0, \mu(a t+a u)+(1-\mu)(b t+b u)) \in\left(\bar{B}_{1}(0) \times U\right)$, i.e. $\mu a+(1-\mu) b \in V(U)$.

Let us present now a stronger version of Michael's theorem, which has however the advantage of providing a less technical and so more manageable sufficient condition for a topology to be lmc. This more convenient condition actually identifies an entire class of TA: the so-called $A$-convex algebras introduced by Cochran, Keown and Williams in the early seventies [5].

Definition 2.5.2. $A \mathbb{K}$-algebra $X$ is called $A$-convex if it is endowed with the topology induced by an absorbing family of seminorms on $X$.

Definition 2.5.3. A seminorm $p$ on $a \mathbb{K}$-algebra $X$ is called:

- left absorbing if $\forall a \in X, \exists \lambda>0$ s.t. $p(a x) \leq \lambda p(x), \forall x \in X$.
- right absorbing if $\forall a \in X, \exists \lambda>0$ s.t. $p(x a) \leq \lambda p(x), \forall x \in X$.
- absorbing if it is both left and right absorbing.

Proposition 2.5.4. Every $A$-convex algebra is a lc algebra.

Proof.
Let $(X, \tau)$ be an A-convex algebra. Then by definition $\tau=\tau_{\mathcal{P}}$ where $\mathcal{P}$ is a family of absorbing seminorm. Hence, by Theorem $2.2 .12,(X, \tau)$ is an lc TVS. It remains to show that it is a TA. Let $a \in X$ and consider the left multiplication $\ell_{a}: X \rightarrow X, x \mapsto a x$. Since any $p \in \mathcal{P}$ is left absorbing, we have that there exists $\lambda>0$ such that $p(a x) \leq \lambda p(x)$ for all $x \in X$ and so that $\frac{1}{\lambda} U_{p} \subseteq \ell_{a}^{-1}\left(U_{p}\right)$. Hence, $\ell_{a}$ is $\tau$-continuous. Similarly, one can prove the continuity of the right multiplication. We can then conclude that $(X, \tau)$ is an lc algebra.

Note that not every lc algebra is A-convex (see Sheet 4) but every lmc algebra is A-convex as the submultiplicativity of the generating seminorms implies that they are absorbing. Let us focus now on the inverse question of establishing when an A-convex algebra is lmc.
Theorem 2.5.5. Every m-barrelled $A$-convex algebra is an lmc algebra.

## Proof.

Let $(X, \tau)$ be an m-barrelled A-convex algebra. By the previous proposition, we have that $(X, \tau)$ is an lc algebra. Denote by $\mathcal{P}:=\left\{p_{i}: i \in I\right\}$ a family of absorbing seminorm generating $\tau$. Then, by Proposition 2.2.17, $\mathcal{Q}:=\left\{\max _{i \in B} p_{i}: \emptyset \neq B \subseteq I\right.$ with $B$ finite $\}$ is a directed family of seminorms such that $\tau=\tau_{\mathcal{Q}}$. Also, each $q \in \mathcal{Q}$ is absorbing. Indeed, $q=\max _{i \in B} p_{i}$ for some $\emptyset \neq B \subseteq I$ with $B$ finite and so for any $i \in B$ and any $a \in X$ we have that there exists $\lambda_{i}>0$ such that $p_{i}(a x) \leq \lambda_{i} p(x)$ for all $x \in X$. Hence, for any $a \in X$ we get

$$
q(a x)=\max _{i \in B} p_{i}(a x) \leq \max _{i \in B} \lambda_{i} p_{i}(x) \leq \lambda \max _{i \in B} p_{i}(x)=q(x), \forall x \in X,
$$

where $\lambda:=\max _{i \in B} \lambda_{i}$. Then $\mathcal{M}:=\left\{\varepsilon U_{q}: q \in \mathcal{Q}, 0<\varepsilon \leq 1\right\}$ is a basis of neighbourhoods of the origin for $(X, \tau)$ and for each $a \in A, q \in \mathcal{Q}$ and $0<\varepsilon \leq 1$ we have that if $x \in a \varepsilon U_{q}$ then $x=a \varepsilon y$ for some $y \in U_{q}$ and so $q(x)=q(a \varepsilon y) \leq \varepsilon q(a y) \leq \lambda \varepsilon q(y) \leq \lambda \varepsilon$ i.e. $x \in \lambda \varepsilon U_{q}$. Hence, we proved that $\forall a \in A, \forall q \in \mathcal{Q}, \forall 0<\varepsilon \leq 1, a \varepsilon U_{q} \subseteq \lambda \varepsilon U_{q}$ which means that $\mathcal{M}$ fulfills condition b) in Theorem 2.5.1. Then ( $X, \tau$ ) satisfies all the assumptions of Theorem 2.5.1 which guarantees that it is an lmc algebra.

To conclude this section let us just restate the result by Gel'fand mentioned in the beginning in one of the many formulation which reveals the analogy with Michael's theorem.

Theorem 2.5.6. If $X$ is a $\mathbb{K}$-algebra endowed with a norm which makes it into a Banach space and a TA, then there exists an equivalent norm on $X$ which makes it into a Banach algebra.

## Chapter 3

## Further special classes of topological algebras

### 3.1 Metrizable and Fréchet algebras

Definition 3.1.1. A metrizable algebra $X$ is a $T A$ which is in particular a metrizable TVS, i.e. a TVS whose topology is induced by a metric.

We recall that a metric $d$ on a set $X$ is a mapping $d: X \times X \rightarrow \mathbb{R}^{+}$with the following properties:

1. $d(x, y)=0$ if and only if $x=y$ (identity of indiscernibles);
2. $d(x, y)=d(y, x)$ for all $x, y \in X$ (symmetry);
3. $d(x, z) \leq d(x, y)+d(y, z)$ for all $x, y, z \in X$ (triangular inequality).

Saying that the topology of a TVS $(X, \tau)$ is induced by a metric $d$ means that for any $x \in X$ the sets of all open (or equivalently closed) balls:

$$
B_{r}(x):=\{y \in X: d(x, y)<r\}, \quad \forall r>0
$$

forms a basis of neighbourhoods of $x$ for $\tau$.
There exists a completely general characterization of metrizable TVS.
Theorem 3.1.2. A TVS $X$ is metrizable if and only if $X$ is Hausdorff and has a countable basis of neighbourhoods of the origin.

Note that one direction is quite straightforward. Indeed, suppose that $X$ is a metrizable TVS and that $d$ is a metric defining the topology of $X$, then the collection of all $B_{\frac{1}{n}}(o)$ with $n \in \mathbb{N}$ is a countable basis of neighbourhoods of the origin $o$ in $X$. Moreover, the intersection of all these balls is just the singleton $\{o\}$, which proves that the TVS $X$ is also Hausdorff (see Proposition 1.3.2) The other direction requires more work and we are not going to prove it in full generality as it would go beyond the aim of this course (see e.g. [23, Chapter I, Section 6.1] or [16, proof of Theorem 1.1] for a proof for locally convex TVS).

However, we are going to use Theorem 3.1.2 to give a characterization of all metrizable lmc algebras.

Theorem 3.1.3. Let $A$ be $a \mathbb{K}$-algebra. Then the following are equivalent:
a) $A$ is a metrizable lmc algebra
b) A is a TVS having a decreasing sequence of m-barrels with trivial intersection as a basis of neighbourhoods of the origin.
c) A is a TVS whose topology is generated by an increasing sequence of submultiplicative seminorms which form a separating family.

The obvious analogous statement is true for metrizable lc algebras.

## Proof.

$\mathrm{a}) \Rightarrow \mathrm{b})$ Suppose that $(A, \tau)$ is a metrizable lmc algebra. Then in particular $(A, \tau)$ is a metrizable TVS and so by Theorem 3.1.2 it is Hausdorff and has a countable basis $\left\{U_{n}: n \in \mathbb{N}\right\}$ of neighbourhoods of the origin. As $(A, \tau)$ is an lmc algebra, by Theorem 2.1.11, we can assume w.l.o.g. that each $U_{n}$ is an m-barrel. Now for each $n \in \mathbb{N}$ set $V_{n}:=U_{1} \cap \cdots \cap U_{n}$. Then one can easily verify that each $V_{n}$ is still an m-barrel and clearly $V_{n+1} \subseteq V_{n}$. Hence, the decreasing sequence $\left\{V_{n}: n \in \mathbb{N}\right\}$ is a basis of neighbourhoods of the origin in $(A, \tau)$ consisting of m-barrels. The Hausdorfness of $(A, \tau)$ implies, by Proposition 1.3.2, that $\bigcap_{n \in \mathbb{N}} V_{n}=\{o\}$.
$\mathrm{b}) \Rightarrow \mathrm{c})$ Suppose that $(A, \tau)$ is a TVS and that $\left\{V_{n}: n \in \mathbb{N}\right\}$ is a basis of neighbourhoods of the origin such that, for any $n \in \mathbb{N}, V_{n}$ is an m-barrel, $V_{n+1} \subseteq V_{n}$ and $\bigcap_{n \in \mathbb{N}} V_{n}=\{o\}$. Then Theorem 2.1.11 guarantees that $(A, \tau)$ is an $\operatorname{lmc}$ algebra and the family $\mathcal{S}:=\left\{p_{V_{n}}: n \in \mathbb{N}\right\}$ is a family of submultiplicative seminorms generating $\tau$ (see proof of Theorem 2.2.12). Actually, $\mathcal{S}$ is an increasing sequence, since $V_{n+1} \subseteq V_{n}$ implies that $p_{V_{n}} \leq p_{V_{n+1}}$. Moreover, we
 i.e. $\mathcal{S}$ is separating (c.f. (2.9)).
c) $\Rightarrow \mathrm{a})$ Suppose that $(A, \tau)$ is a TVS and that $\mathcal{P}:=\left\{p_{n}: n \in \mathbb{N}\right\}$ is a separating increasing sequence of submultiplicative seminorm generating $\tau$. By Theorem 2.2.12 and Proposition 2.3.3, $(A, \tau)$ is a Hausdorff lmc algebra. W.l.o.g. we can assume that $\mathcal{P}$ is directed and so, by using Exercise 3 in Sheet 3, we have that $\left\{\frac{1}{n} U_{p_{n}}: n \in \mathbb{N}\right\}$ is a countable basis of neighbourhoods of the origin. Then Theorem 3.1.2 ensures that $(A, \tau)$ is also a metrizable TVS and, hence, a metrizable lmc algebra.

A special class of metrizable algebras are the so-called Fréchet algebras.

Definition 3.1.4. A Fréchet algebra is a $T A$ which is in particular a Fréchet TVS, i.e. a complete metrizable lc TVS.

It is clear that every Fréchet algebra is a Hausdorff complete lc algebra whose topology is induced by an increasing family of seminorms, but these are not necessarily submultiplicative. If this is the case, we speak of Fréchet lmc algebras.

Definition 3.1.5. A Fréchet lmc algebra is a complete metrizable lmc algebra.
As completeness is fundamental to understand the structure of a Fréchet algebra, let us recall here some of the most important properties of complete TVS (for a more detailed exposition about complete TVS see e.g. [15, Section 2.5] or [24, Part I, Section 5]).

## Definition 3.1.6.

A TVS X is said to be complete if every Cauchy filter on $X$ converges to $a$ point $x$ of $A$.

It is important to distinguish between completeness and sequentially completeness.

## Definition 3.1.7.

A TVS $X$ is said to be sequentially complete if any Cauchy sequence in $X$ converges to a point in $A$.

Clearly, a TA is complete (resp. sequentially complete) if it is in particular a complete (resp. sequentially complete) TVS. Remind that

Definition 3.1.8. A filter $\mathcal{F}$ on a TVS $(X, \tau)$ is said to be a Cauchy filter if

$$
\begin{equation*}
\forall U \in \mathcal{F}(o) \text { in } X, \exists M \in \mathcal{F}: M-M \subset U, \tag{3.1}
\end{equation*}
$$

where $\mathcal{F}(o)$ denotes the filter of neighbourhoods of the origin o in $(X, \tau)$.
Definition 3.1.9. A sequence $S:=\left\{x_{n}\right\}_{n \in \mathbb{N}}$ of points in a TVS $(X, \tau)$ is said to be a Cauchy sequence if

$$
\begin{equation*}
\forall U \in \mathcal{F}(o) \text { in } X, \exists N \in \mathbb{N}: x_{m}-x_{n} \in U, \forall m, n \geq N \text {, } \tag{3.2}
\end{equation*}
$$

where $\mathcal{F}(o)$ denotes the filter of neighbourhoods of the origin o in $(X, \tau)$.

## Proposition 3.1.10.

The filter associated with a Cauchy sequence in a TVS X is a Cauchy filter.

Proof.
Let $S$ be a Cauchy sequence. Then, recalling that the collection $\mathcal{B}:=\left\{S_{m}\right.$ : $m \in \mathbb{N}\}$ with $S_{m}:=\left\{x_{n} \in S: n \geq m\right\}$ is a basis of the filter $\mathcal{F}_{S}$ associated with $S$, we can easily rewrite (3.2) as

$$
\forall U \in \mathcal{F}(o) \text { in } X, \exists N \in \mathbb{N}: S_{N}-S_{N} \subset U
$$

This immediately gives that $\mathcal{F}_{S}$ fulfills (3.2) and so that it is a Cauchy filter.

It is then not hard to prove that

## Proposition 3.1.11.

If a TVS $X$ is completem then $A$ is sequentially complete.
Proof.
Let $S:=\left\{x_{n}\right\}_{n \in \mathbb{N}}$ be a Cauchy sequence of points in $A$. Then Proposition 3.1.10 guarantees that the filter $\mathcal{F}_{S}$ associated to $S$ is a Cauchy filter in $A$. By the completeness of $A$ we get that there exists $x \in A$ such that $\mathcal{F}_{S}$ converges to $x$. This is equivalent to say that the sequence $S$ is convergent to $x \in A$ (see [15, Proposition 1.1.29]). Hence, $A$ is sequentially complete.

The converse is false in general (see [15, Example 2.5.9]). However, the two notions coincide in metrizable TVS, and so we have that

Proposition 3.1.12. A metrizable lc algebra is a Fréchet algebra if and only if it is sequentially complete.

Another important property of Fréchet algebras is that they are Baire spaces, i.e. topological spaces in which the union of any countable family of closed sets, none of which has interior points, has no interior points itself (or, equivalently, the intersection of any countable family of everywhere dense open sets is an everywhere dense set). This is actually a consequence of the following more general result:

Proposition 3.1.13. A complete metrizable TVS $X$ is a Baire space.
Proof. (see [16, Proposition 1.1.8])
Example 3.1.14. An example of Baire space is $\mathbb{R}$ with the euclidean topology. Instead $\mathbb{Q}$ with the subset topology given by the euclidean topology on $\mathbb{R}$ is not a Baire space. Indeed, for any $q \in \mathbb{Q}$ the subset $\{q\}$ is closed and has empty
interior in $\mathbb{Q}$, but $\cup_{q \in \mathbb{Q}}\{q\}=\mathbb{Q}$ which has interior points in $\mathbb{Q}$ (actually its interior is the whole $\mathbb{Q})$.

Corollary 3.1.15. Every Fréchet TVS is barrelled. In particular, every Fréchet algebra is m-barrelled.

Proof.
Let $(X, \tau)$ be a Fréchet TVS and $V$ a barrelled subset of $X$. Then $V$ is absorbing and closed, so $X=\bigcup_{n \in \mathbb{N}} n V$ is a countable union of closed sets. Hence, as Proposition 3.1.13 ensures that $(X, \tau)$ is a Baire space, we have that there exists $k \in \mathbb{N}$ such that $(k V) \neq \emptyset$. This implies that there exists $x \in(k V)$, i.e. there exists a neighbourhood $N$ of the origin in $(X, \tau)$ such that $x+N \subseteq V$. As ( $X, \tau$ ) is in particular an lc TVS, we can assume that $N$ is absolutely convex. Then we get

$$
N=\frac{1}{2} N-\frac{1}{2} N=\frac{1}{2}(x+N)+\frac{1}{2}(-x-N) \subseteq \frac{1}{2} V+\frac{1}{2}(-V)=V,
$$

where in the last equality we used that $V$ is a barrel and so absolutely convex. Hence, we can conclude that $V$ is a neighbourhood of the origin and so $(X, \tau)$ is barrelled.

If $(X, \tau)$ is a Fréchet algebra, then it is in particular a Fréchet TVS and so the previous part of the proof guarantees that every m-barrelled subset of $X$ is a neighbourhood of the origin, i.e. $(X, \tau)$ is an m-barrelled algebra.

This result together with Theorem 2.5.1 (resp. Theorem 2.5.5) clearly provides that every Fréchet algebra having a basis of neighbourhoods of the origin which satifies (2.10) (resp. every A-convex Fréchet algebra) is lmc. Proposition 3.1.13 plays also a fundamental role in proving the following general property of complete metrizable algebras and so of Fréchet algebras.

Proposition 3.1.16. Every complete metrizable algebra is a $T A$ with continuous multiplication.

Proof.
Let $A$ be a complete metrizable algebra. The metrizability provides the existence of a countable basis $\mathcal{B}:=\left\{W_{n}: n \in \mathbb{N}\right\}$ of neighbourhoods of the origin. We aim to show that for any $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $W_{m} W_{m} \subseteq W_{n}$.

Fixed $n \in \mathbb{N}$, as we are in a TVS, there always exists a closed neighbourhood $V$ of the origin such that $V-V \subseteq W_{n}$. As for any $b \in A$ the right multiplication $r_{b}: A \rightarrow A, a \mapsto a b$ is continuous we have that
$r_{b}^{-1}(V):=\{a \in A: a b \in V\}$ is closed. For any $k \in \mathbb{N}$, set $U_{k}:=\bigcap_{b \in W_{k}} r_{b}^{-1}(V)$. Then each $U_{k}$ is closed and ${ }^{1} A=\bigcup_{k \in \mathbb{N}} U_{k}$.

Since $A$ is a Baire space by Proposition 3.1.13, we have that there exists $h \in \mathbb{N}$ such that $\dot{U}_{h} \neq \emptyset$. Therefore, there exists $x \in \stackrel{\circ}{U}_{h}$, i.e. there exists $j \in \mathbb{N}$ such that $x+W_{j} \subseteq U_{h}$. This in turn provides that

$$
U_{h}-U_{h} \supseteq x+W_{j}-x-W_{j}=W_{j}-W_{j} \supseteq W_{j} .
$$

Since $\mathcal{B}$ is a basis for the filter of neighbourhoods of the origin, we can find $m \in \mathbb{N}$ such that $W_{m} \subseteq W_{j} \cap W_{h}$ and therefore

$$
W_{m} W_{m} \subseteq W_{j} W_{h} \subseteq\left(U_{h}-U_{h}\right) W_{h}=U_{h} W_{h}-U_{h} W_{h} \subseteq V-V \subseteq W_{n}
$$

where in the last inclusion we have just used the definition of $U_{h}$. Hence, the multiplication in $A$ is jointly continuous.

## Example 3.1.17.

1) Let $\mathcal{C}^{\infty}([0,1])$ be the space of all real valued infinitely differentiable functions on $[0,1]$ equipped with pointwise operations. We endow the algebra $\mathcal{C}^{\infty}([0,1])$ with the topology $\tau_{\mathcal{P}}$ generated by $\mathcal{P}:=\left\{p_{n}: n \in \mathbb{N}_{0}\right\}$ with $p_{n}(f):=\sup _{x \in[0,1]}\left|\left(D^{(n)} f\right)(x)\right|$ for any $f \in \mathcal{C}^{\infty}([0,1])$ (here $D^{(n)} f$ denotes the $n$-th derivative of $f) . \mathcal{P}$ is a countable separating family of seminorms so that $\left(\mathcal{C}^{\infty}([0,1]), \tau_{\mathcal{P}}\right)$ is a metrizable lc algebra but the seminorms in $\mathcal{P}$ are not submultiplicative since if for example we take $f(t):=t$ then $p_{1}\left(f^{2}\right)=2>1=p_{1}(f) p_{1}(f)$. However, we are going to show that $\tau_{\mathcal{P}}$ can be in fact generated by a countable separating family of submultiplicative seminorms and so it is actually an lmc algebra. First, let us consider the family $\mathcal{R}:=\left\{r_{n}:=\max _{j=0, \ldots, n} p_{j}: n \in \mathbb{N}_{0}\right\}$. As each $p_{n} \leq r_{n}$, we have that $\tau_{\mathcal{P}}=\tau_{\mathcal{R}}$ and also for all $n \in \mathbb{N}_{0}, f, g \in \mathcal{C}^{\infty}([0,1])$ the following holds:

$$
\begin{aligned}
r_{n}(f g) & =\max _{j=0, \ldots, n} p_{j}(f g)=\max _{j=0, \ldots, n} \sup _{x \in[0,1]}\left|\left(D^{(j)} f g\right)(x)\right| \\
& \leq \max _{j=0, \ldots, n} \sum_{k=0}^{j}\binom{j}{k} \sup _{x \in[0,1]}\left|\left(D^{(j-k)} f\right)(x)\right| \sup _{x \in[0,1]}\left|\left(D^{(k)} g\right)(x)\right| \\
& \leq \max _{j=0, \ldots, n} \sum_{k=0}^{j}\binom{j}{k} p_{j-k}(f) p_{k}(g) \\
& \leq\left(\max _{j=0, \ldots, n} \sum_{k=0}^{j}\binom{j}{k}\right) r_{n}(f) r_{n}(g)=2^{n} r_{n}(f) r_{n}(g) .
\end{aligned}
$$

[^7]Therefore, setting $q_{n}(f):=2^{n} r_{n}(f)$ for any $n \in \mathbb{N}_{0}$ and $f \in \mathcal{C}^{\infty}([0,1])$, we have that the family $\mathcal{Q}:=\left\{q_{n}: n \in \mathbb{N}_{0}\right\}$ is a countable family of submultiplicative seminorms such that $\tau_{\mathcal{Q}}=\tau_{\mathcal{R}}=\tau_{\mathcal{P}}$. Hence, $\left(\mathcal{C}^{\infty}([0,1]), \tau_{\mathcal{P}}\right)$ is a metrizable lmc algebra. Actually, it is also complete and so a Fréchet lmc algebra. Indeed, as it is metrizable we said it is enough to show sequentially complete. Let $\left(f_{n}\right)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\left(\mathcal{C}^{\infty}([0,1]), \tau_{\mathcal{P}}\right)$, i.e. $\forall U \in \tau_{\mathcal{P}} \exists N \in \mathbb{N}$ s.t. $f_{n}-f_{m} \in U \forall n, m \geq N$. Thus,

$$
\begin{equation*}
\forall \varepsilon>0 \forall k \in \mathbb{N}_{0} \exists N \in \mathbb{N}: p_{k}\left(f_{n}-f_{m}\right) \leq \varepsilon \quad \forall n, m \geq N \tag{3.3}
\end{equation*}
$$

which yields

$$
\begin{equation*}
\forall \varepsilon>0 \exists N \in \mathbb{N}: \sup _{x \in[0,1]}\left|f_{n}(x)-f_{m}(x)\right| \leq \varepsilon \quad \forall n, m \geq N \tag{3.4}
\end{equation*}
$$

so that $\left(f_{n}(x)\right)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ for all $x \in[0,1]$. Since $\mathbb{R}$ is complete for all $x \in[0,1]$ there exists $y_{x} \in \mathbb{R}$ s.t. $f_{n}(x) \rightarrow y_{x}$ as $n \rightarrow \infty$. Set $f(x):=y_{x}$ for all $x \in[0,1]$, then $\left(f_{n}\right)_{n \in \mathbb{N}}$ converges pointwise to $f$. The latter combined with (3.4) yields that $\left(f_{n}\right)_{n \in \mathbb{N}}$ converges uniformly to $f$ which implies that $f \in \mathcal{C}([0,1])$ by [16, Lemma 1.2.2]. By (3.4) for $k=1$, we get $\left(\left(D^{(1)} f_{n}\right)(x)\right)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ for all $x \in[0,1]$ and reasoning as above $\left(D^{(1)} f_{n}\right)_{n \in \mathbb{N}}$ uniformly converges to some $g$ on $[0,1]$. By [16, Lemma 1.2.3], $g=D^{(1)} f$ and so $f \in \mathcal{C}^{1}([0,1])$. Proceeding by induction, we can show that $\left(D^{(j)} f_{n}\right)_{n \in \mathbb{N}}$ converges uniformly to $D^{(j)} f$ on $[0,1]$ and $f \in \mathcal{C}^{j}([0,1])$ for all $j \in \mathbb{N}_{0}$, i.e.

$$
\forall \varepsilon>0 \exists N \in \mathbb{N}: \sup _{x \in[0,1]}\left|D^{(j)} f_{n}(x)-D^{(j)} f(x)\right| \leq \varepsilon \quad \forall n, m \geq N .
$$

Therefore, $\left(f_{n}\right)_{n \in \mathbb{N}}$ converges to $f \in \mathcal{C}^{\infty}([0,1])$ in $\tau_{\mathcal{P}}$. Hence, completeness is proven.

