Topological Algebras

Maria Infusino University of Konstanz

Summer Semester 2018

Contents

 \mathbf{v}

Introduction

1	General Concepts			
	1.1	Brief reminder about algebras over a field	1	
	1.2	Definition and main properties of a topological algebra	3	
	1.3	Hausdorffness and unitizations of a TA	12	
	1.4	Subalgebras and quotients of a TA	15	
2 Locally multiplicative convex algebras				
2	Loc	ally multiplicative convex algebras	21	
2	Loc 2.1	ally multiplicative convex algebras Neighbourhood definition of lmc algebras	21 21	
2	Loc 2.1 2.2	ally multiplicative convex algebrasNeighbourhood definition of lmc algebrasSeminorm characterization of lmc algebras	21 21 26	
2	Loc 2.1 2.2 2.3	ally multiplicative convex algebrasNeighbourhood definition of lmc algebrasSeminorm characterization of lmc algebrasHausdorff lmc algebras	21 21 26 36	
2	Loc 2.1 2.2 2.3 2.4	ally multiplicative convex algebrasNeighbourhood definition of lmc algebrasSeminorm characterization of lmc algebrasHausdorff lmc algebrasThe finest lmc topology	21 26 36 39	
2	Loc 2.1 2.2 2.3 2.4 2.5	ally multiplicative convex algebrasNeighbourhood definition of lmc algebrasSeminorm characterization of lmc algebrasHausdorff lmc algebrasThe finest lmc topologyTopological algebras admitting lmc topologies	21 26 36 39 41	

3	Fur	ther special classes of topological algebras	47	
	3.1	Metrizable and Fréchet algebras	47	
	3.2	Locally bounded algebras	54	
	3.3	Projective limit algebras	61	
		3.3.1 Projective topology	61	
		3.3.2 Projective systems of TAs and their projective limit	65	
		3.3.3 Arens-Michael decomposition	69	
4	Syn	nmetric tensor algebras	79	
	4.1	Tensor product of vector spaces	79	
	4.2	The π -topology on the tensor product of lc TVS	81	
	4.3	Tensor algebra and symmetric tensor algebra of a vs	83	
	4.4	An lmc topology on the symmetric algebra of a lc TVS	87	
5	Sho	ort overview on the moment problem	93	
	5.1	The classical finite-dimensional moment problem	93	
	5.2	Moment problem for commutative \mathbb{R} -algebras	95	
	5.3	Moment problem for submultiplicative seminormed \mathbb{R} -algebras	97	
	5.4	Moment problem for symmetric algebras of lc spaces	99	
Bibliography 1				

The primary sources for these notes are [18] and [3]. However, we also referred to [2] and [25]. The references to results from the theory of topological vector spaces appear in the following according to the enumeration used in [15] and [16].

Introduction

The theory of topological algebras has its first roots in the famous works by Gelfand on "normed rings" of 1939 (see [6, 7, 8, 9]) followed by about fifteen years of successful activity on this subject which culminated in the publication of the book dealing with the commutative theory and its applications. From there the theory of normed and Banach algebras gained more and more importance (see [10] for a thorough account) until, with the development of the theories of topological rings and topological vector spaces, the investigation of general topological algebras became unavoidable. On the one hand, there was a great interest in better understanding which are the advantages of having in the same structure both the properties of topological rings and topological vector spaces. On the other hand it was desirable to understand how far one can go beyond normed and Banach algebras still retaining their distinguished features. The need for such an extension has been apparent since the early days of the theory of general topological algebras, more precisely with the introduction of locally multiplicative convex algebras by Arens in [1] and Michael in [21] (they introduced the notion independently). Moreover, it is worth noticing that the previous demand was due not only to a theoretical interest but also to concrete applications of this general theory to a variety of other disciplines (such as quantum filed theory and more in general theoretical physics). This double impact of the theory of topological algebras is probably the reason for which, after almost 80 years from its foundation, this is still an extremely active subject which is indeed recently enjoying very fast research developments.

Chapter 1

General Concepts

In this chapter we are going to consider vector spaces over the field \mathbb{K} of real or complex numbers which is given the usual euclidean topology defined by means of the modulus.

1.1 Brief reminder about algebras over a field

Let us first recall the basic vocabulary needed to discuss about algebras.

Definition 1.1.1. A \mathbb{K} -algebra A is a vector space over \mathbb{K} equipped with an additional binary operation which is bilinear:

$$\begin{array}{rrrr} A \times A & \to & A \\ (a,b) & \mapsto & a \cdot b \end{array}$$

called vector multiplication.

In other words, $(A, +, \cdot)$ is a ring such that the vector operations are both compatible with the multiplication by scalars in \mathbb{K} .

If a \mathbb{K} -algebra has an associative (resp. commutative) vector multiplication then it is said to be an *associative* (resp. *commutative*) \mathbb{K} -algebra. Furthermore, if a \mathbb{K} -algebra A has an identity element for the vector multiplication (called the unity of A), then A is referred to as *unital*.

Examples 1.1.2.

- 1. The real numbers form a unital associative commutative \mathbb{R} -algebra.
- 2. The complex numbers form a unital associative commutative \mathbb{R} -algebra.
- 3. Given $n \in \mathbb{N}$, the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$ (real coefficients and n variables) equipped with pointwise addition and multiplication is a unital associative commutative \mathbb{R} -algebra.