Note that we could have first proved completeness and then used Corollary 3.1.15 to show that $\left(\mathcal{C}^{\infty}([0,1]), \tau_{\mathcal{P}}\right)$ is $m$-barrelled. Then, observing that $\tau_{\mathcal{P}}=\tau_{\mathcal{R}}$ and that the seminorms in $\mathcal{R}$ are all absorbing, we could have applied Theorem 2.5.5 and concluded that $\left(\mathcal{C}^{\infty}([0,1]), \tau_{\mathcal{P}}\right)$ is an lmc algebra.
2) Let $\mathbb{K}^{\mathbb{N}}=\left\{\underline{a}=\left(a_{n}\right)_{n \in \mathbb{N}}: a_{n} \in \mathbb{K}\right\}$ be the space of all $\mathbb{K}$-valued sequences endowed $\mathbb{K}^{\mathbb{N}}$ with the topology $\tau_{\mathcal{P}}$ generated by $\mathcal{P}:=\left\{p_{n}: n \in \mathbb{N}\right\}$ with $p_{n}(\underline{a}):=\max _{k \leq n}\left|a_{k}\right|$ for any $\underline{a} \in \mathbb{K}^{\mathbb{N}}(n \in \mathbb{N})$. Since $\mathcal{P}$ is an increasing family of submultiplicative seminorms and separating, $\left(\mathbb{K}^{\mathbb{N}}, \tau_{\mathcal{P}}\right)$ is a
metrizable lmc algebra by Theorem 3.1.3. Indeed, if $\underline{a}, \underline{b} \in \mathbb{K}^{\mathbb{N}}$, then

$$
p_{n}(\underline{a b})=\max _{k \leq n}\left|a_{k} b_{k}\right| \leq \max _{k \leq n}\left|a_{k}\right| \max _{k \leq n}\left|b_{k}\right|=p_{n}(\underline{a}) p_{n}(\underline{b})
$$

for all $n \in \mathbb{N}$. Further, if $p_{n}(\underline{a})=0$ for all $n \in \mathbb{N}$, then

$$
\max _{k \leq n}\left|a_{k}\right|=0, \forall n \in \mathbb{N} \Rightarrow\left|a_{k}\right|=0, \forall k \in \mathbb{N} \Rightarrow a \equiv 0
$$

Moreover, $\left(\mathbb{K}^{\mathbb{N}}, \tau_{\mathcal{P}}\right)$ is sequentially complete and so complete (prove it yourself). Hence, it is a Fréchet lmc algebra.
3) The Arens-algebra $L^{\omega}([0,1]):=\bigcap_{p \geq 1} L^{p}([0,1])$ endowed with the topology $\tau_{\mathcal{P}}$ generated by $\mathcal{P}:=\left\{\|\cdot\|_{p}: p \in \mathbb{N}\right\}$ is a Fréchet lc algebra which is not lmc. We have already showed that it is an lc algebra but not lmc. Metrizability comes from the fact that the family of seminorms is countable and increasing (Hölder-inequality). Proving completeness is more complicated which we will maybe see it later on.

### 3.2 Locally bounded algebras

The TAs we are going to study in this section were first introduced by W. Zelazko in the 1960's and provide non-trivial examples of TAs whose underlying space is not necessarily locally convex (so they are neither necessarily lc algebras nor lmc algebras) but they still share several nice properties of Banach and/or lmc algebras.

Definition 3.2.1. A TA is locally bounded (lb) if there exists a neighbourhood of the origin which is bounded. Equivalently, a locally bounded algebra is a TA which is in particular a locally bounded TVS (i.e. the space has a bounded neighbourhood of the origin).

## Recall that:

Definition 3.2.2. A subset $B$ of a TVS $X$ is bounded if for any neighbourhood $U$ of the origin in $X$ there exists $\lambda>0$ s.t. $B \subseteq \lambda U$ (i.e. $B$ can be swallowed by any neighbourhood of the origin).

This generalizes the concept of boundedness we are used to in the theory of normed and metric spaces, where a subset is bounded whenever we can find a ball large enough to contain it.

Example 3.2.3. The subset $Q:=[0,1]^{2}$ is bounded in $\left(\mathbb{R}^{2},\|\cdot\|\right)$ as for any $\varepsilon>0$ there exists $\lambda>0$ s.t. $Q \subseteq \lambda B_{\varepsilon}(o)$ namely, if $\varepsilon \geq \sqrt{2}$ take $\lambda=1$, otherwise take $\lambda=\frac{\sqrt{2}}{\varepsilon}$.

Proposition 3.2.4. Every Hausdorff locally bounded algebra is metrizable.
Proof.
Let $(A, \tau)$ be a Hausdorff locally bounded algebra and $\mathcal{F}(o)$ its filter of neighbourhoods of the origin. Then there exists $U \in \mathcal{F}(o)$ bounded. W.l.o.g. we can assume that $U$ is balanced. Indeed, if this is not the case, then we can replace it by some $V \in \mathcal{F}(o)$ balanced s.t. $V \subseteq U$. Then the boundedness of $U$ provides that $\forall N \in \mathcal{F}(o) \exists \lambda>0$ s.t. $U \subseteq \lambda N$ and so $V \subseteq \lambda N$, i.e. $V$ is bounded and balanced.

The collection $\left\{\frac{1}{n} U: n \in \mathbb{N}\right\}$ is a countable basis of neighbourhoods of the origin $o$. In fact, for any $N \in \mathcal{F}(o)$ there exists $\lambda>0$ s.t. $U \subseteq \lambda N$, i.e. $\frac{1}{\lambda} U \subseteq N$, and so $\frac{1}{n} U \subseteq \frac{1}{\lambda} U$ for all $n \geq \lambda$ as $U$ is balanced. Hence, we obtain that for any $N \in \mathcal{F}(o)$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} U \subseteq N$. Then we can apply Theorem 3.1.2 which gives that $(A, \tau)$ is a metrizable algebra.

The converse is not true in general as for example the countable product of 1-dimensional metrizable TVS is metrizable but not locally bounded.

Corollary 3.2.5. Every complete Hausdorff lb algebra has continuous multiplication.

Proof. Since local boundedness and Hausdorffness imply metrizability, Proposition 3.1.16 ensures that the multiplication is continuous.

The concept of lb TVS and so of lb TA can be characterized through extensions of the notion of norm, which will allow us to see how some results can be extended from Banach algebras to complete lb algebras.

Definition 3.2.6. Let $X$ be $a \mathbb{K}$-vector space. $A$ map $\|\cdot\|: X \rightarrow \mathbb{R}^{+}$is said to be a quasi-norm if

1. $\forall x \in X:\|x\|=0 \Longleftrightarrow x=0$,
2. $\forall x \in X \forall \lambda \in \mathbb{K}:\|\lambda x\|=|\lambda|\|x\|$,
3. $\exists k \geq 1:\|x+y\| \leq k(\|x\|+\|y\|), \forall x, y \in X$.

If $k=1$ this coincides with the notion of norm.

## Example 3.2.7.

Let $0<p<1$ and consider the space $L^{p}([0,1])$ with $\|\cdot\|_{p}: L^{p}([0,1]) \rightarrow \mathbb{R}^{+}$ defined by $\|f\|_{p}:=\left(\int_{0}^{1}|f(x)|^{p} d x\right)^{\frac{1}{p}}$ for all $f \in L^{p}([0,1])$. Then the Minkowski
inequality does not hold but we still have that $\|f+g\|_{p} \leq 2^{\frac{1-p}{p}}(\|f\|+\|g\|)$ for all $f, g \in L^{p}([0,1])$ and so that $\|\cdot\|_{p}$ is a quasi-norm.

Proposition 3.2.8. Let $(X, \tau)$ be a Hausdorff TVS. Then $(X, \tau)$ is lb if and only if $\tau$ is induced by a quasi-norm on $X$.

## Proof.

Assume that $(X, \tau)$ is lb and $\mathcal{F}(o)$ is its filter of neighbourhoods of the origin. Then there exists balanced and bounded $U \in \mathcal{F}(o)$ and $\mathcal{B}:=\{\alpha U: \alpha>0\}$ is a basis of neighbourhoods of the origin in $(X, \tau)$ because for any $N \in \mathcal{F}(o)$ there exists $\lambda>0$ s.t. $U \subseteq \lambda N \Rightarrow \mathcal{B} \ni \frac{1}{\lambda} U \subseteq N$. Consider the Minkowski functional $p_{U}(x):=\inf \{\alpha>0: x \in \alpha U\}$. In the proof of Lemma 2.2.7 we have already seen that if $U$ is absorbing and balanced, then $0 \leq p_{U}(x)<\infty$ and $p_{U}(\lambda x)=|\lambda| p_{U}(x)$ for all $x \in X$ and all $\lambda \in \mathbb{K}$. If $p_{U}(x)=0$, then $x \in \alpha U$ for all $\alpha>0$ and so $x \in \bigcap_{\alpha>0} \alpha U=\{o\}$, i.e. $x=o$. Since $X$ is a TVS, $\exists V \in \mathcal{F}(o)$ s.t. $V+V \subseteq U$ and also $\exists \alpha>0$ s.t. $\alpha U \subseteq V$ as $\mathcal{B}$ is a basis of neighbourhoods. Therefore, $\alpha U+\alpha U \subseteq V+V \subseteq U$ and taking $k \geq \max \left\{1, \frac{1}{\alpha}\right\}$, we obtain $U+U \subseteq \frac{1}{\alpha} U \subseteq k U$ as $U$ is balanced.

Let $x, y \in X$ and $\rho>p_{U}(x), \delta>p_{U}(y)$, then $x \in \rho U, y \in \delta U$ since $U$ is balanced, and so $\frac{x}{\rho}, \frac{y}{\delta} \in U$. Thus,

$$
\frac{x+y}{\rho+\delta}=\frac{\rho}{\rho+\delta} \frac{x}{\rho}+\frac{\delta}{\rho+\delta} \frac{y}{\delta} \in U+U \subseteq k U .
$$

and we obtain $x+y \in k(\rho+\delta) U$ which implies $p_{U}(x+y) \leq k(\rho+\delta)$. As $\rho>p_{U}(x)$ and $\delta>p_{U}(y)$ were chosen arbitrarily, we conclude $p_{U}(x+y) \leq$ $k\left(p_{U}(x)+p_{U}(y)\right)$. Hence, $p_{U}$ is a quasi-norm.

Let $B_{1}^{p_{U}}:=\left\{x \in X: p_{U}(x) \leq 1\right\}$. Then we have $U \subseteq B_{1}^{p_{U}} \subseteq(1+\varepsilon) U$ for all $\varepsilon>0$. Indeed, if $x \in U$, then $p_{U}(x) \leq 1$ and so $x \in B_{1}^{p_{U}}$. If $x \in B_{1}^{p_{U}}$, then $p_{U}(x) \leq 1$ and so $\forall \varepsilon>0 \exists \alpha$ with $\alpha \leq 1+\varepsilon$ s.t. $x \in \alpha U$. This gives that $x \in(1+\varepsilon) U$ as $U$ is balanced and so $\alpha U \subseteq(1+\varepsilon) U$. Since $\left\{\varepsilon B_{1}^{p_{U}}: \varepsilon>0\right\}$ is a basis of $\tau_{p_{U}}$, this implies $\tau=\tau_{p_{U}}$.
Conversely, assume that $\tau=\tau_{q}$ for a quasi-norm $q$ on $X$ and $\mathcal{F}^{q}(o)$ its filter of neighbourhoods of the origin. The collection $\mathcal{B}:=\left\{\varepsilon B_{1}^{q}: \varepsilon>0\right\}$ is a basis of neighbourhoods of the origin in $(X, \tau)$ (by Theorem 1.2.6). Let us just show that $\forall N \in \mathcal{F}^{q}(o) \exists V \in \mathcal{F}^{q}(o)$ s.t. $V+V \subseteq N$. Indeed, $\frac{1}{2 k} B_{1}^{q}+\frac{1}{2 k} B_{1}^{q} \subseteq B_{1}^{q}$ because if $x, y \in B_{1}^{q}$, then

$$
q\left(\frac{x+y}{2 k}\right)=\frac{1}{2 k} q(x+y) \leq \frac{k(q(x)+q(y))}{2 k} \leq \frac{2 k}{2 k}=1
$$

and so $\frac{x+y}{2 k} \in B_{1}^{q}$. Then for all $N \in \mathcal{F}^{q}(o)$ there is some $\varepsilon>0$ s.t. $\varepsilon B_{1}^{q} \subseteq N$ and so $\frac{\varepsilon}{2 k} B_{1}^{q}+\frac{\varepsilon}{2 k} B_{1}^{q} \subseteq \varepsilon B_{1}^{q} \subseteq N$. Since $\mathcal{B}$ is a basis for $\tau_{q}$, for any $N \in \mathcal{F}^{q}(o)$ there exists $\varepsilon>0$ s.t. $\varepsilon B_{1}^{q} \subseteq N$, which implies $B_{1}^{q} \subseteq \frac{1}{\varepsilon} N$. Therefore, $B_{1}^{q}$ is bounded and so $\tau_{q}$ is a lb TVS.

Using the previous proposition and equipping the space in Example 3.2.7 with pointwise multiplication, we get an example of lb but not lc algebra (see Sheet 5). An example of lc but not lb algebra is given by the following.

Example 3.2.9. Let $K$ be any compact subset of $(\mathbb{R},\|\cdot\|)$ and let us consider the algebra $\mathcal{C}^{\infty}(K)$ of all real valued infinitely differentiable functions on $K$ equipped with pointwise operations. Using the same technique as in Example 3.1.17, we can show that $\mathcal{C}^{\infty}(K)$ endowed with the topology $\tau_{K}$, generated by the family $\left\{r_{n}: n \in \mathbb{N}_{0}\right\}$ where $r_{n}(f):=\sup _{j=0, \ldots, n} \sup _{x \in K}\left|\left(D^{(j)} f\right)(x)\right|$ for any $f \in \mathcal{C}^{\infty}(K)$, is a Fréchet lmc algebra, i.e. an lc metrizable and complete algebra.

Denote now by $\mathcal{C}^{\infty}(\mathbb{R})$ the space of all real valued infinitely differentiable functions on $\mathbb{R}$ and by $\mathcal{C}_{c}^{\infty}(K)$ its subset consisting of all the functions $f \in$ $\mathcal{C}^{\infty}(\mathbb{R})$ whose support lies in $K$, i.e.

$$
\mathcal{C}_{c}^{\infty}(K):=\left\{f \in \mathcal{C}^{\infty}(\mathbb{R}): \operatorname{supp}(f) \subseteq K\right\},
$$

where $\operatorname{supp}(f)$ denotes the support of the function $f$, that is the closure in $(\mathbb{R},\|\cdot\|)$ of the subset $\{x \in \mathbb{R}: f(x) \neq 0\}$. Then it is easy to see that $\mathcal{C}_{c}^{\infty}(K)=\mathcal{C}^{\infty}(K)$ and this is a linear subspace of $\mathcal{C}^{\infty}(\mathbb{R})$. Indeed, for any $f, g \in \mathcal{C}_{c}^{\infty}(K)$ and any $\lambda \in \mathbb{R}$, we clearly have $f+g \in \mathcal{C}^{\infty}(\mathbb{R})$ and $\lambda f \in \mathcal{C}^{\infty}(\mathbb{R})$ but also $\operatorname{supp}(f+g) \subseteq \operatorname{supp}(f) \cup \operatorname{supp}(g) \subseteq K$ and $\operatorname{supp}(\lambda f)=\operatorname{supp}(f) \subseteq K$, which gives $f+g, \lambda f \in \mathcal{C}_{c}^{\infty}(K)$.

Let $\mathcal{C}_{c}^{\infty}(\mathbb{R})$ be the union of the subspaces $\mathcal{C}_{c}^{\infty}(K)$ as $K$ varies in all possible ways over the family of compact subsets of $\mathbb{R}$, i.e. $\mathcal{C}_{c}^{\infty}(\mathbb{R})$ consists of all the functions belonging to $\mathcal{C}^{\infty}(\mathbb{R})$ having compact support (this is what is actually encoded in the subscript " $c$ "). In particular, the space $\mathcal{C}_{c}^{\infty}(\mathbb{R})$ is usually called space of test functions and plays an essential role in the theory of distributions.

Consider a sequence $\left(K_{j}\right)_{j \in \mathbb{N}}$ of compact subsets of $\mathbb{R}$ s.t. $K_{j} \subseteq K_{j+1}, \forall j \in$ $\mathbb{N}$ and $\bigcup_{j=1}^{\infty} K_{j}=\mathbb{R}$. Then $\mathcal{C}_{c}^{\infty}(\mathbb{R})=\bigcup_{j=1}^{\infty} \mathcal{C}_{c}^{\infty}\left(K_{j}\right)$, as an arbitrary compact subset $K$ of $\mathbb{R}$ is contained in $K_{j}$ for some sufficiently large $j$, and we have that $\mathcal{C}_{c}^{\infty}\left(K_{j}\right) \subseteq \mathcal{C}_{c}^{\infty}\left(K_{j+1}\right)$. For any $j \in \mathbb{N}$, we endow $\mathcal{C}_{c}^{\infty}\left(K_{j}\right)$ with the topology $\tau_{j}:=\tau_{K_{j}}$ defined as above. Then $\left(\mathcal{C}_{c}^{\infty}\left(K_{j}\right), \tau_{K_{j}}\right)$ is a Fréchet lmc algebra and $\tau_{j+1} \upharpoonright_{\mathcal{C}_{c}^{\infty}\left(K_{j}\right)}=\tau_{j}$. Denote by $\tau_{\text {ind }}$ the finest lc topology on $\mathcal{C}_{c}^{\infty}(\mathbb{R})$ such that all the inclusions $\mathcal{C}_{c}^{\infty}\left(K_{j}\right) \subseteq \mathcal{C}_{c}^{\infty}(\mathbb{R})$ are continuous ( $\tau_{\text {ind }}$ does not depend on
the choice of the sequence of compact sets $K_{j}$ 's provided they fill $\left.\mathbb{R}\right)$. Then it is possible to show that $\left(\mathcal{C}_{c}^{\infty}(\mathbb{R}), \tau_{\text {ind }}\right)$ is a complete lc algebra but not Baire. Hence, Proposition 3.1.13 provides that $\left(\mathcal{C}_{c}^{\infty}(\mathbb{R}), \tau_{\text {ind }}\right)$ is not metrizable and so not lb by Proposition 3.2.4.

Definition 3.2.10. Let $X$ be a $\mathbb{K}$-vector space and $0<\alpha \leq 1$. A map $q: X \rightarrow \mathbb{R}^{+}$is an $\alpha$-norm if

1. $\forall x \in X: q(x)=0 \Longleftrightarrow x=0$,
2. $\forall x \in X \forall \lambda \in \mathbb{K}: q(\lambda x)=|\lambda|^{\alpha} q(x)$,
3. $\forall x, y \in X: q(x+y) \leq q(x)+q(y)$.

If $\alpha=1$, this coincides with the notion of norm.
Definition 3.2.11. A TVS $(X, \tau)$ is $\alpha$-normable if $\tau$ can be induced by an $\alpha$-norm for some $0<\alpha \leq 1$.

In order to understand how $\alpha$-norms relates to lb spaces we need to introduce a generalization of the concept of convexity.

Definition 3.2.12. Let $0<\alpha \leq 1$ and $X$ a $\mathbb{K}$-vector space.

- A subset $V$ of $X$ is $\alpha$-convex if for any $x, y \in V$ we have $t x+s y \in V$ for all $t, s>0$ such that $t^{\alpha}+s^{\alpha}=1$.
- A subset $V$ of $X$ is absolutely $\alpha$-convex if for any $x, y \in V$ we have $t x+s y \in V$ for all $t, s \in \mathbb{K}$ such that $|t|^{\alpha}+|s|^{\alpha} \leq 1$.
- For any $W \subseteq X, \Gamma_{\alpha}(W)$ denotes the smallest absolutely $\alpha$-convex subset of $X$ containing $W$, i.e.

$$
\Gamma_{\alpha}(W):=\left\{\sum_{i=1}^{n} \lambda_{i} w_{i}: n \in \mathbb{N}, w_{i} \in W, \lambda_{i} \in \mathbb{K} \text { s.t. } \sum_{i=1}^{n}\left|\lambda_{i}\right|^{\alpha} \leq 1\right\}
$$

Proposition 3.2.13. Let $(X, \tau)$ be a TVS and $0<\alpha \leq 1$. Then $(X, \tau)$ is $\alpha$-normable if and only if there exists an $\alpha$-convex, bounded neighbourhood of the origin.

Proof.
Suppose that $\tau$ is induced by an $\alpha$-norm $q$, i.e. the collection of all $B_{r}^{q}:=$ $\{x \in X: q(x) \leq r\}$ for all $r>0$ is a basis of neighbourhoods of the origin for $\tau$. Then for any $x, y \in B_{1}^{q}$ and any $t, s \in \mathbb{K}$ such that $|t|^{\alpha}+|s|^{\alpha} \leq 1$ we have that

$$
q(t x+s y) \leq|t|^{\alpha} q(x)+|s|^{\alpha} q(y) \leq|t|^{\alpha}+|s|^{\alpha} \leq 1
$$

i.e. $B_{1}^{q}$ is absolutely $\alpha$-convex. Also, the definition of $\alpha$-norm easily implies that

$$
\forall \rho>0, \forall x \in B_{1}^{q}, q\left(\rho^{\frac{1}{\alpha}}\right)=\rho q(x) \leq \rho
$$

and so that $B_{1}^{q} \subseteq \rho^{-\frac{1}{\alpha}} B_{\rho}^{q}$. Hence, $B_{1}^{q}$ is a bounded absolutely $\alpha$-convex neighbourhood of the origin.

Conversely, suppose that $V$ is an $\alpha$-convex bounded neighbourhood of the origin in $(X, \tau)$.

Claim 1: W.l.o.g. we can always assume that $V$ is absolutely $\alpha$-convex. Then, as we showed in the proof of Proposition 3.2.8, the Minkowski functional $p_{V}$ of $V$ is a quasi-norm generating $\tau$. Hence, defining $q(x):=p_{V}(x)^{\alpha}, \forall x \in X$ we can prove that

Claim 2: $q$ is an $\alpha$-norm.
Now $V \subseteq B_{1}^{q}$ because for any $x \in V$ we have that $q(x) \leq 1$. Also, for any $x \in B_{1}^{q}$ we have that $p_{V}(x) \leq 1$ and so for any $\varepsilon>0$ there exists $\rho>0$ s.t. $x \in \rho V$ and $\rho<p_{V}(x)+\varepsilon \leq 1+\varepsilon$. Then $x \in \rho V \subseteq(1+\varepsilon) V$ as $V$ is balanced. Then we have just showed that

$$
\forall \varepsilon>0, V \subseteq B_{1}^{q} \subseteq(1+\varepsilon) V,
$$

which in turn provides that $\tau$ is generated by $q$.
Let us now complete the proof by showing both claims.

## Proof. of Claim 1

By assumption $V$ is $\alpha$-convex bounded neighbourhood of the origin in $(X, \tau)$. If $V$ is also balanced, then there is nothing to prove as $V$ is already absolutely $\alpha$-convex. If $V$ is not balanced, then we can replace it with $\Gamma_{\alpha}(W)$ for some $W$ balanced neighbourhood of the origin in $X$ such that $W \subseteq V$ (the existence of such a $W$ is given by Theorem 1.2.6 as $(X, \tau)$ is a TVS). In fact, we can show that $\Gamma_{\alpha}(W) \subseteq V$, which provides in turn that $\Gamma_{\alpha}(W)$ is both bounded and absolutely $\alpha$-convex.

Let $z \in \Gamma_{\alpha}(W)$. Then $z=\sum_{i=1}^{n} \lambda_{i} w_{i}$ for some $n \in \mathbb{N}, w_{i} \in W$, and some $\lambda_{i} \in \mathbb{K}$ s.t. $\sum_{i=1}^{n}\left|\lambda_{i}\right|^{\alpha} \leq 1$. Take $\rho>0$ such that $\rho^{\alpha}=\sum_{i=1}^{n}\left|\lambda_{i}\right|^{\alpha}$ and for each $\in\{1, \ldots, n\}$ set $\varepsilon_{i}:=\frac{\bar{\lambda}_{i}}{\left|\lambda_{i}\right|} \rho$. Then

$$
\begin{equation*}
z=\sum_{i=1}^{n} \lambda_{i} w_{i}=\sum_{i=1}^{n} \frac{\left|\lambda_{i}\right|}{\rho} \varepsilon_{i} w_{i} . \tag{3.5}
\end{equation*}
$$

As $\rho^{\alpha} \leq 1$, we have $\rho \leq 1$ and so $\left|\varepsilon_{i}\right| \leq 1$. Then by the balancedness of $W$, for each $i \in\{1, \ldots, n\}$, we get that $\varepsilon_{i} w_{i} \in W \subset V$. Since $\sum_{i=1}^{n}\left(\frac{\left|\lambda_{i}\right|}{\rho}\right)^{\alpha}=1$ and $V$ is $\alpha$-convex, (3.5) provides that $z \in V$.

Proof. of Claim 2
Since $p_{V}$ is a quasi-norm on $X$, we have that $\forall x \in X, p_{V}(x) \geq 0$, which clearly
implies that $\forall x \in X, q(x)=p_{V}(x)^{\alpha} \geq 0$. Moreover, we have that $x=0$ if and only if $p_{V}(x)=0$, which is equivalent to $q(x)=0$. The positive homogeneity of $p_{V}$ gives in turn that

$$
\begin{equation*}
\forall x \in X, \forall \lambda \in \mathbb{K}, q(\lambda x)=p_{V}(\lambda x)^{\alpha}=|\lambda|^{\alpha} p_{V}(x)^{\alpha}=|\lambda|^{\alpha} q(x) . \tag{3.6}
\end{equation*}
$$

To show the triangular inequality for $q$, let us fix $x, y \in X$ and choose $\rho, \sigma \in \mathbb{R}^{+}$ such that $\rho>p_{V}(x)$ and $\sigma>p_{V}(y)$. Then there exist $\lambda, \mu>0$ such that $x \in \lambda V, \lambda<\rho$ and $y \in \mu V, \mu<\sigma$. These together with the balancedness of $V$ imply that $x \in \rho V$ and $y \in \sigma V$. Hence, we have $\frac{x}{\rho}, \frac{y}{\sigma} \in V$ and so, by the $\alpha-$ convexity of $V$ we can conclude that

$$
\frac{x+y}{\left(\rho^{\alpha}+\sigma^{\alpha}\right)^{\frac{1}{\alpha}}}=\frac{\rho}{\left(\rho^{\alpha}+\sigma^{\alpha}\right)^{\frac{1}{\alpha}}} \cdot \frac{x}{\rho}+\frac{\sigma}{\left(\rho^{\alpha}+\sigma^{\alpha}\right)^{\frac{1}{\alpha}}} \cdot \frac{y}{\sigma} \in V .
$$

Then $p_{V}\left(\frac{x+y}{\left(\rho^{\alpha}+\sigma^{\alpha}\right)^{\frac{1}{\alpha}}}\right) \leq 1$ and so $q\left(\frac{x+y}{\left(\rho^{\alpha}+\sigma^{\alpha}\right)^{\frac{1}{\alpha}}}\right) \leq 1$. Then, by using (3.6), we get that $\left(\frac{1}{\left(\rho^{\alpha}+\sigma^{\alpha}\right)^{\frac{1}{\alpha}}}\right)^{\alpha} \cdot q(x+y) \leq 1$, that is $q(x+y) \leq \rho^{\alpha}+\sigma^{\alpha}$. Since this holds for all $\rho, \sigma \in \mathbb{R}^{+}$such that $\rho>p_{V}(x)$ and $\sigma>p_{V}(y)$, we obtain that $q(x+y) \leq p_{V}(x)^{\alpha}+p_{V}(y)^{\alpha}$.

Corollary 3.2.14. Every $\alpha$-normable TVS is lb.
The converse also holds and in proving it the following notion turns out to be very useful.

Definition 3.2.15. If $(X, \tau)$ is an lb TVS, then for any balanced, bounded, neighbourhood $U$ of the origin in $X$ we define

$$
C(U):=\inf \{\lambda: U+U \subseteq \lambda U\} .
$$

The concavity module $C(X)$ of $X$ is defined as follows

$$
C(X):=\inf \{C(U): U \text { balanced, bounded, neighbourhood of o in } X\} .
$$

Theorem 3.2.16. Let $(X, \tau)$ be a TVS. Then $(X, \tau)$ is lb if and only if $\tau$ is induced by some $\alpha$-norm for some $0<\alpha \leq 1$.

Proof. The sufficiency is given by the previous corollary. As for the necessity, it is possible to show that if $(X, \tau)$ is lb then there exists a bounded $\alpha$-convex neighbourhood of the origin for all $0<\alpha<\alpha_{0}$, where $\alpha_{0}:=\frac{\log 2}{\log C(X)}$ (see Sheet 5). Hence, the conclusion follows by Proposition 3.2.13.

In the context of lb algebras, it might happen that the $\alpha$-norm defining the topology is actually submultiplicative. This is actually the case if the considered algebra is complete.

Definition 3.2.17. An $\alpha$-normed algebra is a $\mathbb{K}$-algebra endowed with the topology induced by a submultiplicative $\alpha$-norm.

Theorem 3.2.18. Any lb Hausdorff complete algebra can be made into an $\alpha$-normed algebra for some $0<\alpha \leq 1$.

Proof. Sketch
Let $(X, \tau)$ be a Hausdorff complete lb algebra. For convenience let us assume that $X$ is unital but the proof can be adapted also to the non-unital case.

As $(X, \tau)$ is lb, Theorem 3.2.16 ensures that the exists $0<\alpha \leq 1$ such that $\tau$ is induced by an $\alpha-$ norm $q$. Consider the space $L(X)$ of all linear continuous operators on $X$ equipped with pointwise addition and scaler multiplication and with the composition as multiplication. Then the operator norm on $L(X)$ defined by $\|\ell\|:=\sup _{x \in X \backslash\{o\}} \frac{q(\ell(x))}{q(x)}$ for all $\ell \in L(X)$ is a submultiplicative $\alpha-$ norm. Since $(X, q)$ is complete, it is possible to show that it is topologically isomorphic to $(L(X),\|\cdot\|)$. If we denote by $\varphi$ such an isomorphism, we then get that $(X, p)$ with $p(x):=\|\varphi(x)\|$ for all $x \in X$ is an $\alpha-$ normed algebra.

Proposition 3.2.19. Let $(X, \tau)$ be an lb Hausdorff TA. Show that if $(X, \tau)$ has jointly continuous multiplication, then $(X, \tau)$ is $\alpha$-normable.
Proof. (see Sheet 6)

### 3.3 Projective limit algebras

The class of topological algebras which we are going to introduce in this section consisits of algebras obtained as a projective limit of a family of TAs and then endowed with the so-called projective topology associated to the natural system of maps given by the projective limit construction. Therefore, we are first going to introduce in general the notion of projective topology w.r.t. a family of maps, then we will focus on the projective limit construction from both an algebraic and topological point of view.

### 3.3.1 Projective topology

Let $\left\{\left(E_{\alpha}, \tau_{\alpha}\right): \alpha \in I\right\}$ be a family of TVSs over $\mathbb{K}(I$ is an arbitrary index set). Let $E$ be a vector space over the same field $\mathbb{K}$ and, for each $\alpha \in I$, let $f_{\alpha}: E \rightarrow E_{\alpha}$ be a linear mapping. The projective topology $\tau_{\text {proj }}$ on $E$ w.r.t.
the family $\left\{\left(E_{\alpha}, \tau_{\alpha}\right), f_{\alpha}, I\right\}$ is the coarsest topology on $E$ for which all the mappings $f_{\alpha}(\alpha \in I)$ are continuous.

It is easy to check that $\left(E, \tau_{\text {proj }}\right)$ is a TVS and that a basis of neighbourhoods of the origin is given by:

$$
\begin{equation*}
\mathcal{B}_{\text {proj }}:=\left\{\bigcap_{\alpha \in F} f_{\alpha}^{-1}\left(U_{\alpha}\right): F \subseteq I \text { finite, } U_{\alpha} \in \mathcal{B}_{\alpha}, \forall \alpha \in F\right\} \tag{3.7}
\end{equation*}
$$

where $\mathcal{B}_{\alpha}$ is a basis of neighbourhoods of the origin in $\left(E_{\alpha}, \tau_{\alpha}\right)$.
Remark 3.3.1. Note that the projective topology $\tau_{\text {proj }}$ coincides with the initial topology given by the map

$$
\begin{aligned}
& \varphi: E \rightarrow\left(\prod_{\alpha \in I} E_{\alpha}\right) \\
& x \mapsto\left(f_{\alpha}(x)\right)_{\alpha \in I} .
\end{aligned}
$$

Recall that the initial topology is defined as the coarsest topology on E such that $\varphi$ is continuous or equivalently as the topology on $E$ generated by the collection of all $\varphi^{-1}(U)$ when $U$ is a neighbourhood of the origin in $\left(\prod_{\alpha \in I} E_{\alpha}, \tau_{\text {prod }}\right)$.

Let us first introduce some properties of the projective topology in the TVS setting.

Lemma 3.3.2. Let $E$ be a vector space over $\mathbb{K}$ endowed with the projective topology $\tau_{\text {proj }}$ w.r.t. the family $\left\{\left(E_{\alpha}, \tau_{\alpha}\right), f_{\alpha}, I\right\}$, where each $\left(E_{\alpha}, \tau_{\alpha}\right)$ is a TVS over $\mathbb{K}$ and each $f_{\alpha}$ a linear mapping from $E$ to $E_{\alpha}$. Let $(F, \tau)$ be an arbitrary $T V S$ and $g$ a linear mapping from $F$ into $E$. The mapping $g: F \rightarrow E$ is continuous if and only if, for each $\alpha \in I, f_{\alpha} \circ g: F \rightarrow E_{\alpha}$ is continuous.
Proof.
Suppose that $g: F \rightarrow E$ is continuous. Since by definition of $\tau_{\text {proj }}$ all $f_{\alpha}$ 's are continuous, we have that for each $\alpha \in I, f_{\alpha} \circ g: F \rightarrow E_{\alpha}$ is continuous.

Conversely, suppose that for each $\alpha \in I$ the map $f_{\alpha} \circ g: F \rightarrow E_{\alpha}$ is continuous and let $U$ be a neighbourhood of the origin in $\left(E, \tau_{\text {proj }}\right)$. Then there exists a finite subset $F$ of $I$ and for each $\alpha \in F$ there exists $U_{\alpha} \in \mathcal{B}_{\alpha}$ such that $\bigcap_{\alpha \in F} f_{\alpha}^{-1}\left(U_{\alpha}\right) \subseteq U$. Therefore, we obtain

$$
g^{-1}(U) \supseteq g^{-1}\left(\bigcap_{\alpha \in F} f_{\alpha}^{-1}\left(U_{\alpha}\right)\right)=\bigcap_{\alpha \in F} g^{-1}\left(f_{\alpha}^{-1}\left(U_{\alpha}\right)\right)=\bigcap_{\alpha \in F}\left(f_{\alpha} \circ g\right)^{-1}\left(U_{\alpha}\right),
$$

which yields that $g^{-1}(U)$ is a neighbourhood of the origin in $(F, \tau)$ since the continuity of all $f_{\alpha} \circ g$ 's ensures that $\left(f_{\alpha} \circ g\right)^{-1}\left(U_{\alpha}\right)$ is a neighbourhood of the origin in $(F, \tau)$.

Proposition 3.3.3. Let $E$ be a vector space over $\mathbb{K}$ endowed with the projective topology $\tau_{\text {proj }}$ w.r.t. the family $\left\{\left(E_{\alpha}, \tau_{\alpha}\right), f_{\alpha}, I\right\}$, where each $\left(E_{\alpha}, \tau_{\alpha}\right)$ is a TVS over $\mathbb{K}$ and each $f_{\alpha}$ a linear mapping from $E$ to $E_{\alpha}$. Then $\tau_{\text {proj }}$ is Hausdorff if and only if for each $0 \neq x \in E$, there exists an $\alpha \in I$ and $a$ neighbourhood $U_{\alpha}$ of the origin in $\left(E_{\alpha}, \tau_{\alpha}\right)$ such that $f_{\alpha}(x) \notin U_{\alpha}$.

Proof.
Suppose that $\left(E, \tau_{\text {proj }}\right)$ is Hausdorff and let $0 \neq x \in E$. By Proposition 1.3.2, there exists a neighbourhood $U$ of the origin in $E$ not containing $x$. Then, by (3.7), there exists a finite subset $F \subseteq I$ and, for any $\alpha \in F$, there exists $U_{\alpha}$ neighbourhood of the origin in $\left(E_{\alpha}, \tau_{\alpha}\right)$ s.t. $\bigcap_{\alpha \in F} f_{\alpha}^{-1}\left(U_{\alpha}\right) \subseteq U$. Hence, as $x \notin$ $U$, there exists $\alpha \in F$ s.t. $x \notin f_{\alpha}^{-1}\left(U_{\alpha}\right)$, i.e. $f_{\alpha}(x) \notin U_{\alpha}$. Conversely, suppose that there exists $\alpha \in I$ and a neighbourhood $V_{\alpha}$ of the origin in $\left(E_{\alpha}, \tau_{\alpha}\right)$ such that $f_{\alpha}(x) \notin V_{\alpha}$. Let $\mathcal{B}_{\alpha}$ be a basis of neighbourhoods of the origin in $\left(E_{\alpha}, \tau_{\alpha}\right)$. Then there exists $U_{\alpha} \in \mathcal{B}_{\alpha}$ such that $U_{\alpha} \subseteq V_{\alpha}$. Hence, $x \notin f_{\alpha}^{-1}\left(U_{\alpha}\right)$ and $f_{\alpha}^{-1}\left(U_{\alpha}\right) \in \mathcal{B}_{\text {proj }}$ (see (3.7)), that is, we have found a neighbourhood of the origin in $\left(E, \tau_{p r o j}\right)$ not containing $x$. This implies, by Proposition 1.3.2, that $\tau_{p r o j}$ is a Hausdorff topology.

Coming back to the context of TAs, we have the following result.
Theorem 3.3.4. Let $E$ be $a \mathbb{K}$-algebra endowed with the projective topology $\tau_{\text {proj }}$ w.r.t. the family $\left\{\left(E_{\alpha}, \tau_{\alpha}\right), f_{\alpha}, I\right\}$, where each $\left(E_{\alpha}, \tau_{\alpha}\right)$ is a TA over $\mathbb{K}$ (resp. a TA with continuous multiplication) and each $f_{\alpha}$ a homomorphism from $E$ to $E_{\alpha}$. Then $\left(E, \tau_{p r o j}\right)$ is a TA (resp. a TA with continuous multiplication).

Proof.
As each $\left(E_{\alpha}, \tau_{\alpha}\right)$ is a TVS, it is easy to verify that $\left(E, \tau_{p r o j}\right)$ is a TVS. Therefore, it remains to show that left and right multiplication are both continuous. For any $x \in E$, consider the left multiplication $\ell_{x}: E \rightarrow E$. For each $\alpha \in I$ we get that:
$\forall y \in E,\left(f_{\alpha} \circ \ell_{x}\right)(y)=f_{\alpha}(x y)=f_{\alpha}(x) f_{\alpha}(y)=\ell_{f_{\alpha}(x)}\left(f_{\alpha}(y)\right)=\left(\ell_{f_{\alpha}(x)} \circ f_{\alpha}\right)(y)$.
Since $f_{\alpha}(x) \in E_{\alpha}$ and $\left(E_{\alpha}, \tau_{\alpha}\right)$ is a TA, we have that $\ell_{f_{\alpha}(x)}: E_{\alpha} \rightarrow E_{\alpha}$ is continuous and so $\ell_{f_{\alpha}(x)} \circ f_{\alpha}$ is continuous. Hence, by (3.8), we have that $f_{\alpha} \circ \ell_{x}$ is continuous for all $\alpha \in I$ and so by the previous lemma we have that $\ell_{x}$ is continuous. Similarly, we get the continuity of the right multiplication in $E$. Hence, $\left(E, \tau_{\text {proj }}\right)$ is a TA.

If each $\left(E_{\alpha}, \tau_{\alpha}\right)$ is a TA with continuous multiplication, then by combining Remark 3.3.1 and Proposition 1.4.1 we can conclude that ( $E, \tau_{\text {proj }}$ ) is a TA.

Proposition 3.3.5. Let $E$ be $a \mathbb{K}$-algebra endowed with the projective topology $\tau_{\text {proj }}$ w.r.t. the family $\left\{\left(E_{\alpha}, \tau_{\alpha}\right), f_{\alpha}, I\right\}$, where each $\left(E_{\alpha}, \tau_{\alpha}\right)$ is an lc (resp. lmc) algebra over $\mathbb{K}$ and each $f_{\alpha}$ a homomorphism from $E$ to $E_{\alpha}$. Then $\left(E, \tau_{p r o j}\right)$ is an lc (resp. lmc) algebra.
Proof.
By assumption, we know that each $\left(E_{\alpha}, \tau_{\alpha}\right)$ is a TA and so Theorem 3.3.4 ensures that $\left(E, \tau_{p r o j}\right)$ is a TA, too. Moreover, as each $\left(E_{\alpha}, \tau_{\alpha}\right)$ is an lc (resp. lmc) algebra, there exists a basis $\mathcal{B}_{\alpha}$ of convex (resp. m-convex) neighbourhoods of the origin in $\left(E_{\alpha}, \tau_{\alpha}\right)$. Then the corresponding $\mathcal{B}_{\text {proj }}$ (see (3.7)) also consists of convex (resp. m-convex) neighbourhoods of the origin in $\left(E, \tau_{p r o j}\right)$. In fact, any $B \in \mathcal{B}_{\text {proj }}$ is of the form $B=\bigcap_{\alpha \in F} f_{\alpha}^{-1}\left(U_{\alpha}\right)$ for some $F \subseteq I$ finite and $U_{\alpha} \in \mathcal{B}_{\alpha}, \forall \alpha \in F$. Since all the $U_{\alpha}$ 's are convex (resp. m-convex) and the preimage of a convex (resp. m-convex) set under a homomorphism is convex (resp. m-convex by Proposition 2.1.3-d)), we get that $B$ is a finite intersection of convex (resp. m-convex) sets and so it is convex (resp. m-convex).

Corollary 3.3.6. Let $(A, \tau)$ be an lc (resp. lmc) algebra and $M$ a subalgebra of $A$. If we endow $M$ with the relative topology $\tau_{M}$ induced by $A$, then $\left(M, \tau_{M}\right)$ is an lc (resp. lmc) algebra.
Proof.
Recalling that $\tau_{M}$ coincides with the projective topology on $M$ induced by id : $M \rightarrow A$ (see Corollary 1.4.2), the conclusion easily follows from the previous proposition (applied for $I=\{1\}, E_{1}=A$ and $\tau_{1}=\tau, E=M$ and $f_{1}=\mathrm{id}$ ).

Corollary 3.3.7. Any subalgebra of a Hausdorff TA is a Hausdorff TA.
Proof. This is a direct application of Proposition 3.3.3 and Corollary 1.4.2.
Example 3.3.8. Let $\left\{\left(E_{\alpha}, \tau_{\alpha}\right): \alpha \in I\right\}$ be a family of TAs over $\mathbb{K}$. Then the Cartesian product $F=\prod_{\alpha \in I} E_{\alpha}$ equipped with coordinatewise operation is a $\mathbb{K}$-algebra. Consider the canonical projections $\pi_{\alpha}: F \rightarrow E_{\alpha}$ defined by $\pi_{\alpha}(x):=x_{\alpha}$ for any $x=\left(x_{\beta}\right)_{\beta \in I} \in F$, which are all homomorphisms. Then the product topology $\tau_{\text {prod }}$ on $F$ is the coarsest topology for which all the canonical projections are continuous and so coincides with the projective
topology on $F$ w.r.t. $\left\{\left(E_{\alpha}, \tau_{\alpha}\right), \pi_{\alpha}: \alpha \in I\right\}^{2}$. Hence, by Theorem 3.3.4 we have that $\left(F, \tau_{\text {prod }}\right)$ is a TA.

Recalling that a cartesian product of complete Hausdorff TAs endowed with the product topology is a complete Hausdorff TA and applying Proposition 3.3.5, Corollary 3.3.6 and Proposition 3.3.3 to the previous example, we can easily prove the following properties

- any Cartesian product of lc (resp. lmc) algebras endowed with the product topology is an lc (resp. 1 mc ) algebra
- any subalgebra of a Cartesian product of lc (resp. lmc) endowed with the relative topology is a TA of the same type
- a cartesian product of Hausdorff TAs endowed with the product topology is a Hausdorff TA.


### 3.3.2 Projective systems of TAs and their projective limit

In this section we are going to discuss the concept of projective system (resp. projective limit) first for just $\mathbb{K}$-algebras and then for TAs.

Definition 3.3.9. Let $(I,<)$ be a directed partially ordered set (i.e. for all $\alpha, \beta \in I$ there exists $\gamma \in I$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$ ). A projective system of algebras $\left\{E_{\alpha}, f_{\alpha \beta}, I\right\}$ is a family of $\mathbb{K}$-algebras $\left\{E_{\alpha}, \alpha \in I\right\}$ together with a family of homomorphisms $f_{\alpha \beta}: E_{\beta} \rightarrow E_{\alpha}$ defined for all $\alpha \leq \beta$ in $I$ such that $f_{\alpha \alpha}$ is the identity on $E_{\alpha}$ and $f_{\alpha \beta} \circ f_{\beta \gamma}=f_{\alpha \gamma}$ for all $\alpha \leq \beta \leq \gamma$, i.e. the diagram

commutes.

[^8]Definition 3.3.10. Given a projective system of algebras $\mathcal{S}:=\left\{E_{\alpha}, f_{\alpha \beta}, I\right\}$, we define the projective limit of $\mathcal{S}$ (or the projective limit algebra associated with $\mathcal{S}$ ) to be the triple $\left\{E_{\mathcal{S}}, f_{\alpha}, I\right\}$, where

$$
E_{\mathcal{S}}:=\left\{x:=\left(x_{\alpha}\right)_{\alpha \in I} \in \prod_{\alpha \in I} E_{\alpha}: x_{\alpha}=f_{\alpha \beta}\left(x_{\beta}\right), \forall \alpha \leq \beta \text { in } I\right\}
$$

and, for any $\alpha \in I, f_{\alpha}: E_{\mathcal{S}} \rightarrow E_{\alpha}$ is defined by $f_{\alpha}:=\pi_{\alpha} \upharpoonright_{E_{\mathcal{S}}}$ (where $\pi_{\alpha}: \prod_{\beta \in I} E_{\beta} \rightarrow E_{\alpha}$ is the canonical projection, see Example 3.3.8).