- The space C(X) of K−valued continuous function on a topological space X equipped with pointwise addition and multiplication is a unital associative commutative K−algebra.
- 5. Given $n \in \mathbb{N}$, the ring $\mathbb{R}^{n \times n}$ of real square matrices of order n equipped with the standard matrix addition and matrix multiplication is a unital associative \mathbb{R} -algebra but not commutative.
- 6. The set of quaternions $\mathbb{H} := \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$ equipped with the componentwise addition and scalar multiplication is a real vector space with basis $\{1, i, j, k\}$. Let us equip \mathbb{H} with the Hamilton product which is defined first on the basis elements by setting

$$i \cdot 1 = 1 = 1 \cdot i$$
, $j \cdot 1 = 1 = 1 \cdot j$, $k \cdot 1 = 1 = 1 \cdot k$, $i^2 = j^2 = k^2 = -1$

$$ij=k, \quad ji=-k, \quad jk=i, \quad kj=-i, \quad ki=j, \quad ik=-j,$$

and then it is extended to all quaternions by using the distributive property and commutativity with real quaternions. Note that the multiplication formulas are equivalent to $i^2 = j^2 = k^2 = ijk = -1$.

Then \mathbb{H} is a unital, associative but non-commutative \mathbb{R} -algebra since e.g. ij = k but ji = -k.

7. The three-dimensional Euclidean space \mathbb{R}^3 equipped with componentwise addition and scalar multiplication and with the vector cross product \wedge as multiplication is a non-unital, non-associative, non-commutative \mathbb{R} -algebra. Non-associative since e.g. $(i \wedge j) \wedge j = k \wedge j = -i$ but $i \wedge (j \wedge j) = i \wedge 0 = 0$, non-commutative since e.g. $i \wedge j = k$ but $j \wedge i = -k$ and non-unital because if there was a unit element u then for any $x \in \mathbb{R}^3$ we would have $u \wedge x = x = x \wedge u$, which is equivalent to say that x is perpendicular to itself and so that x = 0. (Here i = (1,0,0), j = (0,1,0) and k = (0,0,1)).

If we replace the vector cross product by the componentwise multiplication then \mathbb{R}^3 becomes a unital associative commutative \mathbb{R} -algebra with unity (1, 1, 1).

Recall that:

Definition 1.1.3. Let A be a \mathbb{K} -algebra. Then

- 1. A subalgebra B of A is a linear subspace of A closed under vector multiplication, i.e. $\forall b, b' \in B$, $bb' \in B$.
- 2. A left ideal (resp. right ideal) I of A is a linear subspace of A such that $\forall a \in A, \forall b \in I, ab \in I$ (resp. $\forall a \in A, \forall b \in I, ba \in I$. An ideal of A is a linear subspace of A which is simultaneously left and right ideal of A.

- 3. A homomorphism between two K−algebras (A, ·) and (B, *) is a linear map φ : A → B such that φ(a · b) = φ(a) * φ(b) for all a, b ∈ A. Its kernel Ker(φ) is an ideal of A and its image φ(A) is a subalgebra of B. A homomorphism between two unital K−algebras has the additional property that φ(1_A) = 1_B where 1_A and 1_B are respectively the unit element in A and the unit element in B.
- 4. The vector space $A_1 = \mathbb{K} \times A$ equipped with the following operations:

$$\begin{split} (\lambda, a) + (\mu, b) &:= (\lambda + \mu, a + b), \ \forall \lambda, \mu \in \mathbb{K}, a, b \in A \\ \mu(\lambda, a) &:= (\mu\lambda, \mu a), \ \forall \lambda, \mu \in \mathbb{K}, a \in A \\ (\lambda, a) \cdot (\mu, b) &:= (\lambda\mu, \lambda b + \mu a + ab), \ \forall \lambda, \mu \in \mathbb{K}, a, b \in A \end{split}$$

is called the unitization of A.

Proposition 1.1.4. $A \mathbb{K}$ -algebra A can be always embedded in its unitization A_1 which is a unital algebra.

Proof. It is easy to check that A_1 fulfils the assumptions of \mathbb{K} -algebra and that the map

$$e: A \to A_1, a \mapsto (0, a)$$

is an injective homomorphism, i.e. a monomorphism. The unit element of A_1 is given by (1, o) as $(\lambda, a) \cdot (1, o) = (\lambda, a) = (1, o) \cdot (\lambda, a), \forall \lambda \in \mathbb{K}, a \in A$. Identifying a and e(a) for any $a \in A$, we can see A as a subalgebra of A_1 . \Box

1.2 Definition and main properties of a topological algebra

Definition 1.2.1. A \mathbb{K} -algebra A is called a topological algebra (TA) if A is endowed with a topology τ which makes the vector addition and the scalar multiplication both continuous and the vector multiplication separately continuous. (Here \mathbb{K} is considered with the euclidean topology and, $A \times A$ and $\mathbb{K} \times A$ with the corresponding product topologies.)

If the vector multiplication in a TA is jointly continuous then we just speak of a *TA with a continuous multiplication*. Recall that jointly continuous implies separately continuous but the converse is false in general. In several books, the definition of TA is given by requiring a jointly continuous vector multiplication but we prefer here the more general definition according to [18].