It is easy to see from the previous definitions that $E_{\mathcal{S}}$ is a subalgebra of $\prod_{\alpha \in I} E_{\alpha}$. Indeed, for any $x, y \in E_{\mathcal{S}}$ and for any $\lambda \in \mathbb{K}$ we have that for all $\alpha \leq \beta$ in $I$ the following hold

$$
\lambda x_{\alpha}+y_{\alpha}=\lambda f_{\alpha \beta}\left(x_{\beta}\right)+f_{\alpha \beta}\left(y_{\beta}\right)=f_{\alpha \beta}\left(\lambda x_{\beta}+y_{\beta}\right)
$$

and

$$
x_{\alpha} y_{\alpha}=f_{\alpha \beta}\left(x_{\beta}\right) f_{\alpha \beta}\left(y_{\beta}\right)=f_{\alpha \beta}\left(x_{\beta} y_{\beta}\right),
$$

i.e. $\lambda x+y, x y \in E_{\mathcal{S}}$. Note that the $f_{\alpha}$ 's are not necessarily surjective and also that

$$
f_{\alpha}=f_{\alpha \beta} \circ f_{\beta}, \forall \alpha \leq \beta \text { in } I,
$$

since for all $x:=\left(x_{\alpha}\right)_{\alpha \in I} \in E_{\mathcal{S}}$ we have $f_{\alpha}(x)=x_{\alpha}=f_{\alpha \beta}\left(x_{\beta}\right)=f_{\alpha \beta}\left(f_{\beta}(x)\right)$.
Also, we can show that $\left\{E_{\mathcal{S}}, f_{\alpha}, I\right\}$ satisfies the following universal property: given a $\mathbb{K}$-algebra $A$ and a family of homomorphism $\left\{g_{\alpha}: A \rightarrow E_{\alpha}, \alpha \in\right.$ $I\}$ such that $g_{\alpha}=f_{\alpha \beta} \circ g_{\beta}$ for all $\alpha \leq \beta$ in $I$, there exists a unique homomorphism $\varphi: A \rightarrow E_{\mathcal{S}}$ such that $g_{\alpha}=f_{\alpha} \circ \varphi$ for all $\alpha \in I$, i.e. the diagram

commutes. In fact, the map $\varphi: A \rightarrow E_{\mathcal{S}}$ defined by $\varphi(a):=\left(g_{\alpha}(a)\right)_{\alpha \in I}$ for all $a \in A$ is a homomorphism such that $\left(f_{\alpha} \circ \varphi\right)(a)=(\varphi(a))_{\alpha}=g_{\alpha}(a)$, for all $a \in A$. Moreover, if there exists $\varphi^{\prime}: A \rightarrow E_{\mathcal{S}}$ such that $g_{\alpha}=f_{\alpha} \circ \varphi^{\prime}$ for all $\alpha \in I$, then for all $a \in A$ we get

$$
\varphi(a)=\left(g_{\alpha}(a)\right)_{\alpha \in I}=\left(\left(f_{\alpha} \circ \varphi^{\prime}\right)(a)\right)_{\alpha \in I}=\left(\left(\varphi^{\prime}(a)\right)_{\alpha}\right)_{\alpha \in I}=\varphi^{\prime}(a),
$$

i.e. $\varphi^{\prime} \equiv \varphi$ on $A$.

These considerations allows to easily see that one can give the following more general definition of projective limit algebra.
Definition 3.3.11. Given a projective system of algebras $\mathcal{S}:=\left\{E_{\alpha}, f_{\alpha \beta}, I\right\}$, a projective limit of $\mathcal{S}$ (or a projective limit algebra associated with $\mathcal{S}$ ) is a triple $\left\{E, h_{\alpha}, I\right\}$, where $E$ is a $\mathbb{K}$-algebra; for any $\alpha \in I, h_{\alpha}: E \rightarrow E_{\alpha}$ is a homomorphisms such that $h_{\alpha}=f_{\alpha \beta} \circ h_{\beta}, \forall \alpha \leq \beta$ in $I$; and the following universal property holds: for any $\mathbb{K}$-algebra $A$ and any family of homomorphism $\left\{g_{\alpha}: A \rightarrow E_{\alpha}, \alpha \in I\right\}$ such that $g_{\alpha}=f_{\alpha \beta} \circ g_{\beta}$ for all $\alpha \leq \beta$ in $I$, there exists a unique homomorphism $\varphi: A \rightarrow E$ such that $g_{\alpha}=h_{\alpha} \circ \varphi$ for all $\alpha \in I$.

It is easy to show that $\left\{E, h_{\alpha}, I\right\}$ is unique up to (algebraic) isomorphisms, i.e. if $\left\{\tilde{E}, \tilde{h}_{\alpha}, I\right\}$ fulfills Definition 3.3.11 then there exists a unique isomorphism between $E$ and $\tilde{E}$. This justifies why in Definition 3.3.10 we have called $\left\{E_{\mathcal{S}}, f_{\alpha}, I\right\}$ the projective limit of $\mathcal{S}$. (Indeed, we have already showed that $\left\{E_{\mathcal{S}}, f_{\alpha}, I\right\}$ fulfills Definition 3.3.11.)

The definitions introduced above for algebras can be easily adapted to the category of TAs.

Definition 3.3.12. Let $(I,<)$ be a directed partially ordered set. A projective system of TAs $\left\{\left(E_{\alpha}, \tau_{\alpha}\right), f_{\alpha \beta}, I\right\}$ is a family of $\mathbb{K}$-algebras $\left\{\left(E_{\alpha}, \tau_{\alpha}\right): \alpha \in I\right\}$ together with a family of continuous homomorphisms $f_{\alpha \beta}: E_{\beta} \rightarrow E_{\alpha}$ defined for all $\alpha \leq \beta$ in $I$ such that $f_{\alpha \alpha}$ is the identity on $E_{\alpha}$ and $f_{\alpha \beta} \circ f_{\beta \gamma}=f_{\alpha \gamma}$ for all $\alpha \leq \beta \leq \gamma$, i.e. the diagram

commutes. Equivalently, a projective system of TAs is a projective system of algebras $\left\{E_{\alpha}, f_{\alpha \beta}, I\right\}$ in which each $E_{\alpha}$ is endowed with a topology $\tau_{\alpha}$ making $\left(E_{\alpha}, \tau_{\alpha}\right)$ into a TA and all the homomorphisms $f_{\alpha \beta}$ continuous.
Definition 3.3.13. Given a projective system $\mathcal{S}:=\left\{\left(E_{\alpha}, \tau_{\alpha}\right), f_{\alpha \beta}, I\right\}$ of TAs, we define the projective limit of $\mathcal{S}$ (or the projective limit TA associated with $\mathcal{S})$ to be the triple $\left\{\left(E_{\mathcal{S}}, \tau_{\text {proj }}\right), f_{\alpha}, I\right\}$ where $\left\{E_{\mathcal{S}}, f_{\alpha}, I\right\}$ is the projective limit algebra associated with $\left\{E_{\alpha}, f_{\alpha \beta}, I\right\}$ and $\tau_{\text {proj }}$ is the projective topology on $E_{\mathcal{S}}$ w.r.t. the family $\left\{\left(E_{\alpha}, \tau_{\alpha}\right), f_{\alpha}: \alpha \in I\right\}$.

Similarly, to the algebraic case, one could give the following more general definition of projective limit TA.

Definition 3.3.14. Given a projective system of $T A s \mathcal{S}:=\left\{\left(E_{\alpha}, \tau_{\alpha}\right), f_{\alpha \beta}, I\right\}$, a projective limit of $\mathcal{S}$ (or a projective limit TA associated with $\mathcal{S}$ ) is a triple $\left\{(E, \tau), h_{\alpha}, I\right\}$ where $(E, \tau)$ is a TA; for any $\alpha \in I, h_{\alpha}: E \rightarrow E_{\alpha}$ is a continuous homomorphism such that $h_{\alpha}=f_{\alpha \beta} \circ h_{\beta}$, for all $\alpha \leq \beta$ in $I$; and the following universal property holds: for any $T A(A, \omega)$ and any family of continuous homomorphism $\left\{g_{\alpha}: A \rightarrow E_{\alpha}, \alpha \in I\right\}$ such that $g_{\alpha}=f_{\alpha \beta} \circ g_{\beta}$ for all $\alpha \leq \beta$ in $I$, there exists a unique continuous homomorphism $\varphi: A \rightarrow E$ such that $g_{\alpha}=h_{\alpha} \circ \varphi$ for all $\alpha \in I$.

It is easy to show that $\left\{(E, \tau), h_{\alpha}, I\right\}$ is unique up to topological isomorphisms. We have already showed that $E_{\mathcal{S}}$ is an algebra such that the family of all $f_{\alpha}:=\pi_{\alpha} \upharpoonright_{E_{\mathcal{S}}}(\alpha \in I)$ fulfills $f_{\alpha}=f_{\alpha \beta} \circ f_{\beta}, \forall \alpha \leq \beta$ in $I$. Endowing $E_{\mathcal{S}}$ with the projective topology $\tau_{\text {proj }}$ w.r.t. $\left\{\left(E_{\alpha}, \tau_{\alpha}\right), f_{\alpha}, I\right\}$, we get by Theorem 3.3.4 that $\left(E_{\mathcal{S}}, \tau_{p r o j}\right)$ is a TA and that all $f_{\alpha}$ 's are continuous. Also, for any TA $(A, \omega)$ and any family of continuous homomorphism $\left\{g_{\alpha}: A \rightarrow E_{\alpha}, \alpha \in I\right\}$ such that $g_{\alpha}=f_{\alpha \beta} \circ g_{\beta}$ for all $\alpha \leq \beta$ in $I$, we have already showed that $\varphi: A \rightarrow E_{\mathcal{S}}$ defined by $\varphi(a):=\left(g_{\alpha}(a)\right)_{\alpha \in I}$ for all $a \in A$ is the unique homomorphism such that $g_{\alpha}=f_{\alpha} \circ \varphi$ for all $\alpha \in I$. But $\varphi$ is also continuous because for any $U \in \mathcal{B}_{\text {proj }}$ we have $U=\bigcap_{\alpha \in F} f_{\alpha}^{-1}\left(U_{\alpha}\right)$ for some $F \subset I$ finite and some $U_{\alpha} \in \mathcal{B}_{\alpha}$ for all $\alpha \in F$ and so $\varphi^{-1}(U)=\bigcap_{\alpha \in F} \varphi^{-1}\left(f_{\alpha}^{-1}\left(U_{\alpha}\right)\right)=$ $\bigcap_{\alpha \in F}\left(f_{\alpha} \circ \varphi\right)^{-1}\left(U_{\alpha}\right)=\bigcap_{\alpha \in F} g_{\alpha}^{-1}\left(U_{\alpha}\right) \in \mathcal{B}_{\omega}$. Hence, $\left\{\left(E_{\mathcal{S}}, \tau_{p r o j}\right), f_{\alpha}, I\right\}$ satisfies Definiton 3.3.14.

## Remark 3.3.15.

From the previous definitions one can easily deduce the following:
a) the projective limit of a projective system of Hausdorff TAs is a Hausdorff TA (easily follows by Proposition 3.3.3).
b) the projective limit of a projective system of Hausdorff TAs $\left\{\left(E_{\alpha}, \tau_{\alpha}\right), f_{\alpha, \beta}, I\right\}$ is a closed subalgebra of $\left(\prod_{\alpha \in I} E_{\alpha}, \tau_{\text {prod }}\right)$ (see Sheet 6).
c) the projective limit of a projective system of complete Hausdorff TAs is a complete Hausdorff TA (see Sheet 6).

Corollary 3.3.16. A projective limit of lmc algebras is an lmc algebra.
Proof.
Let $\left\{\left(E_{\alpha}, \tau_{\alpha}\right), f_{\alpha, \beta}, I\right\}$ be a projective system of lmc algebras. Then its projective limit $\left\{\left(E_{\mathcal{S}}, \tau_{\text {proj }}\right), f_{\alpha}, I\right\}$ is an lmc algebra by Proposition 3.3.5.

This easy corollary brings us to a very natural but fundamental question: can any lmc algebra be written as a projective limit of a projective system of lmc algebras or at least as a subalgebra of such a projective limit? The whole next section will be devoted to show a positive answer to this question.

### 3.3.3 Arens-Michael decomposition

This section will be devoted to the Arens-Michael decomposition theorem, which was independently conceived by Arens and Michael in the early days of the theory of TAs (1952). This result is so important because it provides a device to reduce basic questions about lmc algebra to analogous ones for the corresponding factor Banach algebras. Since the theory of Banach algebras has been heavily studied, being able to reduce to Banach algebras is very advantageous and so much desirable.

Before stating the Arens-Michael decomposition theorem, let us recall the completion theorem for TVS and two useful lemmas about projective limit algebras.

## Theorem 3.3.17.

Let $X$ be a Haudorff TVS. Then there exists a complete Hausdorff TVS $\hat{X}$ and a mapping $i: X \rightarrow \hat{X}$ with the following properties:
a) The mapping $i$ is a topological monomorphism.
b) The image of $X$ under $i$ is dense in $\hat{X}$.
c) For every complete Hausdorff TVS Y and for every continuous linear map $f: X \rightarrow Y$, there is a continuous linear map $\hat{f}: \hat{X} \rightarrow Y$ such that the diagram

is commutative. Furthermore:
I) Any other pair $\left(\hat{X}_{1}, i_{1}\right)$, consisting of a complete Hausdorff TVS $\hat{X}_{1}$ and of a mapping $i_{1}: X \rightarrow \hat{X}_{1}$ such that properties (a) and (b) hold substituting $\hat{X}$ with $\hat{X}_{1}$ and $i$ with $i_{1}$, is topologically isomorphic to $(\hat{X}, i)$. This means that there is a topological isomorphism $j$ of $\hat{X}$ onto $\hat{X}_{1}$ such that the diagram

is commutative.
II) Given $Y$ and $f$ as in property (c), the continuous linear map $\hat{f}$ is unique.

Lemma 3.3.18. Let $\left\{\left(A_{\mathcal{S}}, \tau_{\text {proj }}\right), g_{\alpha}, J\right\}$ be the projective limit of the projective system $\mathcal{S}:=\left\{\left(A_{\alpha}, \tau_{\alpha}\right), g_{\alpha \beta}, J\right\}$ of TAs. Then a basis of neighbourhoods of the
origin in $\left(A_{\mathcal{S}}, \tau_{\text {proj }}\right)$ is given by

$$
\widetilde{\mathcal{B}}_{\text {proj }}:=\left\{g_{\alpha}^{-1}\left(V_{\alpha}\right): V_{\alpha} \in \mathcal{B}_{\alpha}, \alpha \in J\right\},
$$

where each $\mathcal{B}_{\alpha}$ is a basis of neighbourhoods of the origin in $\left(A_{\alpha}, \tau_{\alpha}\right)$.

## Proof.

By the continuity of the $g_{\alpha}$ 's, we know that $\widetilde{\mathcal{B}}_{\text {proj }}$ is a collection of neighbourhoods of the origin in $\left(A_{\mathcal{S}}, \tau_{\text {proj }}\right)$. We want to show that it is a basis.

By (3.7), we have that a basis of neighbourhoods of the origin in $\left(A_{\mathcal{S}}, \tau_{\text {proj }}\right)$ is given by

$$
\mathcal{B}_{\text {proj }}:=\left\{\bigcap_{\beta \in F} g_{\beta}^{-1}\left(V_{\beta}\right): F \subseteq I \text { finite, } V_{\beta} \in \mathcal{B}_{\beta}, \forall \beta \in F\right\} .
$$

As $J$ is directed, for any finite subset $F$ of $I$ there exists $\alpha \in J$ such that $\beta \leq \alpha$ for all $\beta \in F$. Then we have that $g_{\beta}=g_{\beta \alpha} \circ g_{\alpha}$ for all $\beta \in F$ and so

$$
\bigcap_{\beta \in F} g_{\beta}^{-1}\left(V_{\beta}\right)=\bigcap_{\beta \in F}\left(g_{\beta \alpha} \circ g_{\alpha}\right)^{-1}\left(V_{\beta}\right)=\bigcap_{\beta \in F} g_{\alpha}^{-1}\left(g_{\beta \alpha}^{-1}\left(V_{\beta}\right)\right)=g_{\alpha}^{-1}\left(\bigcap_{\beta \in F} g_{\beta \alpha}^{-1}\left(V_{\beta}\right)\right) .
$$

Set $W_{\alpha}:=\bigcap_{\beta \in F} g_{\beta \alpha}^{-1}\left(V_{\beta}\right)$. Since for all $\beta \in F$ the map $g_{\beta \alpha}: A_{\alpha} \rightarrow A_{\beta}$ is continuous, we get that for all $\beta \in F$ the set $g_{\beta \alpha}^{-1}\left(V_{\beta}\right)$ is a neighbourhood of the origin in $\left(A_{\alpha}, \tau_{\alpha}\right)$ and so is $W_{\alpha}$. Then there exists $V_{\alpha} \in \mathcal{B}_{\alpha}$ such that $V_{\alpha} \subseteq W_{\alpha}$. Hence, we obtain

$$
\bigcap_{\beta \in F} g_{\beta}^{-1}\left(V_{\beta}\right)=g_{\alpha}^{-1}\left(W_{\alpha}\right) \supseteq g_{\alpha}^{-1}\left(V_{\alpha}\right)
$$

and so we have showed that for any $M \in \mathcal{B}_{\text {proj }}$ there exists $\widetilde{M} \in \widetilde{\mathcal{B}}_{\text {proj }}$ such that $\widetilde{M} \subseteq M$, i.e. $\widetilde{\mathcal{B}}_{\text {proj }}$ is a basis of neighbourhoods of the origin in $\left(A_{\mathcal{S}}, \tau_{p r o j}\right)$.

Lemma 3.3.19. Let $\left\{\left(A_{\mathcal{S}}, \tau_{\text {proj }}\right), g_{\alpha}, J\right\}$ be the projective limit of the projective system $\mathcal{S}:=\left\{\left(A_{\alpha}, \tau_{\alpha}\right), g_{\alpha \beta}, J\right\}$ of TAs and $W$ a linear subspace of $A_{\mathcal{S}}$. Then

$$
\bar{W}^{\tau_{p r o j}}=\bigcap_{\alpha \in J} g_{\alpha}^{-1}\left({\overline{g_{\alpha}(W)}}^{\tau_{\alpha}}\right)=\operatorname{proj} \lim \left(\mathcal{S}_{1}\right),
$$

where $\mathcal{S}_{1}$ denotes the projective system $\left\{{\overline{g_{\alpha}(W)}}^{\tau_{\alpha}}, g_{\alpha \beta} \upharpoonright_{\overline{g_{\beta}(W)}}, J\right\}$ of TAs (here ${\overline{g_{\alpha}(W)}}^{\tau_{\alpha}}$ is intended as endowed with the relative topology induced by $\tau_{\alpha}$ ).

In particular, if $W$ is closed in $\left(A_{\mathcal{S}}, \tau_{\text {proj }}\right)$ then

$$
W=\operatorname{projlim}\left(\mathcal{S}_{2}\right)=\operatorname{projlim}\left(\mathcal{S}_{1}\right),
$$

where $\mathcal{S}_{2}$ denotes the projective system $\left\{g_{\alpha}(W), g_{\alpha \beta} \upharpoonright_{g_{\beta}(W)}, J\right\}$ of TAs (here $g_{\alpha}(W)$ is intended as endowed with the relative topology induced by $\left.\tau_{\alpha}\right)$.

Proof.
Since $\mathcal{S}$ is a projective system of TAs, by Definition 3.3.12, we have that for any $\alpha \leq \beta$ the map $g_{\alpha \beta}: A_{\beta} \rightarrow A_{\alpha}$ is a continuous homomorphism fulfilling

$$
\begin{equation*}
g_{\alpha \alpha}=\mathrm{id}, \forall \alpha \in J \tag{3.9}
\end{equation*}
$$

and

$$
\begin{equation*}
g_{\alpha \beta} \circ g_{\beta \gamma}=g_{\alpha \gamma}, \forall \alpha \leq \beta \leq \gamma \text { in } J . \tag{3.10}
\end{equation*}
$$

Also, by Definition 3.3.14 we have that for any $\alpha \in J$ the map $g_{\alpha}: A_{\mathcal{S}} \rightarrow A_{\alpha}$ is a continuous homomorphism such that

$$
\begin{equation*}
g_{\alpha}=g_{\alpha \beta} \circ g_{\beta}, \forall \alpha \leq \beta \text { in } J . \tag{3.11}
\end{equation*}
$$

For any $\alpha \in J$, we have that $g_{\alpha}(W) \subseteq A_{\alpha}$ and so (3.9) provides that $g_{\alpha \alpha} \upharpoonright_{g_{\alpha}(W)}=\mathrm{id} \upharpoonright_{g_{\alpha}(W)}$. Moreover, for any $\alpha \leq \beta \leq \gamma$ in $J$, the relation (3.11) implies that $g_{\beta \gamma}\left(g_{\gamma}(W)\right) \subseteq g_{\beta}(W)$, which in turn gives that for any $x \in g_{\gamma}(W)$ :

$$
g_{\alpha \beta} \upharpoonright_{g_{\beta}(W)}\left(g_{\beta \gamma} \upharpoonright_{g_{\gamma}(W)}(x)\right)=g_{\alpha \beta}\left(g_{\beta \gamma}(x)\right) \stackrel{(3.10)}{=} g_{\alpha \gamma}(x)=g_{\alpha \gamma} \upharpoonright_{g_{\gamma}(W)}(x) .
$$

Endowing each $g_{\beta}(W)$ with the subspace topology induced by $\tau_{\beta}$, we have that $g_{\alpha \beta} \upharpoonright_{g_{\beta}(W)}$ is continuous for any $\alpha \leq \beta$ in $J$. Hence, we have showed that $\mathcal{S}_{2}$ is a projective system of TAs.

By the continuity of the $g_{\alpha \beta}$ 's for all $\alpha \leq \beta$ in $J$, we also get that

$$
\begin{equation*}
g_{\beta \gamma}\left(\overline{g_{\gamma}(W)}\right) \subseteq \overline{g_{\beta \gamma}\left(g_{\gamma}(W)\right)} \stackrel{(3.11)}{=} \overline{g_{\beta}(W)}, \forall \beta \leq \gamma \text { in } J \tag{3.12}
\end{equation*}
$$

Therefore, for any $\alpha \leq \beta \leq \gamma$ in $J$ and for any $x \in \overline{g_{\gamma}(W)}$ we obtain that

$$
g_{\alpha \beta} \upharpoonright \overline{g_{\beta}(W)}\left(g_{\beta \gamma} \upharpoonright_{\overline{g_{\gamma}(W)}}(x)\right)=g_{\alpha \beta}\left(g_{\beta \gamma}(x)\right) \stackrel{(3.10)}{=} g_{\alpha \gamma}(x)=g_{\alpha \gamma} \upharpoonright_{\overline{g_{\gamma}(W)}}(x) .
$$

Hence, we have showed that $\mathcal{S}_{1}$ is a projective system of TAs, too.