An alternative definition of TA can be given in connection to TVS. Let us recall the definition:

Definition 1.2.2. A vector space X over \mathbb{K} is called a topological vector space (TVS) if X is provided with a topology τ which is compatible with the vector space structure of X, i.e. τ makes the vector addition and the scalar multiplication both continuous. (Here \mathbb{K} is considered with the euclidean topology and, $X \times X$ and $\mathbb{K} \times X$ with the corresponding product topologies.)

Then it is clear that

Definition 1.2.3. A topological algebra over \mathbb{K} is a TVS over \mathbb{K} equipped with a separately continuous vector multiplication.

Therefore, TAs inherit all the advantageous properties of TVS. In the following we will try to characterize topologies which make a \mathbb{K} -algebra into a TA. To do that we will make use of the results already available from the theory of TVS and see the further properties brought in by the additional structure of being a TA.

In this spirit, let us first recall that the topology of a TVS is always translation invariant that means, roughly speaking, that any TVS topologically looks about any point as it does about any other point. More precisely:

Proposition 1.2.4.

The filter ¹ $\mathcal{F}(x)$ of neighbourhoods of x in a TVS X coincides with the family of the sets O + x for all $O \in \mathcal{F}(o)$, where $\mathcal{F}(o)$ is the filter of neighbourhoods of the origin o (i.e. neutral element of the vector addition).

(see [15, Corollary 2.1.9]]). This result easily implies that:

Proposition 1.2.5. Let X, Y be two t.v.s. and $f : X \to Y$ linear. The map f is continuous if and only if f is continuous at the origin o.

Proof. (see [15, Corollary 2.1.15-3]]).

Thus, the topology of a TVS (and in particular the one of a TA) is completely determined by the filter of neighbourhoods of any of its points, in particular by the filter of neighbourhoods of the origin o or, more frequently, by a base of neighbourhoods of the origin o. We would like to derive a criterion on a collection of subsets of a \mathbb{K} -algebra A which ensures that it is a basis of neighbourhoods of the origin o for some topology τ making (A, τ) a TA. To this aim let us recall the following result from TVS theory:

¹ A *filter* on a set X is a family \mathcal{F} of subsets of X which fulfils the following conditions:

⁽F1) the empty set \emptyset does not belong to \mathcal{F}

⁽F2) \mathcal{F} is closed under finite intersections

⁽F3) any subset of X containing a set in \mathcal{F} belongs to \mathcal{F}

⁽c.f. [15, Section 1.1.1]]).

Theorem 1.2.6. A filter \mathcal{F} of a vector space X over \mathbb{K} is the filter of neighbourhoods of the origin for some topology τ making X into a TVS iff

- 1. The origin belongs to every set $U \in \mathcal{F}$ 2. $\forall U \in \mathcal{F}, \exists V \in \mathcal{F} \text{ s.t. } V + V \subset U$
- 3. $\forall U \in \mathcal{F}, \forall \lambda \in \mathbb{K} \text{ with } \lambda \neq 0 \text{ we have } \lambda U \in \mathcal{F}$
- 4. $\forall U \in \mathcal{F}, U \text{ is absorbing.}$
- 5. $\forall U \in \mathcal{F}, \exists V \in \mathcal{F} \text{ balanced s.t. } V \subset U.$

Proof. (see [15, Theorem 2.1.10]).

Recall that:

Definition 1.2.7. Let U be a subset of a vector space X.

- 1. U is absorbing (or radial) if $\forall x \in X \exists \rho > 0 \text{ s.t. } \forall \lambda \in \mathbb{K} \text{ with } |\lambda| < \rho$ we have $\lambda x \in U$.
- 2. U is balanced (or circled) if $\forall x \in U, \forall \lambda \in \mathbb{K}$ with $|\lambda| \leq 1$ we have $\lambda x \in U.$

(see also [15, Examples 2.1.12, Proposition 2.1.13]).

A first interesting consequence of Theorem 1.2.6 for TA is that

Lemma 1.2.8. For a TVS to be a TA with continuous multiplication it is necessary and sufficient that the vector multiplication is jointly continuous at the point (o, o).

Proof.

If A is a TA with continuous multiplication, then clearly the multiplication is jointly continuous everywhere and so in particular at (o, o). Conversely, let A be a TVS with multiplication M jointly continuous at the point (o, o) and denote by $\mathcal{F}(o)$ the filter of neighbourhoods of the origin in A. Let $(o, o) \neq i$ $(a,b) \in A \times A$ and $U \in \mathcal{F}(o)$. Then Theorem 1.2.6 guarantees that there exists $V \in \mathcal{F}(o)$ balanced and such that $V + V + V \subset U$. Moreover, the joint continuity of the multiplication at (o, o) gives that there exists $U_1, U_2 \in \mathcal{F}(0)$ such that $U_1U_2 \subset V$. Taking $W := U_1 \cap U_2$ we have $WW \subseteq V$. Also, since W is absorbing, there exists $\rho > 0$ s.t. for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq \rho$ we have $\lambda a \in W, \, \lambda b \in W. \text{ For } \theta := \left\{ \begin{array}{ll} \rho & \text{if } \rho \leq 1 \\ \frac{1}{\rho} & \text{if } \rho > 1 \end{array} \right. \text{, we have both } |\theta| \leq 1 \text{ and } |\theta| \leq \rho.$ Hence,

$$(a + \theta W)(b + \theta W) \subseteq ab + a\theta W + W\theta b + \theta^2 WW \subseteq ab + WW + WW + \theta^2 V \subseteq ab + V + V + V \subseteq ab + U.$$

We showed that $\exists N \in \mathcal{F}(o)$ such that $M^{-1}(ab+U) \supseteq (a+N) \times (b+N)$ which proves that joint continuity of M at the point (a, b).