Then

$$
\begin{aligned}
& \operatorname{proj} \lim \left(\mathcal{S}_{1}\right)=\left\{x:=\left(x_{\alpha}\right)_{\alpha \in J}: x_{\alpha} \in \overline{g_{\alpha}(W)}, \forall \alpha \in J\right. \text { and } \\
&\left.x_{\alpha}=g_{\alpha \beta}\left(x_{\beta}\right), \forall \alpha \leq \beta \text { in } J\right\} \\
& \stackrel{(3.12)}{=}\left\{x:=\left(x_{\alpha}\right)_{\alpha \in J}: x_{\alpha} \in \overline{g_{\alpha}(W)}, \alpha \in J\right\} \\
&=\left\{x \in A_{\mathcal{S}}: g_{\alpha}(x) \in \overline{g_{\alpha}(W)}, \forall \alpha \in J\right\}=\bigcap_{\alpha \in J} g_{\alpha}^{-1}\left(\overline{g_{\alpha}(W)}\right)
\end{aligned}
$$

and similarly projlim $\left(\mathcal{S}_{2}\right)=\bigcap_{\alpha \in J} g_{\alpha}^{-1}\left(g_{\alpha}(W)\right)$.
For any $\alpha \in J$, the continuity of $g_{\alpha}$ provides that $g_{\alpha}(\bar{W}) \subseteq \overline{g_{\alpha}(W)}$ and so $\left.\bar{W} \subseteq g_{\alpha}^{-1}\left(\overline{g_{\alpha}(W)}\right)\right)$. Hence,

$$
\left.\bar{W} \subseteq \bigcap_{\alpha \in J} g_{\alpha}^{-1}\left(\overline{g_{\alpha}(W)}\right)\right)=\operatorname{projlim}\left(\mathcal{S}_{1}\right)
$$

Conversely, suppose that $x \in \operatorname{projlim}\left(\mathcal{S}_{1}\right)$. Then $\left.x \in g_{\alpha}^{-1}\left(\overline{g_{\alpha}(W)}\right)\right)$ for all $\alpha \in J$, that means $g_{\alpha}(x) \in \overline{g_{\alpha}(W)}$ for all $\alpha \in J$. Hence, for each $\alpha \in J$, we have that for any neighbourhood $V_{\alpha}$ of the origin in $\left(A_{\alpha}, \tau_{\alpha}\right)$, the following holds $\left(g_{\alpha}(x)+V_{\alpha}\right) \cap g_{\alpha}(W) \neq \emptyset$ and so $\left(x+g_{\alpha}^{-1}\left(V_{\alpha}\right)\right) \cap W \neq \emptyset$. This gives by Lemma 3.3.18 that for any $U$ neighbourhood of the origin in $\left(A_{\mathcal{S}}, \tau_{p r o j}\right)$ the sets $x+U$ and $W$ have non-empty intersection, i.e. $x \in \bar{W}$. We have therefore showed that $\bar{W}=\operatorname{proj} \lim \left(\mathcal{S}_{1}\right)$.

If $W$ is closed, then $W=\bar{W}$. However, we have

$$
W \subseteq \bigcap_{\alpha \in J} g_{\alpha}^{-1}\left(g_{\alpha}(W)\right) \subseteq \bigcap_{\alpha \in J} g_{\alpha}^{-1}\left(\overline{g_{\alpha}(W)}\right)=\operatorname{proj} \lim \left(\mathcal{S}_{1}\right)=\bar{W}=W
$$

i.e. $W=\operatorname{projlim}\left(\mathcal{S}_{2}\right)=\operatorname{projlim}\left(\mathcal{S}_{1}\right)$.

Suppose now that $(E, \tau)$ is a Hausdorff lmc algebra. Then, by Theorem 2.1.11 there exists a basis $\mathcal{M}:=\left\{U_{\alpha}\right\}_{\alpha \in I}$ of neighbourhoods of the origin in $(E, \tau)$ consisting of m-barrels. For each $\alpha \in I$, let $p_{\alpha}$ be the Minkowski functional of $U_{\alpha}$. Then we have showed in Section 2.2 that $\left\{p_{\alpha}\right\}_{\alpha \in I}$ is a family of submultiplicative seminorms on $E$ generating $\tau$. For each $\alpha \in I$, we define $N_{\alpha}:=\operatorname{ker}\left(p_{\alpha}\right)$ which is a closed ideal in $(E, \tau)$. Then we can take the quotient $E_{\alpha}:=E / N_{\alpha}$ and endow it with the quotient norm $q_{\alpha}\left(\rho_{\alpha}(x)\right):=\inf _{y \in N_{\alpha}} p_{\alpha}(x-y)$ where $\rho_{\alpha}: E \rightarrow E_{\alpha}$ denotes the corresponding quotient map. With a similar proof to the one of Proposition 1.4.9 we can prove that is $\left(E_{\alpha}, q_{\alpha}\right)$ is a normed algebra. Taking the completion $\left(\hat{E}_{\alpha}, \hat{q}_{\alpha}\right)$ of each $\left(E_{\alpha}, q_{\alpha}\right)$, we get a family of Banach algebras. If we denote
by $i_{\alpha}: E_{\alpha} \rightarrow \hat{E}_{\alpha}$ the canonical injection (which is an injective continuous and open homomorphism), then $\overline{\rho_{\alpha}}:=i_{\alpha} \circ \rho_{\alpha}$ is a continuous open homomorphism. For convenience, from now on we will just denote ( $E_{\alpha}, q_{\alpha}$ ) by $E_{\alpha}$ and ( $\hat{E}_{\alpha}, \hat{q}_{\alpha}$ ) by $\hat{E}_{\alpha}$.

We define a partial order on $I$ by setting:

$$
\alpha \leq \beta \Leftrightarrow U_{\beta} \subseteq U_{\alpha} \Leftrightarrow p_{\alpha}(x) \leq p_{\beta}(x), \forall x \in X .
$$

Then $(I, \leq)$ is directed because $\mathcal{M}$ is a basis and so for any $\alpha, \beta \in I$ we have $U_{\alpha} \cap U_{\beta} \in \mathcal{M}$, i.e. there exists $\gamma \in I$ such that $U_{\gamma} \subseteq U_{\alpha} \cap U_{\beta}$ and so $U_{\gamma} \subseteq U_{\alpha}$ and $U_{\gamma} \subseteq U_{\beta}$, i.e. $\alpha \leq \gamma$ and $\beta \leq \gamma$. Also, for any $\alpha \leq \beta$ in $I$ we have $N_{\beta} \subseteq N_{\alpha}$ and hence

$$
\begin{array}{lll}
f_{\alpha \beta}: & E_{\beta} & \rightarrow E_{\alpha} \\
x+N_{\beta} & \mapsto x+N_{\alpha}
\end{array}
$$

is a well-defined surjective homomorphism and the following holds

$$
\begin{equation*}
\rho_{\alpha}=f_{\alpha \beta} \circ \rho_{\beta}, \forall \alpha \leq \beta \text { in } I . \tag{3.13}
\end{equation*}
$$

Then all $f_{\alpha \beta}$ 's are continuous homomorphisms and for any $\alpha \leq \beta \leq \gamma$ in $I$ and any $x \in E_{\beta}$, we have

$$
f_{\alpha \beta}\left(f_{\beta \gamma}\left(x+N_{\gamma}\right)\right)=f_{\alpha \beta}\left(x+N_{\beta}\right)=x+N_{\alpha}=f_{\alpha \gamma}\left(x+N_{\gamma}\right),
$$

i.e. $f_{\alpha \gamma}=f_{\alpha \beta} \circ f_{\beta \gamma}$.

Hence, $\left\{\left(E_{\alpha}, q_{\alpha}\right), f_{\alpha \beta}, I\right\}$ is a projective system of normed algebras. Moreover, for any $\alpha \leq \beta$ in $I$ there exists $\overline{f_{\alpha \beta}}: \hat{E}_{\beta} \rightarrow \hat{E}_{\alpha}$ continuous and linear such that $\overline{f_{\alpha \beta}} \circ i_{\beta}=i_{\alpha} \circ f_{\alpha \beta}$ where $i_{\alpha}$ (resp. $i_{\beta}$ ) denotes the embedding of $E_{\alpha}$ (resp. $E_{\beta}$ ) in $\hat{E}_{\alpha}$ (resp. $\hat{E}_{\beta}$ ). Then it is easy to check that $\left\{\left(\hat{E}_{\alpha}, \hat{q}_{\alpha}\right), \overline{f_{\alpha \beta}}, I\right\}$ is a projective system of Banach algebras.

We are ready now for the Arens-Michael decomposition theorem.
Theorem 3.3.20. Let $(E, \tau)$ be a Hausdorff lmc algebra and $\mathcal{M}:=\left\{U_{\alpha}\right\}_{\alpha \in I}$ a basis of neighbourhoods of the origin in $(E, \tau)$ consisting of m-barrels. Consider the projective system $\left\{E_{\alpha}, f_{\alpha \beta}, I\right\}$ of normed algebras and the projective system $\left\{\hat{E}_{\alpha}, \overline{f_{\alpha \beta}}, I\right\}$ of Banach algebras introduced above. Then there exist the following topological monomorphisms

$$
\begin{equation*}
E \hookrightarrow \operatorname{proj} \lim \left\{\left(E_{\alpha}, q_{\alpha}\right), f_{\alpha \beta}, I\right\} \hookrightarrow \operatorname{proj} \lim \left\{\left(\hat{E}_{\alpha}, q_{\alpha}\right), \overline{f_{\alpha \beta}}, I\right\} \cong \hat{E} . \tag{3.14}
\end{equation*}
$$

If in addition $(E, \tau)$ is complete, then the maps in (3.14) are all topological isomorphisms. In this case, the expression $E=\operatorname{projlim}\left\{\left(\hat{E}_{\alpha}, q_{\alpha}\right), \overline{f_{\alpha \beta}}, I\right\}$ is called the Arens-Michael decomposition of $E$ w.r.t. $\mathcal{M}$.

## Proof.

For convenience, let us denote by $\mathcal{P}$ and $\hat{\mathcal{P}}$ the projective systems $\left\{E_{\alpha}, f_{\alpha \beta}, I\right\}$ and $\left\{\hat{E}_{\alpha}, \overline{f_{\alpha \beta}}, I\right\}$, respectively.

For any $x \in E$, let us define $\phi(x):=\left(\rho_{\alpha}(x)\right)_{\alpha \in I}$. Then $\phi(E) \subseteq \operatorname{projlim}(\mathcal{P})$. Indeed, for any $x \in E$ and $\alpha \leq \beta$ in $I$ we have

$$
(\phi(x))_{\alpha}=\rho_{\alpha}(x) \stackrel{(3.13)}{=} f_{\alpha \beta}\left(\rho_{\beta}(x)\right)=f_{\alpha \beta}\left((\phi(x))_{\beta}\right) .
$$

Then the following hold:

- $\phi$ is a homomorphism, as each $\rho_{\alpha}$ is a homomorphism and $\operatorname{projlim}(\mathcal{P})$ is equipped with coordinatewise operations. Let us just show that $\phi$ is multiplicative: for all $x, y \in E$,
$\phi(x y)=\left(\rho_{\alpha}(x y)\right)_{\alpha \in I}=\left(\rho_{\alpha}(x) \rho_{\alpha}(y)\right)_{\alpha \in I}=\left(\rho_{\alpha}(x)\right)_{\alpha \in I}\left(\rho_{\alpha}(y)\right)_{\alpha \in I}=\phi(x) \phi(y)$.
- $\phi$ is injective, because

$$
\phi(x)=0 \Rightarrow \rho_{\alpha}(x)=0, \forall \alpha \in I \Rightarrow x \in N_{\alpha}, \forall \alpha \in I \Rightarrow p_{\alpha}(x)=0, \forall \alpha \in I \Rightarrow x=0,
$$

where in the last implication we used that $E$ is Hausdorff and so $\left\{p_{\alpha}\right\}_{\alpha \in I}$ is a separating family of seminorms.

- $\phi$ is continuous, because Lemma 3.3.18 guarantees that for any neighbourhood $U$ of the origin in $\operatorname{projlim}(\mathcal{P})$, there exist $\alpha \in I$ and a neighbourhood $V_{\alpha}$ of the origin in $E_{\alpha}$ such that $f_{\alpha}^{-1}\left(V_{\alpha}\right) \subseteq U$. Then $\rho_{\alpha}^{1}\left(V_{\alpha}\right)=\phi^{-1}\left(f_{\alpha}^{-1}\left(V_{\alpha}\right)\right) \subseteq$ $\phi^{-1}(U)$ and so, by the continuity of $\rho_{\alpha}$ we have that $\phi^{-1}(U)$ is a neighborhood of the origin in $E$.
- $\phi$ is an open map. Indeed, recalling that $\mathcal{M}:=\left\{U_{\alpha}\right\}_{\alpha \in I}$ is a basis of neighbourhoods of the origin in $(E, \tau)$, we can show that for any $\alpha \in I$ the set $V:=\left(\rho_{\alpha}\left(\frac{1}{2} U_{\alpha}\right) \times \prod_{\beta \in I \backslash\{\alpha\}} E_{\beta}\right) \cap \phi(E)$ is a neighbourhood of the origin in $\operatorname{projlim}(\mathcal{P})$ such that $V \subseteq \phi\left(U_{\alpha}\right)$. Fix $\alpha \in I$. Then the openness of $\rho_{\alpha}$ implies that $\rho_{\alpha}\left(\frac{1}{2} U_{\alpha}\right)$ is a neighbourhood of the origin in $E_{\alpha}$ and so $\rho_{\alpha}\left(\frac{1}{2} U_{\alpha}\right) \times \prod_{\beta \in I \backslash\{\alpha\}} E_{\beta}$ is a neighbourhood of the origin in $\prod_{\gamma \in I} E_{\gamma}$ endowed with the product topology. Hence, $V$ is a neighbourhood of the origin in projlim $(\mathcal{P})$.

Moreover, for any $x:=\left(x_{\gamma}\right)_{\gamma \in I} \in V$ we have that:
a) $x \in \phi(E)$, i.e. there exists $y \in E$ such that $\phi(y)=x$
b) $x_{\alpha} \in \rho_{\alpha}\left(\frac{1}{2} U_{\alpha}\right)$
c) $x_{\beta} \in E_{\beta}$ for all $\beta \neq \alpha$ in $I$.

Then

$$
\rho_{\alpha}(y)=f_{\alpha}(\phi(y)) \stackrel{(a)}{=} f_{\alpha}(x)=x_{\alpha} \stackrel{(b)}{\in} \rho_{\alpha}\left(\frac{1}{2} U_{\alpha}\right),
$$

which implies that there exists $z \in \frac{1}{2} U_{\alpha}$ such that $\rho_{\alpha}(y)=\rho_{\alpha}(z)$. Therefore, $y=z+w$ for some $w \in N_{\alpha}$, which gives in turn

$$
\left|p_{\alpha}(y)-p_{\alpha}(z)\right| \leq p_{\alpha}(y-z) \leq p_{\alpha}(w)=0
$$

and so $p_{\alpha}(y)=p_{\alpha}(z) \leq \frac{1}{2}<1$, i.e. $y \in U_{\alpha}$. Hence, $x \stackrel{(a)}{=} \phi(y) \in \phi\left(U_{\alpha}\right)$, that gives $V \subseteq \phi\left(U_{\alpha}\right)$.

We have then just showed that $\phi: E \hookrightarrow \operatorname{projlim}(\mathcal{P})$ is a topological monomorphism.

Now, by using Theorem 3.3.17, we get that for any $\alpha \leq \beta$ in $I$ the diagram

commutes, where $i_{\alpha}$ and $i_{\beta}$ are topological monomorphisms such that $\overline{i_{\alpha}\left(E_{\alpha}\right)}=$ $\hat{E}_{\alpha}$ and $\overline{i_{\beta}\left(E_{\beta}\right)}=\hat{E}_{\beta}$. Then [4, E.III.53, Corollary 1] ensures that there exists a unique topological monomorphism $j: \operatorname{projlim}(\mathcal{P}) \hookrightarrow \operatorname{projlim}(\hat{\mathcal{P}})$ such that the following diagram commutes


Setting $\psi=j \circ \phi$ we get a topological monomorphism from $E$ to $\operatorname{projlim}(\hat{\mathcal{P}})$ and so $\psi(E)$ is a linear subspace of $\operatorname{projlim}(\hat{\mathcal{P}})$. Therefore, Lemma 3.3.19 provides that $\overline{\psi(E)}=\operatorname{projlim}(\mathcal{Q})$, where $\mathcal{Q}:=\left\{\overline{\overline{f_{\alpha}}(\psi(E))}, \overline{f_{\alpha \beta}} \upharpoonright_{\overline{\overline{f_{\beta}}(\psi(E))}}, I\right\}$. By the commutativity of the diagram (3.15), we know that

$$
\bar{f}_{\alpha}(\psi(E))=\bar{f}_{\alpha}(j(\phi(E)))=i_{\alpha}\left(f_{\alpha}(\phi(E))\right)=i_{\alpha}\left(\rho_{\alpha}(E)\right)=i_{\alpha}\left(E_{\alpha}\right) .
$$

Hence, $\overline{\overline{f_{\alpha}}(\psi(E))}=\overline{i_{\alpha}\left(E_{\alpha}\right)}=\hat{E}_{\alpha}$ and so

$$
\overline{\psi(E)}=\operatorname{projlim}(\mathcal{Q})=\operatorname{projlim}(\hat{\mathcal{P}}) .
$$

This together with the fact that $\operatorname{projlim}(\hat{\mathcal{P}})$ is complete (see Remark 3.3.15-c)) implies that $\hat{E}$ is topologically isomorphic to $\operatorname{projlim}(\hat{\mathcal{P}})$ by Theorem 3.3.17-I). Therefore, we have proved that

$$
E \stackrel{\phi}{\hookrightarrow} \operatorname{projlim}(\mathcal{P}) \stackrel{j}{\hookrightarrow} \operatorname{projlim}(\hat{\mathcal{P}}) \cong \hat{E} .
$$

If in addition $E$ is complete, then $E=\hat{E}$ and so $\phi$ and $j$ must be also isomorphisms.

Using Remark 3.3.15, we can easily derive from Theorem 3.3.20 the following

## Corollary 3.3.21.

a) Every Hausdorff lmc algebra can be topologically embedded in a cartesian product of Banach algebras.
b) Every Fréchet lmc algebra is topologically isomorphic to the projective limit of a sequence of Banach algebras.

Theorem 3.3.22. Let $(E, \tau)$ be a Hausdorff complete lmc algebra and $\mathcal{M}:=$ $\left\{U_{\alpha}\right\}_{\alpha \in I}$ a basis of neighbourhoods of the origin in ( $E, \tau$ ) consisting of mbarrels. Then:
a) $E$ is unital if and only if each component of its Arens-Michael decomposition w.r.t. $\mathcal{M}$ is a unital Banach algebra.
b) $x \in E$ is invertible if and only if its image into each component of the its Arens-Michael decomposition of $E$ w.r.t. $\mathcal{M}$ is invertible.

Proof.
Let $E=\operatorname{projlim}\left\{\hat{E}_{\alpha}, \overline{f_{\alpha \beta}}, I\right\}$ be the Arens-Michael decomposition of $E$ w.r.t. $\mathcal{M}$ (see Theorem 3.3.20).
a) Suppose that there exists $u \in E$ s.t. for all $y \in E$ we have $u \cdot y=y=y \cdot u$. For any $\alpha \in I$, set $u_{\alpha}:=\bar{\rho}_{\alpha}(u) \in \hat{E}_{\alpha}$. By the surjectivity of $\bar{\rho}_{\alpha}$, we know that for any $x_{\alpha} \in \hat{E}_{\alpha}$ there exists $x \in E$ such that $\bar{\rho}_{\alpha}(x)=x_{\alpha}$ and so we get that:

$$
x_{\alpha} \cdot u_{\alpha}=\bar{\rho}_{\alpha}(x) \bar{\rho}_{\alpha}(u)=\bar{\rho}_{\alpha}(x \cdot u)=\bar{\rho}_{\alpha}(x)=x_{\alpha}
$$

and similarly we obtain $u_{\alpha} x_{\alpha}=x_{\alpha}$, i.e. each $\hat{E}_{\alpha}$ is unital.
Conversely, suppose that for any $\alpha \in I$ there exists $u_{\alpha} \in \hat{E}_{\alpha}$ s.t. $y \cdot u_{\alpha}=$ $y=u_{\alpha} \cdot y$ for all $y \in \hat{E}_{\alpha}$. Then $u:=\left(u_{\alpha}\right)_{\alpha \in I}$ belongs to $\operatorname{projlim}\left\{\hat{E}_{\alpha}, \overline{f_{\alpha \beta}}, I\right\}$ since for all $\alpha \leq \beta$ in $I$ and for all $x_{\alpha} \in \hat{E}_{\alpha}$ we get:

$$
\begin{aligned}
x_{\alpha} \cdot \overline{f_{\alpha \beta}}\left(u_{\beta}\right) & =\bar{\rho}_{\alpha}(x) \cdot \overline{f_{\alpha \beta}}\left(u_{\beta}\right)=\overline{f_{\alpha \beta}}\left(\bar{\rho}_{\beta}(x)\right) \cdot \overline{f_{\alpha \beta}}\left(u_{\beta}\right) \\
& =\overline{f_{\alpha \beta}}\left(\bar{\rho}_{\beta}(x) \cdot u_{\beta}\right)=\overline{f_{\alpha \beta}}\left(\bar{\rho}_{\beta}(x)\right)=\bar{\rho}_{\alpha}(x)=x_{\alpha},
\end{aligned}
$$

i.e. $\overline{f_{\alpha \beta}}\left(u_{\beta}\right)=u_{\alpha}$. As the multiplication in $\operatorname{proj} \lim \left\{\hat{E}_{\alpha}, \overline{f_{\alpha \beta}}, I\right\}$ is defined coordinatewise, it is then clear that $u:=\left(u_{\alpha}\right)_{\alpha \in I}$ is the identity element of the multiplication in projlim $\left\{\hat{E}_{\alpha}, \overline{f_{\alpha \beta}}, I\right\}$, which is therefore a unital algebra.
b) Suppose that $u$ is the identity element of the multiplication in $E$ and that $x \in E$ is invertible, i.e. there exists $y \in E$ s.t. $x \cdot y=u=y \cdot x$. For each $\alpha \in I$, we have already showed that $u_{\alpha}:=\bar{\rho}_{\alpha}(u)$ is the identity element of the multiplication in $\hat{E}_{\alpha}$. Hence, we have

$$
\bar{\rho}_{\alpha}(x) \cdot \bar{\rho}_{\alpha}(y)=\bar{\rho}_{\alpha}(x \cdot y)=\bar{\rho}_{\alpha}(u)=u_{\alpha},
$$

i.e. $\bar{\rho}_{\alpha}(x)$ is invertible in $\hat{E}_{\alpha}$.

Conversely, suppose that $x \in \operatorname{projlim}\left\{\hat{E}_{\alpha}, \overline{f_{\alpha \beta}}, I\right\}$ is s.t. for each $\alpha \in I$ the element $\bar{\rho}_{\alpha}(x)$ is invertible. Then for each $\alpha \in I$ there exists $y_{\alpha} \in \hat{E}_{\alpha}$ s.t. $\bar{\rho}_{\alpha}(x) \cdot y_{\alpha}=u_{\alpha}=y_{\alpha} \cdot \bar{\rho}_{\alpha}(x)$, where $u_{\alpha}$ is the identity element of the multiplication in $\hat{E}_{\alpha}$. Now as we have already showed that $u:=\left(u_{\alpha}\right)_{\alpha \in I}$ is the identity element of the (coordinatewise) multiplication in projlim $\left\{\hat{E}_{\alpha}, \overline{f_{\alpha \beta}}, I\right\}$, it is enough to prove that $\left(y_{\alpha}\right)_{\alpha \in I} \in \operatorname{proj} \lim \left\{\hat{E}_{\alpha}, \overline{f_{\alpha \beta}}, I\right\}$. This is indeed true since for all $\alpha \leq \beta$ in $I$ the following holds

$$
\bar{\rho}_{\alpha}(x) \cdot \overline{f_{\alpha \beta}}\left(y_{\beta}\right)=\overline{f_{\alpha \beta}}\left(\bar{\rho}_{\beta}(x)\right) \cdot \overline{f_{\alpha \beta}}\left(y_{\beta}\right)=\overline{f_{\alpha \beta}}\left(\bar{\rho}_{\beta}(x) \cdot y_{\beta}\right)=\overline{f_{\alpha \beta}}\left(u_{\beta}\right)=y_{\alpha},
$$

and, hence, $\overline{f_{\alpha \beta}}\left(y_{\beta}\right)=y_{\alpha}$.

## Chapter 4

## Symmetric tensor algebras

As usual, we consider only vector spaces over the field $\mathbb{K}$ of real numbers or of complex numbers. The aim of this section is to present a way to explicitly construct an lmc algebra starting from the symmetric tensor algebra of a lc TVS. For this purpose, we will preliminarily introduce the concept of tensor product of vector spaces and then endow it with one of the many topologies which can be defined when the starting space carries an lc structure.

### 4.1 Tensor product of vector spaces

Let us start with a notion which is central in the definition of tensor product.