5

We are now ready to give a characterization for a basis² of neighbourhoods of the origin in a TA (resp. TA with continuous multiplication).

Theorem 1.2.9. A non-empty collection \mathcal{B} of subsets of a \mathbb{K} -algebra A is a basis of neighbourhoods of the origin for some topology making A into a TA if and only if

a) B is a basis of neighbourhoods of o for a topology making A into a TVS.
b) ∀U ∈ B, ∀a ∈ A, ∃V, W ∈ B s.t. aV ⊆ U and Wa ⊆ U.

Proof.

Let (A, τ) be a TA and \mathcal{B} be a basis of neighbourhoods of the origin of A. Then (A, τ) is in particular a TVS and so (a) holds. Also by definition of TA, the multiplication is separately continuous which means for any $a \in A$ the maps $L_a(y) = ay$ and $R_a(y) = ya$ are both continuous everywhere in A. Then by Proposition 1.2.5 they are continuous at o, i.e. $\forall U \in \mathcal{B}, \forall a \in A, \exists V, W \in \mathcal{B}$ s.t. $V \subset L_a^{-1}(U)$ and $W \subset R_a^{-1}(U)$, i.e. $aV \subseteq U$ and $Wa \subseteq U$, that is (b).

Conversely, suppose that \mathcal{B} is a collection of subsets of a \mathbb{K} -algebra A fulfilling (a) and (b). Then (a) guarantees that there exists a topology τ having \mathcal{B} as basis of neighbourhoods of o and such that (A, τ) is a TVS. Hence, as we have already observed, (b) means that both L_a and R_a are continuous at o and so by Proposition 1.2.5 continuous everywhere. This yields that the vector multiplication on A is separately continuous and so that (A, τ) is a TA. \Box

Theorem 1.2.10. A non-empty collection \mathcal{B} of subsets of a \mathbb{K} -algebra A is a basis of neighbourhoods of the origin for some topology making A into a TAwith continuous multiplication if and only if

a) \mathcal{B} is a basis of neighbourhoods of o for a topology making A into a TVS. b') $\forall U \in \mathcal{B}, \exists V \in \mathcal{B} \text{ s.t. } VV \subseteq U.$

Proof. (Sheet 1).

Examples 1.2.11.

1. Every \mathbb{K} -algebra A endowed with the trivial topology τ (i.e. $\tau = \{\emptyset, A\}$) is a TA.

1. $\mathcal{B} \subseteq \mathcal{F}$

²A family \mathcal{B} of subsets of X is called a *basis of a filter* \mathcal{F} if

^{2.} $\forall A \in \mathcal{F}, \exists B \in \mathcal{B} \text{ s.t. } B \subseteq A$

or equivalently if $\forall A, B \in \mathcal{B}, \exists C \in \mathcal{B} \text{ s.t. } C \subseteq A \cap B \text{ (c.f. [15, Section 1.1.1])}$

2. Let S be a non-emptyset and \mathbb{K}^S be the set of all functions from S to \mathbb{K} equipped with pointwise operations and the topology ω of pointwise convergence (or simple convergence), i.e. the topology generated by

$$\mathcal{B} := \{ W_{\varepsilon}(x_1, \dots, x_n) : n \in \mathbb{N}, x_1, \dots, x_n \in S, \varepsilon > 0 \},\$$

where $W_{\varepsilon}(x_1, \ldots, x_n) := \{ f \in \mathbb{K}^S : f(x_i) \in B_{\varepsilon}(0), i = 1, \ldots, n \}$ and $B_{\varepsilon}(0) = \{ k \in \mathbb{K} : |k| \leq \varepsilon \}$. Then (\mathbb{K}^S, ω) is a TA with continuous multiplication. Indeed, for any $n \in \mathbb{N}, x_1, \ldots, x_n \in S, \varepsilon > 0$ we have that

$$W_{\sqrt{\varepsilon}}(x_1, \dots, x_n) W_{\sqrt{\varepsilon}}(x_1, \dots, x_n) = \{ fg : f(x_i), g(x_i) \in B_{\sqrt{\varepsilon}}(0), i = 1, \dots, n \}$$
$$\subseteq \{ h : h(x_i) \in B_{\varepsilon}(0), i = 1, \dots, n \}$$
$$= W_{\varepsilon}(x_1, \dots, x_n).$$

As it is also easy to show that (\mathbb{K}^S, ω) is a TVS, the conclusion follows by Theorem 1.2.10.

Two fundamental classes of TA are the following ones:

Definition 1.2.12 (Normed Algebra). A normed algebra is a \mathbb{K} -algebra A endowed with the topology induced by a submultiplicative norm $\|\cdot\|$, i.e. $\|xy\| \leq \|x\| \|y\|$, $\forall x, y \in A$.

Definition 1.2.13 (Banach Algebra). A normed algebra whose underlying space is Banach (i.e. complete normed space) is said to be a Banach algebra.

Proposition 1.2.14. Any normed algebra is a TA with continuous multiplication.

Proof.

Let $(A, \|\cdot\|)$ be a normed algebra. It is easy to verify that the topology τ induced by the norm $\|\cdot\|$ (i.e. the topology generated by the collection $\mathcal{B} := \{B_{\varepsilon}(o) : \varepsilon > 0\}$, where $B_{\varepsilon}(o) := \{x \in A : \|x\| \le \varepsilon\}$) makes A into a TVS. Moreover, the submultiplicativity of the norm $\|\cdot\|$ ensures that for any $\varepsilon > 0$ we have: $B_{\sqrt{\varepsilon}}(o)B_{\sqrt{\varepsilon}}(o) \subseteq B_{\varepsilon}(o)$. Hence, \mathcal{B} fulfills both a) and b') in Theorem 1.2.10 and so we get the desired conclusion.

Examples 1.2.15.

1. Let $n \in \mathbb{N}$. \mathbb{K}^n equipped with the componentwise operations of addition, scalar and vector multiplication, and endowed with the supremum norm $||x|| := \max_{i=1,...,n} |x_i|$ for all $x := (x_1,...,x_n) \in \mathbb{K}^n$ is a Banach algebra. 2. Let $n \in \mathbb{N}$. The algebra $\mathbb{R}^{n \times n}$ of all real square matrices of order n equipped with the following norm is a Banach algebra:

$$||A|| := \sup_{x \in \mathbb{R}^n \setminus \{o\}} \frac{|Ax|}{|x|}, \forall A \in \mathbb{R}^{n \times n},$$

where $|\cdot|$ is the usual euclidean norm on \mathbb{R}^n . Indeed, from the previous example it is easy to see that $(\mathbb{R}^{n \times n}, \|\cdot\|)$ is a Banach space. Also, for any $A, B \in \mathbb{R}^{n \times n}$ we have that:

$$||AB|| = \sup_{x \in \mathbb{R}^n \setminus \{o\}} \frac{|A(Bx)|}{|x|} \le ||A|| \sup_{x \in \mathbb{R}^n \setminus \{o\}} \frac{|Bx|}{|x|} = ||A|| ||B||.$$

3. Let (X, τ) be a topological space and $C_c(X)$ the set of all \mathbb{K} -valued continuous functions with compact support. If we equip $C_c(X)$ with the pointwise operations and the supremum norm $||f|| := \sup_{x \in X} |f(x)|$, then $(C_c(X), ||\cdot||)$ is a Banach algebra.

Before coming back to general TA, let us observe a further nice property of normed and so of Banach algebras, which will allow us to assume w.l.o.g. that in a unital normed algebra the unit has always unitary norm.

Proposition 1.2.16. If (A, p) is a unital normed algebra with unit 1_A , then there always exists a subultiplicative norm q on A equivalent to p and such that $q(1_A) = 1$.

Proof. Suppose that $p(1_A) \neq 1$ and define

$$q(a) := \sup_{x \in A \setminus \{o\}} \frac{p(ax)}{p(x)}, \, \forall a \in A.$$

Immediately from the definition, we see that $q(1_A) = 1$ and $p(ay) \le q(a)p(y)$ for all $a, y \in A$. The latter implies at once that

$$p(a) = p(a1_A) \le q(a)p(1_A), \ \forall a \in A$$

$$(1.1)$$

and

$$q(ab) = \sup_{x \in A \setminus \{o\}} \frac{p(abx)}{p(x)} \le \sup_{x \in A \setminus \{o\}} \frac{q(a)p(bx)}{p(x)} = q(a)q(b), \ \forall a, b \in A.$$
(1.2)

Moreover, since p is submultiplicative, we have that for all $a \in A$

$$q(a) \le \sup_{x \in A \setminus \{o\}} \frac{p(a)p(x)}{p(x)} = p(a)$$

The latter together with (1.1) guarantees that q is equivalent to p, while (1.2) its submultiplicativity.

So far we have seen only examples of TA with continuous multiplication. In the following example, we will introduce a TA whose multiplication is separately continuous but not jointly continuous.

Example 1.2.17.

Let $(H, \langle \cdot, \cdot, \rangle)$ be an infinite dimensional separable Hilbert space over \mathbb{K} . Denote by $\|\cdot\|_H$ the norm on H defined as $\|x\|_H := \sqrt{\langle x, x \rangle}$ for all $x \in H$, and by L(H) the set of all linear and continuous maps from H to H. The set L(H) equipped with the pointwise addition \mathfrak{a} , the pointwise scalar multiplication \mathfrak{m} and the composition of maps \circ as multiplication is a \mathbb{K} -algebra.