## Definition 4.1.1.

Let $E, F, M$ be three vector spaces over $\mathbb{K}$ and $\phi: E \times F \rightarrow M$ be a bilinear map. $E$ and $F$ are said to be $\phi$-linearly disjoint if:
(LD) For any $r, s \in \mathbb{N}, x_{1}, \ldots, x_{r}$ linearly independent in $E$ and $y_{1}, \ldots, y_{s}$ linearly independent in $F$, the set $\left\{\phi\left(x_{i}, y_{j}\right): i=1, \ldots, r, j=1, \ldots, s\right\}$ consists of linearly independent vectors in $M$.
or equivalently if:
(LD)' For any $r \in \mathbb{N}$, any $\left\{x_{1}, \ldots, x_{r}\right\}$ finite subset of $E$ and any $\left\{y_{1}, \ldots, y_{r}\right\}$ finite subset of $F$ s.t. $\sum_{i=1}^{r} \phi\left(x_{i}, y_{j}\right)=0$, we have that both the following conditions hold:

- if $x_{1}, \ldots, x_{r}$ are linearly independent in $E$, then $y_{1}=\cdots=y_{r}=0$
- if $y_{1}, \ldots, y_{r}$ are linearly independent in $F$, then $x_{1}=\cdots=x_{r}=0$.

Definition 4.1.2. A tensor product of two vector spaces $E$ and $F$ over $\mathbb{K}$ is a pair $(M, \phi)$ consisting of a vector space $M$ over $\mathbb{K}$ and of a bilinear map $\phi: E \times F \rightarrow M$ (canonical map) s.t. the following conditions are satisfied:
(TP1) The image of $E \times F$ spans the whole space $M$.
(TP2) $E$ and $F$ are $\phi$-linearly disjoint.

The following theorem guarantees that the tensor product of any two vector spaces always exists, satisfies the "universal property" and it is unique up to isomorphisms. For this reason, the tensor product of $E$ and $F$ is usually denoted by $E \otimes F$ and the canonical map by $(x, y) \mapsto x \otimes y$.

Theorem 4.1.3. Let $E, F$ be two vector spaces over $\mathbb{K}$.
(a) There exists a tensor product of $E$ and $F$.
(b) Let $(M, \phi)$ be a tensor product of $E$ and $F$. Let $G$ be any vector space over $\mathbb{K}$, and $b$ any bilinear mapping of $E \times F$ into $G$. There exists a unique linear map $\tilde{b}: M \rightarrow G$ such that the diagram

is commutative.
(c) If $\left(M_{1}, \phi_{1}\right)$ and $\left(M_{2}, \phi_{2}\right)$ are two tensor products of $E$ and $F$, then there is a bijective linear map $u$ such that the diagram

is commutative.
Proof. (see [16, Theorem 4.1.4])

## Examples 4.1.4.

1. Let $n, m \in \mathbb{N}, E=\mathbb{K}^{n}$ and $F=\mathbb{K}^{m}$. Then $E \otimes F=\mathbb{K}^{n m}$ is a tensor product of $E$ and $F$ whose canonical bilinear map $\phi$ is given by:

$$
\begin{array}{rlll}
\phi: & E \times F & \rightarrow \mathbb{K}^{n m} \\
& \left(\left(x_{i}\right)_{i=1}^{n},\left(y_{j}\right)_{j=1}^{m}\right) & \mapsto\left(x_{i} y_{j}\right)_{1 \leq i \leq n, 1 \leq j \leq m}
\end{array}
$$

2. Let $X$ and $Y$ be two sets. For any functions $f: X \rightarrow \mathbb{K}$ and $g: Y \rightarrow \mathbb{K}$, we define:

$$
\begin{array}{rll}
f \otimes g: & X \times Y & \rightarrow \mathbb{K} \\
& (x, y) & \mapsto f(x) g(y)
\end{array}
$$

Let $E$ (resp. $F)$ be the linear space of all functions from $X$ (resp. Y) to $\mathbb{K}$ endowed with the pointwise addition and multiplication by scalars. We
denote by $M$ the linear subspace of the space of all functions from $X \times Y$ to $\mathbb{K}$ spanned by the elements of the form $f \otimes g$ for all $f \in E$ and $g \in F$. Then $M$ is actually a tensor product of $E$ and $F$, i.e. $M=E \otimes F$.

Similarly to how we defined the tensor product of two vector spaces we can define the tensor product of an arbitrary number of vector spaces.

Definition 4.1.5. Let $n \in \mathbb{N}$ with $n \geq 2$ and $E_{1}, \ldots, E_{n}$ vector spaces over $\mathbb{K}$. A tensor product of $E_{1}, \ldots, E_{n}$ is a pair $(M, \phi)$ consisting of a vector space $M$ over $\mathbb{K}$ and of a multilinear map $\phi: E_{1} \times \cdots \times E_{n} \rightarrow M$ (canonical map) s.t. the following conditions are satisfied:
(TP1) The image of $E_{1} \times \cdots \times E_{n}$ spans the whole space $M$.
(TP2) $E_{1}, \ldots, E_{n}$ are $\phi$-linearly disjoint, i.e. for any $r_{1}, \ldots, r_{n} \in \mathbb{N}$ and for any $x_{1}^{(i)}, \ldots, x_{r_{i}}^{(i)}$ linearly independent in $E_{i}(i=1, \ldots, n)$, the set

$$
\left\{\phi\left(x_{j_{1}}^{(1)}, \ldots, x_{j_{n}}^{(n)}\right): j_{1}=1, \ldots, r_{1}, \ldots, j_{n}=1, \ldots, r_{n}\right\}
$$

consists of linearly independent vectors in $M$.
Recall that a map is multilinear if it is linear in each of its variables. As for the case $n=2$ it is possible to show that:
(a) There always exists a tensor product of $E_{1}, \ldots, E_{n}$.
(b) The universal property holds for $E_{1} \otimes \cdots \otimes E_{n}$.
(c) $E_{1} \otimes \cdots \otimes E_{n}$ is unique up to isomorphisms.

### 4.2 The $\pi$-topology on the tensor product of Ic TVS

Given two locally convex TVS $E$ and $F$, there are various ways to construct a topology on the tensor product $E \otimes F$ which makes the vector space $E \otimes F$ in a TVS. Indeed, starting from the topologies on $E$ and $F$, one can define a topology on $E \otimes F$ either relying directly on the seminorms on $E$ and $F$, or using an embedding of $E \otimes F$ in some space related to $E$ and $F$ over which a natural topology already exists. The first method leads to the so-called $\pi$-topology. The second method may lead instead to a variety of topologies, which we are not going to investigate in this course.

Definition 4.2.1 ( $\pi$-topology).
Given two locally convex TVS E and $F$, we define the $\pi$-topology (or projective topology) on $E \otimes F$ to be the finest locally convex topology on this vector space for which the canonical mapping $E \times F \rightarrow E \otimes F$ is continuous. The space $E \otimes F$ equipped with the $\pi$-topology will be denoted by $E \otimes_{\pi} F$.

A basis of neighbourhoods of the origin in $E \otimes_{\pi} F$ is given by the family:

$$
\mathcal{B}:=\left\{\operatorname{conv}_{b}\left(U_{\alpha} \otimes V_{\beta}\right): U_{\alpha} \in \mathcal{B}_{E}, V_{\beta} \in \mathcal{B}_{F}\right\},
$$

where $\mathcal{B}_{E}\left(\right.$ resp. $\left.\mathcal{B}_{F}\right)$ is a basis of neighbourhoods of the origin in $E$ (resp. in $F), U_{\alpha} \otimes V_{\beta}:=\left\{x \otimes y \in E \otimes F: x \in U_{\alpha}, y \in V_{\beta}\right\}$. In fact, on the one hand, the $\pi$-topology is by definition locally convex and so it has a basis $\mathcal{B}$ of convex balanced neighbourhoods of the origin in $E \otimes F$. Then, as the canonical mapping $\phi$ is continuous w.r.t. the $\pi$-topology, we have that for any $C \in \mathcal{B}$ there exist $U_{\alpha} \in \mathcal{B}_{E}$ and $V_{\beta} \in \mathcal{B}_{F}$ s.t. $U_{\alpha} \times V_{\beta} \subseteq \phi^{-1}(C)$. Hence, $U_{\alpha} \otimes V_{\beta}=\phi\left(U_{\alpha} \times V_{\beta}\right) \subseteq C$ and so $\operatorname{conv}_{b}\left(U_{\alpha} \otimes V_{\beta}\right) \subseteq \operatorname{conv}_{b}(C)=C$ which yields that the topology generated by $\mathcal{B}_{\pi}$ is finer than the $\pi$-topology. On the other hand, the canonical map $\phi$ is continuous w.r.t. the topology generated by $\mathcal{B}_{\pi}$, because for any $U_{\alpha} \in \mathcal{B}_{E}$ and $V_{\beta} \in \mathcal{B}_{F}$ we have that $\phi^{-1}\left(\operatorname{conv}_{b}\left(U_{\alpha} \otimes V_{\beta}\right)\right) \supseteq \phi^{-1}\left(U_{\alpha} \otimes V_{\beta}\right)=U_{\alpha} \times V_{\beta}$ which is a neighbourhood of the origin in $E \times F$. Hence, the topology generated by $\mathcal{B}_{\pi}$ is coarser than the $\pi$-topology.

The $\pi$-topology on $E \otimes F$ can be described by means of the seminorms defining the locally convex topologies on $E$ and $F$.

Theorem 4.2.2. Let $E$ and $F$ be two locally convex TVS and let $\mathcal{P}$ (resp. $\mathcal{Q}$ ) be a family of seminorms generating the topology on $E$ (resp. on $F$ ). The $\pi$-topology on $E \otimes F$ is generated by the family of seminorms

$$
\{p \otimes q: p \in \mathcal{P}, q \in \mathcal{Q}\}
$$

where for any $p \in \mathcal{P}, q \in \mathcal{Q}, \theta \in E \otimes F$ we define:
$(p \otimes q)(\theta):=\inf \left\{\sum_{k=1}^{r} p\left(x_{k}\right) q\left(y_{k}\right): \theta=\sum_{k=1}^{r} x_{k} \otimes y_{k},, x_{k} \in E, y_{k} \in F, r \in \mathbb{N}\right\}$.
Proof. (see [16, Proposition 4.3.10 and Theorem 4.3.11])
The seminorm $p \otimes q$ on $E \otimes F$ defined in the previous proposition is called tensor product of the seminorms $p$ and $q$ (or projective cross seminorm)

Proposition 4.2.3. Let $E$ and $F$ be two locally convex TVS. $E \otimes_{\pi} F$ is Hausdorff if and only if $E$ and $F$ are both Hausdorff.

In analogy with the algebraic case (see Theorem 4.1.3-b), we also have a universal property for the space $E \otimes_{\pi} F$.

## Proposition 4.2.4.

Let $E, F$ be locally convex spaces. The $\pi$-topology on $E \otimes_{\pi} F$ is the unique locally convex topology on $E \otimes F$ such that the following property holds:
(UP) For every locally convex space $G$, the algebraic isomorphism between the space of bilinear mappings from $E \times F$ into $G$ and the space of all linear mappings from $E \otimes F$ into $G$ (given by Theorem 4.1.3-b) induces an algebraic isomorphism between $B(E, F ; G)$ and $L(E \otimes F ; G)$, where $B(E, F ; G)$ denotes the space of all continuous bilinear mappings from $E \times F$ into $G$ and $L(E \otimes F ; G)$ the space of all continuous linear mappings from $E \otimes F$ into $G$.

Proof. Let $\tau$ be a locally convex topology on $E \otimes F$ such that the property (UP) holds. Then (UP) holds in particular for $G=(E \otimes F, \tau)$. Therefore, by Theorem 4.1.3-b) the identity id : $E \otimes F \rightarrow E \otimes F$ is the unique linear map such that the diagram

commutes. Hence, we get that $\phi: E \times F \rightarrow E \otimes_{\tau} F$ has to be continuous.
This implies that $\tau \subseteq \pi$ by definition of $\pi$-topology. On the other hand, (UP) also holds for $G=(E \otimes F, \pi)$.


Hence, since by definition of $\pi$-topology $\phi: E \times F \rightarrow E \otimes_{\pi} F$ is continuous, the $i d: E \otimes_{\tau} F \rightarrow E \otimes_{\pi} F$ has to be also continuous. This means that $\pi \subseteq \tau$, which completes the proof.

### 4.3 Tensor algebra and symmetric tensor algebra of a vs

Let $V$ be a vector space over $\mathbb{K}$. For any $k \in \mathbb{N}$, we define the $k$-th tensor power of $V$ as

$$
V^{\otimes k}:=\underbrace{V \otimes \cdots \otimes V}_{\text {k-times }}
$$

and we take by convention $V^{\otimes 0}:=\mathbb{K}$. Then it is possible to show that there exists the following algebraic isomorphism:

$$
\begin{equation*}
\forall n, m \in \mathbb{N}, \quad V^{\otimes n} \otimes V^{\otimes m} \cong V^{\otimes(n+m)} \tag{4.1}
\end{equation*}
$$

We can pack together all tensor powers of $V$ in a unique vector space:

$$
T(V):=\bigoplus_{k=0}^{\infty} V^{\otimes k} .
$$

We define a multiplication over $T(V)$ which makes it into a unital $\mathbb{K}$-algebra.
First of all, let us observe that for any $k \in \mathbb{N}_{0}$ there is a natural embedding $i_{k}: V^{\otimes k} \rightarrow T(V)$. For sake of notational convenience, in the following we will identify each $g \in V^{\otimes k}$ with $i_{k}(g)$. Then every element $f \in T(V)$ can be expressed as $f=\sum_{k=0}^{N} f_{k}$ for some $N \in \mathbb{N}_{0}$ and $f_{k} \in V^{\otimes k}$ for $k=0, \ldots, N$. Using the isomorphism given by (4.1), for any $j, k \in \mathbb{N}$ we can define the following bilinear operation:

$$
\cdot: \begin{array}{ccc}
V^{\otimes k} \times V^{\otimes j} & \rightarrow & V^{\otimes(k+j)} \\
\left(\left(v_{1} \otimes \cdots \otimes v_{k}\right),\left(w_{1} \otimes \cdots \otimes w_{j}\right)\right) & \mapsto & v_{1} \otimes \cdots \otimes v_{k} \otimes w_{1} \otimes \cdots \otimes w_{j} . \tag{4.2}
\end{array}
$$

Hence, we get a multiplication $\cdot: T(V) \times T(V) \rightarrow T(V)$ just by defining for all $f, g \in T(V)$, say $f=\sum_{k=0}^{N} f_{k}$ and $g=\sum_{j=0}^{M} g_{j}$ for some $N, M \in \mathbb{N}_{0}$, $f_{k} \in V^{\otimes k}, g_{j} \in V^{\otimes j}$,

$$
f \cdot g:=\sum_{k=0}^{N} \sum_{j=0}^{M} f_{k} \cdot g_{j},
$$

where $f_{k} \cdot g_{j}$ is the one defined in (4.2). Then we easily see that:
a) . is bilinear on $T(V) \times T(V)$ as it is bilinear on each $V^{\otimes k} \times V^{\otimes j}$ for all $j, k \in \mathbb{N}_{0}$.
b) • is associative, i.e. $\forall f, g, h \in T(V),(f \cdot g) \cdot h=f \cdot(g \cdot h)$. Indeed, if
$f=\sum_{k=0}^{N} f_{k}, g=\sum_{j=0}^{M} g_{j}, h=\sum_{l=0}^{S} h_{l}$ with $N, M, S \in \mathbb{N}_{0}, f_{k} \in V^{\otimes k}$,
$g_{j} \in V^{\otimes j}, h_{l} \in V^{\otimes l}$, then

$$
(f \cdot g) \cdot h=\sum_{k=0}^{N} \sum_{j=0}^{M} \sum_{l=0}^{S}\left(f_{k} \cdot g_{j}\right) \cdot h_{l}=\sum_{k=0}^{N} \sum_{j=0}^{M} \sum_{l=0}^{S} f_{k} \cdot\left(g_{j} \cdot h_{l}\right)=f \cdot(g \cdot h),
$$

where we have just used that $V^{\otimes(k+j)} \otimes V^{\otimes l} \cong V^{\otimes(k+j+l)} \cong V^{\otimes k} \otimes V^{\otimes(j+l)}$ by (4.1).
c) $1 \in \mathbb{K}$ is the identity for the multiplication $\cdot$, since $\mathbb{K}=V^{\otimes 0}$ and for all $f=\sum_{k=0}^{N} f_{k} \in T(V)$ we have $1 \cdot f=\sum_{k=0}^{n}\left(1 \cdot f_{k}\right)=\sum_{k=0}^{N} f_{k}=f$.

Hence, $(T(V), \cdot)$ is a unital $\mathbb{K}$-algebra, which is usually called the tensor algebra of $V$.

Remark 4.3.1. If $\left\{x_{i}\right\}_{i \in \Omega}$ is a basis of the vector space $V$, then each element of $V^{\otimes k}$ can be identified with a polynomial of degree $k$ in the non-commuting variables $\left\{x_{i}\right\}_{i \in \Omega}$ and with coefficients in $\mathbb{K}$. Hence, $T(V)$ is identified with the non-commutative polynomial ring $\mathbb{K}\left\langle x_{i}, i \in \Omega\right\rangle$.

Proposition 4.3.2. Let $V$ be a vector space over $\mathbb{K}$. For any unital $\mathbb{K}$-algebra $(A, *)$ and any linear map $f: V \rightarrow A$, there exists a unique $\mathbb{K}$-algebra homomorphism $\bar{f}: T(V) \rightarrow A$ such that the following diagram commutes

where $i_{1}$ is the natural embedding of $V=V^{\otimes 1}$ into $T(V)$.
Proof.
For any $k \in \mathbb{N}$, we define

$$
\begin{aligned}
f_{k}: \underbrace{V \times \cdots}_{\begin{array}{c}
k \text { times } \\
\left(v_{1}, \ldots, v_{k}\right)
\end{array}} \quad \rightarrow \quad A\left(v_{1}\right) * \cdots * f\left(v_{k}\right)
\end{aligned}
$$

which is multilinear by the linearity of $f$. For $k=0$ we define

$$
\left.\begin{array}{rl}
f_{0}: & \mathbb{K} \\
& \rightarrow \\
& \rightarrow A \\
r & \mapsto
\end{array}\right) r 1_{A} .
$$

By the universal property of $V^{\otimes k}$, we have that there exists a unique linear $\operatorname{map} \sigma_{k}: V^{\otimes k} \rightarrow A$ s.t. $\sigma_{k}\left(v_{1} \otimes \cdots \otimes v_{k}\right)=f_{k}\left(v_{1}, \ldots, v_{k}\right)=f\left(v_{1}\right) * \cdots * f\left(v_{k}\right)$ and for $k=0$ we have $\sigma_{0}(r)=f_{0}(r)=r 1_{A}, \forall r \in \mathbb{K}$. Then, by the universal property of the direct sum, we get that there exists a unique linear map $\bar{f}: T(V) \rightarrow A$ such that $\bar{f}\left(i_{k}\left(v_{1} \otimes \cdots \otimes v_{k}\right)\right)=\sigma_{k}\left(v_{1} \otimes \cdots \otimes v_{k}\right)$


In particular, for $k=1$ we obtain that $\bar{f}\left(i_{1}(v)\right)=\sigma_{1}(v)=f(v)$.
It remains to show that $\bar{f}$ is a $\mathbb{K}$-algebra homomorphism from $T(V)$ to $A$. By construction of $\bar{f}$, we clearly have that $\bar{f}$ is linear and

$$
\bar{f}\left(1_{T(V)}\right)=\bar{f}\left(i_{0}(1)\right)=\sigma_{0}(1)=f_{0}(1)=1_{A}
$$

Let us prove now that for any $x, y \in T(V)$ we get $\bar{f}(x \cdot y)=\bar{f}(x) * \bar{f}(y)$. As $\bar{f}$ is linear, it is enough to show that for any $n, m \in \mathbb{N}$, any $x_{1}, \ldots, x_{n} \in V$ and any $y_{1}, \ldots, y_{m} \in V$, we get

$$
\begin{equation*}
\bar{f}\left(\left(x_{1} \otimes \cdots \otimes x_{n}\right) \cdot\left(y_{1} \otimes \cdots \otimes y_{m}\right)\right)=\bar{f}\left(x_{1} \otimes \cdots \otimes x_{n}\right) * \bar{f}\left(y_{1} \otimes \cdots \otimes y_{m}\right) \tag{4.3}
\end{equation*}
$$

Indeed, by just applying the properties of $\bar{f}$, we obtain that:

$$
\bar{f}\left(x_{1} \otimes \cdots \otimes x_{n}\right)=\sigma_{n}\left(x_{1} \otimes \cdots \otimes x_{n}\right)=f\left(x_{1}\right) * \cdots * f\left(x_{n}\right)
$$

and

$$
\bar{f}\left(y_{1} \otimes \cdots \otimes y_{m}\right)=\sigma_{m}\left(y_{1} \otimes \cdots \otimes y_{m}\right)=f\left(y_{1}\right) * \cdots * f\left(y_{m}\right)
$$

These together with the definition of multiplication in $T(V)$ give that:

$$
\begin{aligned}
\bar{f}\left(x_{1} \otimes \cdots \otimes x_{n}\right) * \bar{f}\left(y_{1} \otimes \cdots \otimes y_{m}\right) & =f\left(x_{1}\right) * \cdots * f\left(x_{n}\right) * f\left(y_{1}\right) * \cdots * f\left(y_{m}\right) \\
& =\bar{f}\left(\left(x_{1} \otimes \cdots \otimes x_{n}\right) \cdot\left(y_{1} \otimes \cdots \otimes y_{m}\right)\right)
\end{aligned}
$$

Consider now the ideal $I$ in $(T(V), \cdot)$ generated by the elements $v \otimes w-$ $w \otimes v$, for all $v, w \in V$. The tensor algebra $T(V)$ factored by this ideal $I$ is denoted by $S(V)$ and called the symmetric (tensor) algebra of $V$. If we denote by $\pi$ the quotient map from $T(V)$ to $S(V)$, then for any $k \in \mathbb{N}_{0}$ and any element $f=\sum_{i=1}^{n} f_{i 1} \otimes \cdots \otimes f_{i k} \in V^{\otimes k}$ (here $n \in \mathbb{N}, f_{i j} \in V$ for $i=1, \ldots, n, j=1, \ldots, k$ and $n \geq 1$ ) we have that

$$
\pi\left(\sum_{i=1}^{n} f_{i 1} \otimes \cdots \otimes f_{i k}\right)=\sum_{i=1}^{n} f_{i 1} \cdots f_{i k}
$$

We define the $k$-th homogeneous component of $S(V)$ to be the image of $V^{\otimes k}$ under $\pi$ and we denoted it by $S(V)_{k}$. Note that $S(V)_{0}=\mathbb{K}$ and $S(V)_{1}=V$. Hence, we have

$$
S(V)=\bigoplus_{k=0}^{\infty} S(V)_{k}
$$

and so every element $f \in S(V)$ can expressed as $f=\sum_{k=0}^{N} f_{k}$ for some $N \in \mathbb{N}$, $f_{k} \in S(V)_{k}$ for $k=0, \ldots, N$.

Remark 4.3.3. If $\left\{x_{i}\right\}_{i \in \Omega}$ is a basis of the vector space $V$, then each element of $S(V)_{k}$ can be identified with a polynomial of degree $k$ in the commuting variables $\left\{x_{i}\right\}_{i \in \Omega}$ and with coefficients in $\mathbb{K}$. Hence, $S(V)$ is identified with the commutative polynomial ring $\mathbb{K}\left[x_{i}: i \in \Omega\right]$.

The universal property of $S(V)$ easily follows from the universal property of $T(V)$.

Proposition 4.3.4. Let $V$ be a vector space over $\mathbb{K}$. For any unital commutative $\mathbb{K}$-algebra $(A, *)$ and any linear map $\psi: V \rightarrow A$, there exists a unique $\mathbb{K}$-algebra homomorphism $\bar{\psi}: S(V) \rightarrow A$ such that the following diagram commutes

i.e. $\bar{\psi} \upharpoonright_{V}=\psi$.