Let τ_w be the **weak operator topology** on L(H), i.e. the coarsest topology on L(H) such that all the maps $E_{x,y} : L(H) \to \mathbb{K}, T \mapsto \langle Tx, y \rangle$ $(x, y \in H)$ are continuous. A basis of neighbourhoods of the origin in $(L(H), \tau_w)$ is given by:

$$\mathcal{B}_w := \left\{ V_{\varepsilon}(x_i, y_i, n) : \varepsilon > 0, n \in \mathbb{N}, x_1, \dots, x_n, y_1, \dots, y_n \in H \right\},\$$

where $V_{\varepsilon}(x_i, y_i, n) := \{ W \in L(H) : |\langle W x_i, y_i \rangle| < \varepsilon, i = 1, \dots, n \}.$

•
$$(L(H), \tau_w)$$
 is a TA.

For any $\varepsilon > 0$, $n \in \mathbb{N}$, $x_1, \ldots, x_n, y_1, \ldots, y_n \in H$, using the bilinearity of the inner product we easily have:

$$V_{\frac{\varepsilon}{2}}(x_i, y_i, n) \times V_{\frac{\varepsilon}{2}}(x_i, y_i, n) = \bigcap_{i=1}^n \left\{ (T, S) : |\langle Tx_i, y_i \rangle| < \frac{\varepsilon}{2}, |\langle Sx_i, y_i \rangle| < \frac{\varepsilon}{2} \right\}$$
$$\subseteq \bigcap_{i=1}^n \left\{ (T, S) : |\langle (T+S)x_i, y_i \rangle| < \varepsilon \right\}$$
$$= \left\{ (T, S) : (T+S) \in V_{\varepsilon}(x_i, y_i, n) \right\}$$
$$= a^{-1}(V_{\varepsilon}(x_i, y_i, n)),$$
$$B_1(0) \times V_{\varepsilon}(x_i, y_i, n) = \bigcap_{i=1}^n \left\{ (\lambda, T) \in \mathbb{K} \times L(H) : |\lambda| < 1, |\langle Tx_i, y_i \rangle| < \varepsilon \right\}$$

$$\subseteq \bigcap_{i=1}^{n} \{ (\lambda, T) : |\langle (\lambda T) x_i, y_i \rangle| < \varepsilon \} = \mathbb{m}^{-1} (V_{\varepsilon}(x_i, y_i, n))$$

which prove that a and m are both continuous. Hence, $(L(H), \tau_w)$ is a TVS.

Furthermore, we can show that the multiplication in $(L(H), \tau_w)$ is separately continuous. For a fixed $T \in L(H)$ denote by T^* the adjoint of T and set $z_i := T^*y_i$ for i = 1, ..., n. Then

$$T \circ V_{\varepsilon}(x_i, z_i, n) = \{T \circ S : |\langle Sx_i, z_i \rangle| < \varepsilon, i = 1, \dots, n\}$$
$$\subseteq \{W \in L(H) : |\langle Wx_i, y_i \rangle| < \varepsilon, i = 1, \dots, n\} = V_{\varepsilon}(x_i, y_i, n),$$

where in the latter inequality we used that

$$|\langle (T \circ S)x_i, y_i \rangle| = |\langle T(Sx_i), y_i \rangle| = |\langle Sx_i, T^*y_i \rangle| = |\langle Sx_i, z_i \rangle| < \varepsilon.$$

Similarly, we can show that $V_{\varepsilon}(x_i, z_i, n) \circ T \subseteq V_{\varepsilon}(x_i, y_i, n)$. Hence, \mathcal{B}_w fulfills a) and b) in Theorem 1.2.9 and so we have that $(L(H), \tau_w)$ is a TA.

• the multiplication in $(L(H), \tau_w)$ is not jointly continuous.

Let us preliminarily observe that a sequence $(W_j)_{j\in\mathbb{N}}$ of elements in L(H)converges to $W \in L(H)$ w.r.t. τ_w , in symbols $W_j \xrightarrow{\tau_w} W$, if and only if for all $x, y \in H$ we have $\langle W_j x, y \rangle \to \langle W x, y \rangle^3$. As H is separable, there exists a countable orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ for H. Define $S \in L(H)$ such that $S(e_1) := o$ and $S(e_k) := e_{k-1}$ for all $k \in \mathbb{N}$ with $k \geq 2$. Then the operator

$$T_n := S^n = \left(\underbrace{S \circ \cdots \circ S}_{n \text{ times}}\right), \quad n \in \mathbb{N}$$
(1.3)

is s.t. $T_n \xrightarrow{\tau_w} o$ as $n \to \infty$. Indeed, $\forall x \in H, \exists ! \lambda_k \in \mathbb{K} : x = \sum_{k=1}^{\infty} \lambda_k e_k^4$ so

$$\|T_n x\| = \left\|\sum_{k=1}^{\infty} \lambda_k T_n(e_k)\right\| = \left\|\sum_{k=n+1}^{\infty} \lambda_k T_n(e_k)\right\| = \left\|\sum_{k=n+1}^{\infty} \lambda_k e_{k-n}\right\|$$
$$= \left\|\sum_{k=1}^{\infty} \lambda_{k+n} e_k\right\| \stackrel{4}{=} \sum_{k=1}^{\infty} |\lambda_{k+n}|^2 = \sum_{k=n+1}^{\infty} |\lambda_k|^2 \to 0, \text{ as } n \to \infty$$

which implies that $\langle T_n x, y \rangle \to 0$ as $n \to \infty$ since $|\langle T_n x, y \rangle| \le ||T_n x|| ||y||$.