Corollary 4.3.5. Let $V$ be a vector space over $\mathbb{K}$. The algebraic dual $V^{*}$ of $V$ is algebraically isomorphic to $\operatorname{Hom}(S(V), \mathbb{K})$.

Proof. For any $\alpha \in \operatorname{Hom}(S(V), \mathbb{K})$ we clearly have $\alpha \upharpoonright_{V} \in V^{*}$. On the other hand, by Proposition 4.3.4, for any $\ell \in V^{*}$ there exists a unique $\bar{\ell} \in$ $\operatorname{Hom}(S(V), \mathbb{K})$ such that $\bar{\ell} \Gamma_{V}=\ell$.

### 4.4 An Imc topology on the symmetric algebra of a Ic TVS

Let $V$ be a vector space over $\mathbb{K}$. In this section we are going to explain how a locally convex topology $\tau$ on $V$ can be naturally extended to a locally convex topology $\bar{\tau}$ on the symmetric algebra $S(V)$ (see [14]). Let us start by considering the simplest possible case, i.e. when $\tau$ is generated by a single seminorm.

Suppose now that $\rho$ is a seminorm on $V$. Starting from the seminorm $\rho$ on $V$, we are going to construct a seminorm $\bar{\rho}$ on $S(V)$ in three steps:

1. For $k \in \mathbb{N}$, let us consider the projective tensor seminorm on $V^{\otimes k}$ see Theorem 4.2.2, i.e.

$$
\begin{aligned}
\rho^{\otimes k}(g) & :=(\underbrace{\rho \otimes \cdots \otimes \rho}_{k \text { times }})(g) \\
& =\inf \left\{\sum_{i=1}^{N} \rho\left(g_{i 1}\right) \cdots \rho_{k}\left(g_{i k}\right): g=\sum_{i=1}^{N} g_{i 1} \otimes \cdots \otimes g_{i k}, g_{i j} \in V, N \in \mathbb{N}\right\} .
\end{aligned}
$$

2. Denote by $\pi_{k}: V^{\otimes k} \rightarrow S(V)_{k}$ the quotient map $\pi$ restricted to $V^{\otimes k}$ and define $\bar{\rho}_{k}$ to be the quotient seminorm on $S(V)_{k}$ induced by $\rho^{\otimes k}$, i.e.

$$
\begin{aligned}
\bar{\rho}_{k}(f) & :=\inf \left\{\rho^{\otimes k}(g): g \in V^{\otimes k}, \pi_{k}(g)=f\right\} \\
& =\inf \left\{\sum_{i=1}^{N} \rho\left(f_{i 1}\right) \cdots \rho\left(f_{i k}\right): f=\sum_{i=1}^{N} f_{i 1} \cdots f_{i k}, f_{i j} \in V, N \in \mathbb{N}\right\} .
\end{aligned}
$$

Define $\bar{\rho}_{0}$ to be the usual absolute value on $\mathbb{K}$.
3. For any $h \in S(V)$, say $h=h_{0}+\cdots+h_{\ell}, f_{k} \in S(V)_{k}, k=0, \ldots, \ell$, define

$$
\bar{\rho}(f):=\sum_{k=0}^{\ell} \bar{\rho}_{k}\left(f_{k}\right) .
$$

We refer to $\bar{\rho}$ as the projective extension of $\rho$ to $S(V)$.
Proposition 4.4.1. $\bar{\rho}$ is a seminorm on $S(V)$ extending the seminorm $\rho$ on $V$ and $\bar{\rho}$ is also submultiplicative i.e. $\bar{\rho}(f \cdot g) \leq \bar{\rho}(f) \bar{\rho}(g), \forall f, g \in S(V)$

To prove this result we need an essential lemma:
Lemma 4.4.2. Let $i, j \in \mathbb{N}, f \in S(V)_{i}$ and $g \in S(V)_{j}$. If $k=i+j$ then $\bar{\rho}_{k}(f g) \leq \bar{\rho}_{i}(f) \bar{\rho}_{j}(g)$.

## Proof.

Let us consider a generic representation of $f \in S(V)_{i}$ and $g \in S(V)_{j}$, i.e. $f=\sum_{p} f_{p 1} \cdots f_{p i}$ with $f_{p k} \in V$ for $k=1, \ldots, i$ and $g=\sum_{q} g_{q 1} \cdots g_{q j}$ with $g_{q l} \in V$ for $l=1, \ldots, j$. Then $f \cdot g=\sum_{p, q} f_{p 1} \cdots f_{p i} g_{q 1} \cdots g_{q j}$, and so

$$
\begin{aligned}
\bar{\rho}_{k}(f \cdot g) & \leq \sum_{p, q} \rho\left(f_{p 1}\right) \cdots \rho\left(f_{p i}\right) \rho\left(g_{q 1}\right) \cdots \rho\left(g_{q j}\right) \\
& =\left(\sum_{p} \rho\left(f_{p 1}\right) \cdots \rho\left(f_{p i}\right)\right)\left(\sum_{q} \rho\left(g_{q 1}\right) \cdots \rho\left(g_{q j}\right)\right) .
\end{aligned}
$$

Since this holds for any representation of $f$ and $g$, we get $\bar{\rho}_{k}(f g) \leq \bar{\rho}_{i}(f) \bar{\rho}_{j}(g)$.

Proof. (of Proposition 4.4.1).
It is quite straightforward to show that $\bar{\rho}$ is a seminorm on $S(V)$. Indeed

- Let $k \in \mathbb{K}$ and $f \in S(V)$. Consider any representation of $f$, say we take $f=\sum_{j=0}^{n} f_{j}$ with $n \in \mathbb{N}$ and $f_{j} \in S(V)_{j}$ for $j=0, \ldots, n$. Then using the definition of $\bar{\rho}$ and the fact that $\bar{\rho}_{k}$ is a seminorm on $S(V)_{k}$ we get:

$$
\bar{\rho}(k f)=\bar{\rho}\left(\sum_{j=0}^{n} k f_{j}\right)=\sum_{j=0}^{n} \bar{\rho}_{j}\left(k f_{j}\right)=|k| \sum_{j=0}^{n} \bar{\rho}_{j}\left(f_{j}\right)=|k| \bar{\rho}(f) .
$$

- Let $f, g \in S(V)$. Consider any representation of $f$ and $g$, say we take $f=\sum_{j=0}^{n} f_{j}, g=\sum_{i=0}^{m} g_{i}$ with $n, m \in \mathbb{N}, f_{j} \in S(V)_{j}$ for $j=0, \ldots, n$ and $g_{i} \in S(V)_{i}$ for $i=0, \ldots, m$. Take $N:=\max \{n, m\}$. Then we can rewrite $f=\sum_{j=0}^{N} f_{j}$ and $g=\sum_{i=0}^{N} g_{i}$, where $f_{j}=0$ for $j=n+1, \ldots, N$ and $g_{i}=0$ for $i=m+1, \ldots, N$. Therefore, using the definition of $\bar{\rho}$ and the fact that $\bar{\rho}_{k}$ is a seminorm on $S(V)_{k}$, we have

$$
\bar{\rho}(f+g)=\bar{\rho}\left(\sum_{j=0}^{N}\left(f_{j}+g_{j}\right)\right) \leq \sum_{j=0}^{N} \bar{\rho}_{j}\left(f_{j}\right)+\sum_{j=0}^{N} \bar{\rho}_{j}\left(g_{j}\right)=\bar{\rho}(f)+\bar{\rho}(g) .
$$

Also, $\bar{\rho}_{1}=\rho$, so $\bar{\rho}$ restricted to $V$ coincides with $\rho$. Let us finally show that $\bar{\rho}$ is submultiplicative. Let $f=\sum_{i=0}^{m} f_{i}, g=\sum_{j=0}^{n} g_{j}, f_{i} \in S(V)_{i}, g_{j} \in S(V)_{j}$ and set $T:=\{0, \ldots, m\} \times\{0, \ldots, n\}$. Then by using the definition of $\bar{\rho}$, the fact that $\bar{\rho}_{k}$ is a seminorm on $S(V)_{k}$ and Lemma 4.4.2 we obtain

$$
\begin{aligned}
\bar{\rho}(f \cdot g) & =\bar{\rho}\left(\sum_{i=0}^{m} \sum_{j=0}^{n} f_{i} g_{j}\right)=\bar{\rho}\left(\sum_{k=0}^{m+n} \sum_{\substack{(i, j) \in T \\
i+j=k}} f_{i} g_{j}\right)=\sum_{k=0}^{m+n} \bar{\rho}_{k}\left(\sum_{\substack{(i, j) \in T \\
i+j=k}} f_{i} g_{j}\right) \\
& \leq \sum_{k=0}^{m+n} \sum_{\substack{(i, j) \in T \\
i+j=k}} \bar{\rho}_{k}\left(f_{i} g_{j}\right) \leq \sum_{k=0}^{m+n} \sum_{\substack{(i, j) \in T \\
i+j=k}} \bar{\rho}_{i}\left(f_{i}\right) \bar{\rho}_{j}\left(g_{j}\right)=\sum_{i=0}^{m} \sum_{j=0}^{n} \bar{\rho}_{i}\left(f_{i}\right) \bar{\rho}_{j}\left(g_{j}\right) \\
& =\left(\sum_{i=0}^{m} \bar{\rho}_{i}\left(f_{i}\right)\right)\left(\sum_{j=0}^{n} \bar{\rho}_{j}\left(g_{j}\right)\right)=\bar{\rho}(f) \bar{\rho}(g) .
\end{aligned}
$$

Let us now consider $(S(V), \bar{\rho})$ and any other submultiplicative seminormed unital commutative $\mathbb{K}$-algebra $(A, \sigma)$. If $\alpha:(S(V), \bar{\rho}) \rightarrow(A, \sigma)$ is linear and continuous, then clearly $\alpha \upharpoonright_{V}:(V, \rho) \rightarrow(A, \sigma)$ is also continuous. However, if $\psi:(V, \rho) \rightarrow(A, \sigma)$ is linear and continuous, then the unique extension $\bar{\psi}$ given by Proposition 4.3.4 need not be continuous. All one can say in general is the following lemma.

Lemma 4.4.3. If $\psi:(V, \rho) \rightarrow(A, \sigma)$ is linear and continuous, namely $\exists$ $C>0$ such that $\sigma(\psi(v)) \leq C \rho(v) \forall v \in V$, then for any $k \in \mathbb{N}$ we have $\sigma(\bar{\psi}(g)) \leq C^{k} \bar{\rho}_{k}(g) \forall g \in S(V)_{k}$.

Proof.
Let $k \in \mathbb{N}$ and $g \in S(V)_{k}$. Suppose $g=\sum_{i=1}^{N} g_{i 1} \cdots g_{i k}$ with $g_{i j} \in V$ for $j=1, \ldots, N$. Then $\bar{\psi}(g)=\sum_{i=1}^{N} \psi\left(g_{i 1}\right) \cdots \psi\left(g_{i k}\right)$, and so

$$
\begin{aligned}
\sigma(\bar{\psi}(g)) & \leq \sigma\left(\sum_{i=1}^{N} \psi\left(g_{i 1}\right) \cdots \psi\left(g_{i k}\right)\right) \leq \sum_{i=1}^{N} \sigma\left(\psi\left(f_{i 1}\right)\right) \cdots \sigma\left(\psi\left(g_{i k}\right)\right) \\
& \leq \sum_{i=1}^{N} C \rho\left(g_{i 1}\right) \cdots C \rho\left(g_{i k}\right)=C^{k} \sum_{i=1}^{N} \rho\left(g_{i 1}\right) \cdots \rho\left(g_{i k}\right) .
\end{aligned}
$$

As this holds for any representation of $g$, we get $\sigma(\bar{\psi}(g)) \leq C^{k} \bar{\rho}_{k}(g)$.

Proposition 4.4.4. If $\psi:(V, \rho) \rightarrow(A, \sigma)$ has operator norm $\leq 1$, then the induced algebra homomorphism $\bar{\psi}:(S(V), \bar{\rho}) \rightarrow(A, \sigma)$ has operator norm $\leq \sigma(1)$.

Recall that given a linear operator $L$ between two seminormed spaces ( $W_{1}, q_{1}$ ) and ( $W_{2}, q_{2}$ ) we define the operator norm of $L$ as follows:

$$
\|L\|:=\sup _{\substack{w \in W_{1} \\ q_{1}(w) \leq 1}} q_{2}(L(w)) .
$$

Proof.
Suppose $\sigma \not \equiv 0$ on $A$ (if this is the case then there is nothing to prove). Then there exists $a \in A$ such that $\sigma(a)>0$. This together with the fact that $\sigma$ is a submultiplicative seminorm gives that

$$
\begin{equation*}
\sigma(1) \geq 1 \text {. } \tag{4.4}
\end{equation*}
$$

Since $\|\psi\| \leq 1$, we have that $\sigma(\psi(v)) \leq \rho(v), \forall v \in V$. Then we can apply Lemma 4.4.3 and get that

$$
\begin{equation*}
\forall k \in \mathbb{N}, g \in S(V)_{k}, \sigma(\bar{\psi}(g)) \leq \bar{\rho}_{k}(g) \tag{4.5}
\end{equation*}
$$

Now let $f \in S(V)$, i.e. $f=\sum_{k=0}^{m} f_{k}$ with $f_{k} \in S(V)_{k}$ for $k=0, \ldots, m$. Then

$$
\begin{aligned}
\sigma(\bar{\psi}(f)) & =\sigma\left(\sum_{k=0}^{m} \bar{\psi}\left(f_{k}\right)\right) \leq \sum_{k=0}^{m} \sigma\left(\bar{\psi}\left(f_{k}\right)\right) \stackrel{(4.5)}{\leq} \sigma\left(\bar{\psi}\left(f_{0}\right)\right)+\sum_{k=1}^{m} \bar{\rho}_{k}\left(f_{k}\right) \\
& =\sigma\left(f_{0}\right)+\sum_{k=1}^{m} \bar{\rho}_{k}\left(f_{k}\right) \leq \sigma(1) \bar{\rho}\left(f_{0}\right)+\sum_{k=1}^{m} \bar{\rho}_{k}\left(f_{k}\right) \\
& \stackrel{(4.4)}{\leq} \sigma(1) \bar{\rho}\left(f_{0}\right)+\sum_{k=1}^{m} \sigma(1) \bar{\rho}_{k}\left(f_{k}\right)=\sigma(1) \sum_{k=0}^{m} \bar{\rho}_{k}\left(f_{k}\right)=\sigma(1) \bar{\rho}(f) .
\end{aligned}
$$

Hence, $\|\bar{\psi}\| \leq \sigma(1)$.
Using the properties we have showed for the projective extension $\bar{\rho}$ of $\rho$ to $S(V)$, we can easily pass to the case when $V$ is endowed with a locally convex topology $\tau$ (generated by more than one seminorm) and to study how to extend this topology to $S(V)$ in a such a way that the latter becomes an lmc TA.

Let $\tau$ be any locally convex topology on a vector space $V$ over $\mathbb{K}$ and let $\mathcal{P}$ be a directed family of seminorms generating $\tau$. Denote by $\bar{\tau}$ the topology on $S(V)$ determined by the family of seminorms $\mathcal{Q}:=\{\overline{n \rho}: \rho \in \mathcal{P}, n \in \mathbb{N}\}$.

Proposition 4.4.5. $\bar{\tau}$ is an lmc topology on $S(V)$ extending $\tau$ and is the finest lmc topology on $S(V)$ having this property.

Proof. By definition of $\bar{\tau}$ and by Proposition 4.4.1, it is clear that $\mathcal{Q}$ is a directed family of submultiplicative seminorms and so that $\bar{\tau}$ is an lmc topology on $S(V)$ extending $\tau$.It remains to show that $\tau$ is the finest lmc topology with extending $\tau$ to $S(V)$. Let $\mu$ an lmc topology on $S(V)$ s.t. $\mu \upharpoonright_{V}=\tau$, i.e. $\mu$ extends $\tau$ to $S(V)$. Suppose that $\mu$ is finer than $\bar{\tau}$. Let $\mathcal{S}$ be a directed family of submultiplicative seminorms generating $\mu$ and consider the identity map id : $(V, \tau) \rightarrow\left(V, \mu \upharpoonright_{V}\right)$. As by assumption $\mu \upharpoonright_{V}=\tau$, we have that id is continuous and so by Theorem 4.6.3-TVS-I (applied for directed families of seminorms) we get that:

$$
\forall s \in \mathcal{S}, \exists n \in \mathbb{N}, \exists \rho \in \mathcal{P}: s(v)=s((i d(v)) \leq n \rho(v), \forall v \in V
$$

Consider the embedding $i:(V, n \rho) \rightarrow(S(V), q)$. Then $\|i\| \leq 1$ and so, by Proposition 4.4.4, the unique extension $\bar{i}:(S(V), \overline{n \rho}) \rightarrow(S(V), s)$ of $i$ is continuous with $\|\bar{i}\| \leq q(1)$. This gives that

$$
s(f) \leq s(1) \overline{n \rho}(f), \forall f \in S(V) .
$$

Hence, all $s \in \mathcal{F}$ are continuous w.r.t. $\bar{\tau}$ and so $\mu$ must be coarser than $\bar{\tau}$.

## Chapter 5

## Short overview on the moment problem

In this chapter we are going to consider always Radon measures on Hausdorff topological spaces, i.e. non-negative Borel measures which are locally finite and inner regular.

### 5.1 The classical finite-dimensional moment problem

Let $\mu$ be a Radon measure on $\mathbb{R}$. We define the $n$-th moment of $\mu$ as

$$
m_{n}^{\mu}:=\int_{\mathbb{R}} x^{n} \mu(d x)
$$

If all moments of $\mu$ exist and are finite, then we can associate to $\mu$ the sequence of real numbers $\left(m_{n}^{\mu}\right)_{n \in \mathbb{N}_{0}}$, which is said to be the moment sequence of $\mu$. The moment problem exactly addresses the inverse question:

Problem 5.1.1 (The one-dimensional $K$-Moment Problem (KMP)).
Given a closed subset $K$ of $\mathbb{R}$ and a sequence $m=\left(m_{n}\right)_{n \in \mathbb{N}_{0}}$ of real numbers, does there exist a Radon measure $\mu$ on $\mathbb{R}$ s.t. for any $n \in \mathbb{N}_{0}$ we have $m_{n}=m_{n}^{\mu}$ and $\mu$ is supported on $K$, i.e.

$$
m_{n}=\underbrace{\int_{\mathbb{R}} x^{n} \mu(d x)}_{n \text {-th moment of } \mu}, \forall n \in \mathbb{N}_{0} \quad \text { and } \quad \operatorname{supp}(\mu) \subseteq K ?
$$

If such a measure $\mu$ does exist we say that $\mu$ is a $K$-representing measure for $m$ or that $m$ is represented by $\mu$ on $K$.

Note that there is a bijective correspondence between the set $\mathbb{R}^{\mathbb{N}_{0}}$ of all sequences of real numbers and the set $(\mathbb{R}[x])^{*}$ of all linear functional from $\mathbb{R}[x]$ to $\mathbb{R}$.

$$
\begin{array}{llrll}
\mathbb{R}^{\mathbb{N}_{0}} & \rightarrow(\mathbb{R}[x])^{*} & & \\
\left(m_{n}\right)_{n \in \mathbb{N}_{0}} & \mapsto & L_{m}: \begin{array}{l}
\mathbb{R}[x] \\
\\
\end{array} & & \rightarrow \mathbb{R} \\
\left.\left(L(x):=\sum_{j}^{n}\right)\right)_{n \in \mathbb{N}_{0}} & \leftarrow & L & & \mapsto
\end{array}
$$

In virtue of this correspondence, we can always reformulate the KMP in terms of linear functionals

Problem 5.1.2 (The one-dimensional $K$-Moment Problem (KMP)).
Given a closed subset $K$ of $\mathbb{R}$ and a linear functional $L: \mathbb{R}[x] \rightarrow \mathbb{R}$, does there exists a Radon measure $\mu$ on $\mathbb{R}$ s.t.

$$
L(p)=\int_{\mathbb{R}} p(x) \mu(d x), \forall p \in \mathbb{R}[x] \quad \text { and } \quad \operatorname{supp}(\mu) \subseteq K ?
$$

As before, if such a measure exists we say that $\mu$ is a $K$-representing measure for $L$ and that it is a solution to the $K$-moment problem for $L$.

Clearly one can generalize the one-dimensional KMP to higher dimension by considering $\mathbb{R}[\mathbf{x}]:=\mathbb{R}\left[x_{1}, \ldots, x_{d}\right]$ for some $d \in \mathbb{N}$ (see [15, Section 5.2.2]).

Problem 5.1.3 (The $d$-dimensional $K$-Moment Problem (KMP)).
Given a closed subset $K$ of $\mathbb{R}^{d}$ and a linear functional $L: \mathbb{R}[\mathbf{x}] \rightarrow \mathbb{R}$, does there exists a Radon measure $\mu$ on $\mathbb{R}^{d}$ s.t.

$$
L(p)=\int_{\mathbb{R}^{d}} p(\mathbf{x}) \mu(d \mathbf{x}), \forall p \in \mathbb{R}[\mathbf{x}] \quad \text { and } \quad \operatorname{supp}(\mu) \subseteq K ?
$$

It is then very natural to ask the following:

## Questions

- What if we have infinitely many variables, i.e. we consider $\mathbb{R}\left[x_{i}: i \in \Omega\right]$ where $\Omega$ is an infinite index set?
- What if instead of real variables we consider variables in a generic $\mathbb{R}$-vector space $V$ (even infinite dimensional)?
- What if instead of the polynomial ring $\mathbb{R}[\mathbf{x}]$ we take any unital commutative $\mathbb{R}$-algebra $A$ ?
All these possible generalization of the moment problem usually go under the name of infinite dimensional moment problem.


### 5.2 Moment problem for commutative $\mathbb{R}$-algebras

In this section we are going to give a formulation of the moment problem general enough to encompass all the possible generalizations addressed in the previous section. Let us start by introducing some notation and terminology.

Given a unital commutative $\mathbb{R}$-algebra $A$, we denote by $\mathcal{X}(A)$ the character space of $A$ (see Definition 2.4.6). For any $a \in A$, we define the Gelfand transform $\hat{a}: \mathcal{X}(A) \rightarrow \mathbb{R}$ as $\hat{a}(\alpha):=\alpha(a), \forall \alpha \in \mathcal{X}(A)$. We endow the character space $\mathcal{X}(A)$ with the weakest topology $\tau_{\mathcal{X}(A)}$ s.t. all Gelfand transforms are continuous, i.e. $\hat{a}$ is continuous for all $a \in A$.

Remark 5.2.1. $\mathcal{X}(A)$ can be seen as a subset of $\mathbb{R}^{A}$ via the embedding:

$$
\begin{aligned}
\pi: \mathcal{X}(A) & \rightarrow \mathbb{R}^{A} \\
\alpha & \mapsto \pi(\alpha):=(\alpha(a))_{a \in A}=(\hat{a}(\alpha))_{a \in A} .
\end{aligned}
$$

If we equip $\mathbb{R}^{A}$ with the product topology $\tau_{\text {prod }}$, then it can be showed (see [19, Section 5.7]) that $\tau_{\mathcal{X}(A)}$ coincides with the topology induced by $\pi$ on $\mathcal{X}(A)$ from $\left(\mathbb{R}^{A}, \tau_{\text {prod }}\right)$, i.e.

$$
\tau_{\mathcal{X}(A)} \equiv\left\{\pi^{-1}(O): O \in \tau_{\text {prod }}\right\}
$$

The space $\left(\mathcal{X}(A), \tau_{\mathcal{X}(A)}\right)$ is therefore Hausdorff.
Problem 5.2.2 (The $K M P$ for unital commutative $\mathbb{R}$-algebras).
Given a closed subset $K \subseteq \mathcal{X}(A)$ and a linear functional $L: A \rightarrow \mathbb{R}$, does there exist a Radon measure $\mu$ on $\mathcal{X}(A)$ s.t. we have

$$
L(a)=\int_{\mathcal{X}(A)} \hat{a}(\alpha) \mu(d \alpha), \forall a \in A \quad \text { and } \quad \operatorname{supp}(\mu) \subseteq K ?
$$

lc

$$
\begin{aligned}
\alpha(p) & =\alpha\left(\sum_{\beta \in \mathbb{N}_{0}^{d}} p_{\beta} \mathbf{x}^{\beta}\right)=\sum_{\beta \in \mathbb{N}_{0}^{d}} \alpha\left(p_{\beta}\right) \alpha\left(x_{1}\right)^{\beta_{1}} \cdots \alpha\left(x_{d}\right)^{\beta_{d}} \\
& =\sum_{\beta \in \mathbb{N}_{0}^{d}} p_{\beta} \alpha\left(x_{1}\right)^{\beta_{1}} \cdots \alpha\left(x_{d}\right)^{\beta_{d}}=p\left(\alpha\left(x_{1}\right), \ldots, \alpha\left(x_{d}\right)\right) .
\end{aligned}
$$

Conversely, for any $\mathbf{y} \in \mathbb{R}^{d}$ we can define the functional $\alpha_{\mathbf{y}}: \mathbb{R}[\mathbf{x}] \rightarrow \mathbb{R}$ by $\alpha_{\mathbf{y}}(p):=p(\mathbf{y})$ for any $p \in \mathbb{R}[\mathbf{x}]$, which is clearly a $\mathbb{R}$-algebra homomorphism. Hence, we have showed that $X(\mathbb{R}[\mathbf{x}]) \cong \mathbb{R}^{d}$ and via this isomorphism we have that, for any $p \in \mathbb{R}[\mathbf{x}]$, the Gelfand transform $\hat{p}$ is identified with the
polynomial $p$ itself. Using these identifications, we get that Problem 5.2.2 for $A=\mathbb{R}[\mathbf{x}]$ is nothing but Problem 5.1.3.