Moreover, the adjoint of S is the continuous linear operator $S^* : H \to H$ such that $S^*(e_k) = e_{k+1}$ for all $k \in \mathbb{N}$. Hence, for any $n \in \mathbb{N}$ we have that $T_n^* = (S^n)^* = (S^*)^n$ and we can easily show that also $T_n^* \xrightarrow{\tau_W} o$. In fact, for any $x, y \in H$ we have that $|\langle T_n^* x, y \rangle| = |\langle x, T_n y \rangle| \leq ||x|| ||T_n y|| \to 0$ as $n \to \infty$. However, we have $S^*S = I$ where I denotes the identity map on H, which gives in turn that $T_n^* \circ T_n = I$ for any $n \in \mathbb{N}$. Hence, for any $n \in \mathbb{N}$ and any $x, y \in H$ we have that $\langle (T_n^* \circ T_n) x, y \rangle = \langle x, y \rangle$ and so that $T_n^* \circ T_n \xrightarrow{\tau_W} o$ as $n \to \infty$, which proves that \circ is not jointly continuous.

³Indeed, we have

$$\begin{array}{ll} W_j \xrightarrow{\tau_W} W & \iff & \forall \varepsilon > 0, n \in \mathbb{N}, x_i, y_i \in H, \exists \overline{j} \in \mathbb{N} : \forall j \geq \overline{j}, W_j - W \in V_{\varepsilon}(x_i, y_i, n) \\ & \iff & \forall \varepsilon > 0, n \in \mathbb{N}, x_i, y_i \in H, \exists \overline{j} \in \mathbb{N} : \forall j \geq \overline{j}, |\langle (W_j - W) x_i, y_i \rangle| < \varepsilon \\ & \iff & \forall n \in \mathbb{N}, x_i, y_i \in H, \langle (W_j - W) x_i, y_i \rangle \to 0, \text{ as } j \to \infty \\ & \iff & \forall x, y \in H, \langle (W_j - W) x, y \rangle \to 0, \text{ as } j \to \infty. \end{array}$$

⁴Recall that if $\{h_i\}_{i \in I}$ is an orthonormal basis of a Hilbert space H then for each $y \in H$ $y = \sum_{i \in I} \langle y, h_i \rangle h_i$ and $\|y\|^2 = \sum_{i \in I} |\langle y, h_i \rangle|^2$ (see e.g. [22, Theorem II.6] for a proof) Let τ_s be the strong operator topology or topology of pointwise convergence on L(H), i.e. the coarsest topology on L(H) such that all the maps $E_x : L(H) \to H, T \mapsto Tx \ (x \in H)$ are continuous. A basis of neighbourhoods of the origin in $(L(H), \tau_s)$ is given by:

$$\mathcal{B}_s := \left\{ U_{\varepsilon}(x_i, n) : \varepsilon > 0, n \in \mathbb{N}, x_1, \dots, x_n \in H \right\},\$$

where $U_{\varepsilon}(x_i, n) := \{T \in L(H) : ||Tx_i||_H < \varepsilon, i = 1, \dots, n\}.$

• $(L(H), \tau_s)$ is a TA.

For any r > 0, denote by $B_r(o)$ (resp. $B_r(0)$) the open unit ball centered at o in H (resp. at 0 in \mathbb{K}). Then for any $\varepsilon > 0$, $n \in \mathbb{N}$, $x_1, \ldots, x_n \in H$ we have:

$$U_{\frac{\varepsilon}{2}}(x_i, n) \times U_{\frac{\varepsilon}{2}}(x_i, n) = \left\{ (T, S) : Tx_i, Sx_i \in B_{\frac{\varepsilon}{2}}(o), i = 1, \dots, n \right\}$$
$$\subseteq \left\{ (T, S) : \| (T + S)x_i \|_H < \varepsilon, i = 1, \dots, n \right\}$$
$$= \left\{ (T, S) : (T + S) \in U_{\varepsilon}(x_i, n) \right\} = a^{-1}(U_{\varepsilon}(x_i, n))$$

$$B_1(0) \times U_{\varepsilon}(x_i, n) = \{ (\lambda, T) \in \mathbb{K} \times L(H) : |\lambda| < 1, ||Tx_i||_H < \varepsilon, i = 1, \dots, n \}$$
$$\subseteq \{ (\lambda, T) : ||(\lambda T)x_i||_H < \varepsilon, i = 1, \dots, n \} = \mathrm{m}^{-1}(U_{\varepsilon}(x_i, n))$$

which prove that a and m are both continuous.

Furthermore, we can show that the multiplication in $(L(H), \tau_s)$ is separately continuous. Fixed $T \in L(H)$, its continuity implies that $T^{-1}(B_{\varepsilon}(o))$ is a neighbourhood of o in H and so that there exists $\eta > 0$ such that $B_{\eta}(o) \subseteq$ $T^{-1}(B_{\varepsilon}(o))$. Therefore, we get:

$$T \circ U_{\eta}(x_i, n) = \{T \circ S : S \in L(H) \text{ with } Sx_i \in B_{\eta}(o), i = 1, \dots, n\}$$
$$\subseteq \{W \in L(H) : Wx_i \in B_{\varepsilon}(o), i = 1, \dots, n\}$$
$$= U_{\varepsilon}(x_i, n),$$

where in the latter inequality we used that

$$(T \circ S)x_i = T(Sx_i) \in T(B_\eta(o)) \subseteq T(T^{-1}(B_\varepsilon(o))) \subseteq B_\varepsilon(o).$$

Similarly, we can show that $U_{\eta}(x_i, n) \circ T \subseteq U_{\varepsilon}(x_i, n)$. Hence, \mathcal{B}_s fulfills a) and b) in Theorem 1.2.9 and so we have that $(L(H), \tau_s)$ is a TA.