Let us come back to the general KMP 5.2.2. Fixed a subset $K$ of $\mathcal{X}(A)$, we denote by

$$
\operatorname{Pos}(K):=\{a \in A: \hat{a} \geq 0 \text { on } K\} .
$$

A necessary condition for the existence of a solution to the KMP 5.2.2 is clearly that $L$ is nonnegative on $\operatorname{Pos}(K)$. In fact, if there exists a $K$-representing measure $\mu$ for $L$ then for all $a \in \operatorname{Pos}(K)$ we have

$$
L(a)=\int_{\mathcal{X}(A)} \hat{a}(\alpha) \mu(d \alpha) \geq 0
$$

since $\mu$ is nonnegative and supported on $K$ and $\hat{a}$ is nonnegative on $K$.
It is then natural to ask if the nonnegativity of $L$ on $\operatorname{Pos}(K)$ is also sufficient. For $A=\mathbb{R}[\mathbf{x}]$ a positive answer is provided by the so-called RieszHaviland theorem (see [15, Theorem 5.2.5]). An analogous result also holds in this general setting:

Theorem 5.2.3 (Generalized Riesz-Haviland Theorem). Let $K \subseteq \mathcal{X}(A)$ closed and $L: A \rightarrow \mathbb{R}$ linear. Suppose there exists $p \in A$ such that $\hat{p} \geq 0$ on $K$ and for all $n \in \mathbb{N}$ the set $\{\alpha \in K: \hat{p}(\alpha) \leq n\}$ is compact. Then $L$ has a $K$-representing measure if and only if $L(\operatorname{Pos}(K)) \subseteq[0,+\infty)$.

This theorem provides a complete solution for the $K$ - moment problem 5.2.2 but it is somehow unpractical! In fact, it reduces the solvability of the $K-$ moment problem to the problem of characterizing $\operatorname{Pos}(K)$. To approach to this problem we will try to approximate elements in $\operatorname{Pos}(K)$ with elements of $A$ whose Gelfand transform is "more evidently" non-negative, e.g. sum of even powers of elements of $A$. In this spirit we consider $2 d$-power modules of the algebra $A$ for $d \in \mathbb{N}$.

Definition 5.2.4 (2d-power module).
Let $d \in \mathbb{N}$. $A 2 d$-power module of $A$ is a subset $M$ of $A$ satisfying $1 \in$ $M, M+M \subseteq M$ and $a^{2 d} M \subseteq M$ for each $a \in A$.

In the case $d=1,2 d$-power modules are referred to as quadratic modules. We denote by $\sum A^{2 d}$ the set of all finite sums $\sum a_{i}^{2 d}, a_{i} \in A . \sum A^{2 d}$ is the smallest $2 d$-power module of $A$.

Definition 5.2.5 (Generated $2 d$-power module).
Let $\left\{p_{j}\right\}_{j \in J}$ be an arbitrary subset of elements in A (J can have also infinite cardinality). The $2 d$-power module of $A$ generated by $\left\{p_{j}\right\}_{j \in J}$ is defined as

$$
M:=\left\{\sigma_{0}+\sigma_{1} p_{j_{1}}+\ldots+\sigma_{s} p_{j_{s}}: s \in \mathbb{N}, j_{1}, \ldots, j_{s} \in J, \sigma_{0}, \ldots, \sigma_{s} \in \sum A^{2 d}\right\} .
$$

For any subset $M$ of $A$, we set

$$
X_{M}:=\{\alpha \in \mathcal{X}(A): \hat{a}(\alpha) \geq 0, \forall a \in M\},
$$

which is a closed subset of $\left(\mathcal{X}(A), \tau_{\mathcal{X}(A)}\right)$. If $M=\sum A^{2 d}$ then $X_{M}=\mathcal{X}(A)$. If $M$ is the $2 d$-power module of $A$ generated by $\left\{p_{j}\right\}_{j \in J}$ then $X_{M}:=\{\alpha \in$ $\left.\mathcal{X}(A): \hat{p}_{j}(\alpha) \geq 0, \forall j \in J\right\}$.

Given a $2 d$-power module $M$, let us consider the $X_{M}$-moment problem for a linear functional $L: A \rightarrow \mathbb{R}$. If there exists a $X_{M}$-representing measure $\mu$ for $L$, then it is clear that $L(M) \subseteq[0,+\infty)$ since $M \subseteq \operatorname{Pos}\left(X_{M}\right)$. Under which assumptions does the converse hold?

The answer is positive when the module $M$ is Archimedean. The main ingredient of the proof of this result is the the so-called Jacobi Positivstellensatz, which holds for Archimedean power modules and provides that $\operatorname{Pos}\left(X_{M}\right) \subseteq$ $\bar{M}^{\varphi}$, where $\varphi$ is the finest locally convex topology on $A$. This inclusion together with Proposition 2.4.8 allows to get the desired conclusion by applying of Hahn-Banach and Riesz-Haviland theorems.

Theorem 5.2.6. Let $M$ be an archimedean $2 d$-power module of $A$ and $L$ : $A \rightarrow \mathbb{R}$ a linear functional. L has a $X_{M}$-representing measure if and only if $L(M) \subseteq[0,+\infty)$.

Proof. See [13, Corollary 2.6]. The conclusion can be also obtained as a consequence of [11, Theorem 5.5].

A $2 d$-power module $M$ in $A$ is said to be archimedean if for each $a \in A$ there exists an integer $N$ such that $N \pm a \in M$. If $M$ is a $2 d-$ power module of $A$ which is archimedean then $X_{M}$ is compact. The converse is false in general (see [19, Section 7.3]).

Does Theorem 5.2.6 still hold when $M$ is not Archimedean? Can we find other topologies $\tau$ rather than the finest lc topology $\varphi$ on $A$ such that $\operatorname{Pos}\left(X_{M}\right) \subseteq \bar{M}^{\tau}$ so that we can get a similar result for $\tau$-continuous linear functionals on $A$ ? In order to attack those questions we are going to investigate the KMP for linear functionals on some special kind of topological $\mathbb{R}$-algebras.

### 5.3 Moment problem for submultiplicative seminormed $\mathbb{R}$-algebras

In this section we are going to present some results about Problem 5.2.2 when $A$ a submultiplicative seminormed $\mathbb{R}$-algebra (for more details see [12]).

Let $A$ be a unital commutative $\mathbb{R}$-algebra and $\sigma$ be a submultiplicative seminorm on an $\mathbb{R}$-algebra $A$, i.e. $\sigma(a \cdot b) \leq \sigma(a) \sigma(b)$ for all $a, b \in A(\cdot$ denotes the multiplication in $A$ ). The algebra $A$ together with such a $\sigma$ is called a submultiplicative seminormed $\mathbb{R}$-algebra and is denoted by $(A, \sigma)$.

We denote the set of all $\sigma$-continuous $\mathbb{R}$-algebra homomorphisms from $A$ to $\mathbb{R}$ by $\mathfrak{s p}(\sigma)$, which we refer to as the Gelfand spectrum of $(A, \sigma)$, i.e.

$$
\mathfrak{s p}(\sigma):=\{\alpha \in \mathcal{X}(A): \alpha \text { is } \sigma-\text { continuous }\} .
$$

We endow $\mathfrak{s p}(\sigma)$ with the subspace topology induced by $\left(\mathcal{X}(A), \tau_{\mathcal{X}(A)}\right)$. Then one can show the following two results (see [12] for a proof).

## Lemma 5.3.1.

For any submultiplicative seminormed $\mathbb{R}$-algebra $(A, \sigma)$ we have:

$$
\mathfrak{s p}(\sigma)=\{\alpha \in \mathcal{X}(A):|\alpha(a)| \leq \sigma(a) \text { for all } a \in A\} .
$$

Corollary 5.3.2. The Gelfand spectrum of any submultiplicative seminormed $\mathbb{R}$-algebra $(A, \sigma)$ is compact.

An important closure result useful for the Problem 5.2.2 when $A$ a submultiplicative seminormed $\mathbb{R}$-algebra was proved by M. Ghasemi, S. Kuhlmann and M. Marshall in [12, Theorem 3.7]. We just state it here but we show in details how this result helps to get better conditions than the ones provided by the Generalized Riesz-Haviland theorem.

Theorem 5.3.3. Let $(A, \sigma)$ be a submultiplicative seminormed $\mathbb{R}$-algebra and $M$ is a $2 d$-power module of $A$ (not necessarily Archimedean). Then

$$
\bar{M}^{\rho}=\operatorname{Pos}\left(X_{M} \cap \mathfrak{s p}(\sigma)\right) .
$$

Corollary 5.3.4. Let $(A, \sigma)$ be a submultiplicative seminormed $\mathbb{R}$-algebra, $M$ is a $2 d$-power module of $A$ and $L: A \rightarrow \mathbb{R}$ a linear functional. $L$ has a representing measure supported on $X_{M} \cap \mathfrak{s p}(\sigma)$ if and only if $L$ is $\sigma$-continuous and $L(M) \subseteq[0,+\infty)$.

Proof.
$(\Leftarrow)$ By our hypothesis and Theorem 5.3.3, $L$ is nonnegative on $\operatorname{Pos}\left(X_{M} \cap\right.$ $\mathfrak{s p}(\sigma))$. Hence, by applying Theorem 5.2.3, $L$ has a ( $\left.X_{M} \cap \mathfrak{s p}(\sigma)\right)$-representing measure. ${ }^{1}$

[^9]$(\Rightarrow)$ Suppose that $L$ has a representing measure $\mu$ supported on $X_{M} \cap \mathfrak{s p}(\sigma)$. Then for all $b \in M$ we have
$$
L(b)=\int_{X_{M} \cap \mathfrak{s p}(\sigma)} \hat{b}(\alpha) \mu(d \alpha) \geq 0
$$
since $\mu$ is nonnegative and supported on a subset of $X_{M}$. Therefore, we have got $L(M) \subseteq[0,+\infty)$. Also, we have that for all $a \in A$ :
\[

$$
\begin{aligned}
|L(a)| & \leq \int_{X_{M} \cap \mathfrak{s p}(\sigma)}|\hat{a}(\alpha)| \mu(d \alpha) \\
& =\int_{X_{M} \cap \mathfrak{s p}(\sigma)}|\alpha(a)| \mu(d \alpha) \\
\stackrel{\text { Lemma 5.3.1 }}{\leq} & \int_{X_{M} \cap \mathfrak{s p}(\sigma)} \sigma(a) \mu(d \alpha)=\sigma(a) \mu\left(X_{M} \cap \mathfrak{s p}(\sigma)\right)
\end{aligned}
$$
\]

Note that $\mu\left(X_{M} \cap \mathfrak{s p}(\sigma)\right)$ is finite since $\mu$ is Radon and $X_{M} \cap \mathfrak{s p}(\sigma)$ compact. Hence, $L$ is $\sigma$-continuous.

### 5.4 Moment problem for symmetric algebras of Ic spaces

In this section we are going to present some results about Problem 5.2.2 when $A$ is the symmetric algebra $S(V)$ of a locally convex space $V$ over $\mathbb{R}$ (for more details see [14]).

Let us start with the simplest case, i.e. when $V$ is a $\mathbb{R}$-vector space endowed with a seminorm $\rho$. In Section 5.2, we have showed how to extend the seminorm $\rho$ to a seminorm $\bar{\rho}$ on $S(V)$, which we proved to be submultiplicative by Proposition 4.4.1. Therefore, $(S(V), \bar{\rho})$ is a submultiplicative seminormed $\mathbb{R}$-algebra and so we can apply Corollary 5.3.4, obtaining the following result.

Proposition 5.4.1. Let $(V, \rho)$ be a seminormed $\mathbb{R}$-vector space, $M$ a $2 d$-power module of $S(V)$ and $L: S(V) \rightarrow \mathbb{R}$ a linear functional. $L$ is $\bar{\rho}$-continuous and $L(M) \subseteq[0,+\infty)$ if and only if $\exists!\mu$ on $V^{*}: L(f)=\int_{V^{*}} \hat{f}(\alpha) \mu(d \alpha)$ and $\operatorname{supp} \mu \subseteq X_{M} \cap \bar{B}_{1}^{\|\cdot\|_{\rho}}$, where $\|\cdot\|_{\rho}$ denotes the operator norm on $V^{*}$, i.e. $\|\beta\|_{\rho}:=\sup _{\substack{v \in)^{\prime} \leq 1}}|\beta(v)|$ and $\bar{B}_{1}^{\|\cdot\|_{\rho}}:=\left\{\beta \in V^{*}:\|\beta\|_{\rho} \leq 1\right\}$.

Proof.
By Proposition 4.4.1, we can apply Corollary 5.3.4 to $(S(V), \bar{\rho})$ and obtain that: $L$ is $\bar{\rho}$-continuous and $L(M) \subseteq[0,+\infty)$ if and only if $\exists!\mu$ on $X(S(V))$ :
$L(f)=\int_{X(S(V))} \hat{f}(\alpha) \mu(d \alpha)$ and $\operatorname{supp} \mu \subseteq X_{M} \cap \mathfrak{s p}(\bar{\rho})$. Now by Corollary 4.3.5 we know that $\operatorname{Hom}(S(V), \mathbb{R}) \cong V^{*}$, i.e. $X(S(V)) \cong V^{*}$. Using this isomorphism we can get that $\mathfrak{s p}(\bar{\rho}) \cong \bar{B}_{1}^{\|\cdot\| \rho}$ and so the desired conclusion.

Let us prove that $\mathfrak{s p}(\bar{\rho}) \cong \bar{B}_{1}^{\|\cdot\|_{\rho}}$. Suppose that $\alpha \in \mathfrak{s p}(\bar{\rho})$. Then by Lemma 5.3.1 we have that $|\alpha(f)| \leq \bar{\rho}(f) \forall f \in S(V)$. Clearly this implies that $|\alpha(v)| \leq \rho(v) \forall v \in V$, so $\left\|\alpha \upharpoonright_{V}\right\|_{\rho} \leq 1$, i.e. $\alpha \in \bar{B}_{1}^{\|\cdot\|_{\rho}}$. Conversely, suppose that $\beta \in V^{*}$ s.t. $\|\beta\|_{\rho} \leq 1$. Denote by $\bar{\beta}$ the unique extension of $\beta$ to an $\mathbb{R}$-algebra homomorphism $\bar{\beta}: S(V) \rightarrow \mathbb{R}$. Then, by Proposition 4.4.4, we get that $\|\bar{\beta}\|_{\bar{\rho}} \leq 1$ and so that $|\bar{\beta}(f)| \leq \bar{\rho}(f) \forall f \in S(V)$. Thus $\bar{\beta} \in \mathfrak{s p}(\bar{\rho})$.

We can generalize this result to $(V, \tau)$ locally convex TVS over $\mathbb{R}$ by using Proposition 4.4.5, which provides an extension of $\tau$ to an lmc topology $\bar{\tau}$ to $S(V)$.

Proposition 5.4.2. Let $(V, \tau)$ be a lmc TVS over $\mathbb{R}$ whose topology is generated by a directed family of seminorms $\mathcal{P}$. Let $M$ be a $2 d-$ power module of $S(V)$ and $L: S(V) \rightarrow \mathbb{R}$ a linear functional. $L$ is $\bar{\tau}$-continuous and $L(M) \subseteq[0,+\infty)$ if and only if $\exists!\mu$ on $V^{*}: L(f)=\int_{V^{*}} \hat{f}(\alpha) \mu(d \alpha)$ and $\operatorname{supp} \mu \subseteq X_{M} \cap \bar{B}_{n}^{\|\cdot\|_{\rho}}$, for some $n \in \mathbb{N}$ and $\rho \in \mathcal{P}$.

Proof.
By Proposition 4.4.5, we know that $\bar{\tau}$ is a lmc topology on $S(V)$ generated by the family $\mathcal{Q}:=\{\overline{n \rho}: \rho \in \mathcal{P}, n \in \mathbb{N}\}$. Then Proposition 4.6.1 in TVS-I guarantees that $L$ is $\bar{\tau}$-continuous if and only if there exists $q \in \mathcal{Q}$ s.t. $L$ is $q$-continuous, i.e. there exists $n \in \mathbb{N}$ and $\rho \in \mathcal{P}$ s.t. $L$ is $\overline{n \rho}$-continuous. Thus we reduced to the case of one single seminorm and so we can apply Proposition 5.4.1 and get that: $L$ is $\bar{\tau}$-continuous and $L(M) \subseteq[0,+\infty)$ if and only if $\exists!\mu$ on $V^{*}: L(f)=\int_{V^{*}} \hat{f}(\alpha) \mu(d \alpha)$ and $\operatorname{supp} \mu \subseteq X_{M} \cap \bar{B}_{1}^{\|\cdot\| \|_{n \rho}}$. This yields the conclusion as $\bar{B}_{1}^{\|\cdot\|_{n \rho}}=\bar{B}_{n}^{\|\cdot\| \|_{\rho}}$.

What happens when the assumption of continuity of $L$ is weakened? Can we get results for this moment problem for measures which are not compactly supported? Some results in this direction have been obtained in [17] for the case when $V=\mathcal{C}_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ endowed with the projective topology introduced in Section 1.4. However, there are still many open questions concerning the moment problem in this general framework and we are still far from a complete understanding of the infinite dimensional moment problem.

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[^0]:    ${ }^{1}$ A filter on a set $X$ is a family $\mathcal{F}$ of subsets of $X$ which fulfils the following conditions: (F1) the empty set $\emptyset$ does not belong to $\mathcal{F}$
    (F2) $\mathcal{F}$ is closed under finite intersections
    (F3) any subset of $X$ containing a set in $\mathcal{F}$ belongs to $\mathcal{F}$ (c.f. [15, Section 1.1.1]]).

[^1]:    ${ }^{2}$ A family $\mathcal{B}$ of subsets of $X$ is called a basis of a filter $\mathcal{F}$ if

    1. $\mathcal{B} \subseteq \mathcal{F}$
    2. $\forall A \in \mathcal{F}, \exists B \in \mathcal{B}$ s.t. $B \subseteq A$
    or equivalently if $\forall A, B \in \mathcal{B}, \exists C \in \mathcal{B}$ s.t. $C \subseteq A \cap B$ (c.f. [15, Section 1.1.1])
[^2]:    ${ }^{5}$ A topological space $X$ is said to be (T1) if, given two distinct points of $X$, each lies in a neighborhood which does not contain the other point; or equivalently if, for any two distinct points, each of them lies in an open subset which does not contain the other point.

[^3]:    ${ }^{6}$ Alternative proof:
    $A$ Hausdorff $\stackrel{1.3 .2}{\Longleftrightarrow}\{o\}$ closed in $A \stackrel{\{0\} \text { closed in } \mathbb{K}}{\Longleftrightarrow}\{(0, o)\}$ closed in $A_{1} \stackrel{1.3 .2}{\Longleftrightarrow}\left(A_{1}, \tau_{\text {prod }}\right)$ Hausdorff.

[^4]:    ${ }^{1}$ Every TVS has basis of closed neighbourhoods of the origin. Proof.
    Let $\mathcal{F}(o)$ be the filter of neighbourhoods of the origin in a TVS $X$ and $N \in \mathcal{F}(o)$. Then Theorem 1.2.6 guarantees that there exists $V \in \mathcal{F}(o)$ balanced such that $V-V \subseteq N$. If $x \in \bar{V}$ then $(V+x) \cap V \neq \emptyset$ and so there exist $u, v \in V$ s.t. $u+x=v$, which gives $x=v-u \in V-V \subseteq N$. Hence, $\bar{V} \subseteq N$.

[^5]:    ${ }^{2}$ Alternative proof By Theorem 2.2.12, we know that $\left(X, \tau_{\mathcal{P}}\right)$ is a TVS and that $\mathcal{B}_{\mathcal{P}}:=$ $\left\{\bigcap_{k=1}^{n} \varepsilon{\stackrel{\circ}{U_{p_{i_{k}}}}}: i_{1}, \ldots, i_{n} \in I, n \in \mathbb{N}, 0<\varepsilon \leq 1\right\}$ is a basis of neighbourhoods of the origin. Then $\bigcap_{B \in \mathcal{B}_{\mathcal{P}}} B=\bigcap_{i \in I, \varepsilon>0} \varepsilon \stackrel{\circ}{U_{p_{i}}} \stackrel{(2.9)}{=}\{o\}$ and so Proposition 1.3 .2 gives that $\left(X, \tau_{\mathcal{P}}\right)$ is Hausdorff.

[^6]:    ${ }^{3}$ Since $\mathcal{B}_{d}$ is a basis of neighbourhoods of the origin, $\exists B \in \mathcal{B}_{d}$ s.t. $B \subseteq V$. If $x$ would belong to all elements of the basis then in particular it would be $x \in B$ and so also $x \in V$, contradiction.

[^7]:    ${ }^{1}$ Clearly, each $U_{k} \subset A$ and so $\bigcup_{k \in \mathbb{N}} U_{k} \subseteq A$. Conversely, if $x \in A$, then the continuity of the left multiplication implies that there exists $j \in \mathbb{N}$ such that $x W_{j} \subseteq V$ and so $x \in r_{b}^{-1}(V)$ for all $b \in W_{j}$, i.e. $x \in \bigcup_{k \in \mathbb{N}} U_{k}$.

[^8]:    ${ }^{2}$ We could have also directly showed that the equivalence of the two topologies using their basis of neighbourhoods of the origin. Indeed

    $$
    \begin{aligned}
    & \mathcal{B}_{\text {proj }} \stackrel{(3.7)}{=} \\
    &=\left\{\bigcap_{\alpha \in F} \pi_{\alpha}^{-1}\left(U_{\alpha}\right): F \subseteq I \text { finite, } U_{\alpha} \in \mathcal{B}_{\alpha}, \forall \alpha \in F\right\} \\
    &=\left\{\prod_{\alpha \in F} U_{\alpha} \times \prod_{\alpha \in I \backslash F} E_{\alpha}: F \subseteq I \text { finite, } U_{\alpha} \in \mathcal{B}_{\alpha}, \forall \alpha \in F\right\}=\mathcal{B}_{\text {prod }} .
    \end{aligned}
    $$

[^9]:    ${ }^{1}$ Note that we can apply the Generalized Riesz-Haviland Theorem since $X_{M} \cap \mathfrak{s p}(\sigma)$ is compact in $\left(\mathcal{X}(A), \tau_{\mathcal{X}(A)}\right)$. (This is a direct consequence of Corollary 5.3.2 and of the fact that $X_{M}$ is a closed subset of $\left.\left(\mathcal{X}(A), \tau_{\mathcal{X}(A)}\right)\right)$.