• the multiplication in $(L(H), \tau_s)$ is not jointly continuous

It is enough to show that there exists a neighbourhood of the origin in $(L(H), \tau_s)$ which does not contain the product of any other two such neighbourhoods. More precisely, we will show $\exists \varepsilon > 0$, $\exists x_0 \in H \ s.t. \ \forall \varepsilon_1, \varepsilon_2 > 0$, $\forall p, q \in \mathbb{N}$, $\forall x_1, \ldots, x_p, y_1, \ldots, y_q \in H \text{ we have } U_{\varepsilon_1}(x_i, p) \circ U_{\varepsilon_2}(y_i, q) \not\subseteq U_{\varepsilon}(x_0), \text{ i.e. there}$ exist $A \in U_{\varepsilon_1}(x_i, p)$ and $B \in U_{\varepsilon_2}(y_i, q)$ with $B \circ A \notin U_{\varepsilon}(x_0)$.

Choose $0 < \varepsilon < 1$ and $x_0 \in H$ s.t. $||x_0|| = 1$. For any $\varepsilon_1, \varepsilon_2 > 0, p, q \in \mathbb{N}$, $x_1, \ldots, x_p, y_1, \ldots, y_q \in H$, take

$$0 < \delta < \frac{\varepsilon_2}{\max_{i=1,\dots,q} \|y_i\|} \tag{1.4}$$

and $n \in \mathbb{N}$ such that

$$||T_n(x_k)|| < \delta \varepsilon_1, \text{ for } k = 1, \dots, p,$$
(1.5)

where T_n is defined as in (1.3). (Note that we can choose such an n as we showed above that $||T_jx|| \to 0$ as $j \to \infty$). Setting $A := \frac{1}{\delta}T_n$ and $B := \delta T_n^*$ we get that:

$$||Ax_k|| = \frac{1}{\delta} ||T_n x_k|| \stackrel{(1.5)}{<} \varepsilon_1, \text{ for } k = 1, \dots, p$$

and

$$||By_i|| = \delta ||T_n^*y_i|| \stackrel{(4)}{=} \delta ||y_i|| \stackrel{(1.4)}{<} \varepsilon_2, \text{ for } i = 1, \dots, q$$

Hence, $A \in U_{\varepsilon_1}(x_i, p)$ and $B \in U_{\varepsilon_2}(y_i, q)$ but $B \circ A \notin U_{\varepsilon}(x_0)$ because

$$||(B \circ A)x_0|| = ||(T_n^*T_n)x_0|| = ||x_0|| = 1 > \varepsilon$$

Note that L(H) endowed with the **operator norm** $\|\cdot\|$ is instead a normed algebra and so has jointly continuous multiplication. Recall that the operator norm is defined by $\|T\| := \sup_{x \in H \setminus \{o\}} \frac{\|Tx\|_H}{\|x\|_H}, \ \forall T \in L(H).$

1.3 Hausdorffness and unitizations of a TA

Topological algebras are in particular topological spaces so their Hausdorfness can be established just by verifying the usual definition of Hausdorff topological space.

Definition 1.3.1. A topological space X is said to be Hausdorff or (T2) if any two distinct points of X have neighbourhoods without common points; or equivalently if two distinct points always lie in disjoint open sets.

However, a TA is more than a mere topological space but it is also a TVS. This provides TAs with the following characterization of their Hausdorfness which holds in general for any TVS.

Proposition 1.3.2. For a TVS X the following are equivalent:

- a) X is Hausdorff.
- b) $\{o\}$ is closed in X.
- c) The intersection of all neighbourhoods of the origin o is just $\{o\}$.
- d) $\forall o \neq x \in X, \exists U \in \mathcal{F}(o) \ s.t. \ x \notin U.$

Since the topology of a TVS is translation invariant, property (d) means that the TVS is a $(T1)^5$ topological space. Recall for general topological spaces (T2) always implies (T1), but the converse does not always hold (c.f. Example 1.1.41-4 in [15]). However, Proposition 1.3.2 ensures that for TVS and so for TAs the two properties are equivalent.

Proof.

Let us just show that (d) implies (a) (for a complete proof see [15, Proposition 2.2.3, Corollary 2.2.4] or even better try it yourself!).

Suppose that (d) holds and let $x, y \in X$ with $x \neq y$, i.e. $x - y \neq o$. Then there exists $U \in \mathcal{F}(o)$ s.t. $x - y \notin U$. By (2) and (5) of Theorem 1.2.6, there exists $V \in \mathcal{F}(o)$ balanced and s.t. $V + V \subset U$. Since V is balanced V = -Vthen we have $V - V \subset U$. Suppose now that $(V + x) \cap (V + y) \neq \emptyset$, then there exists $z \in (V + x) \cap (V + y)$, i.e. z = v + x = w + y for some $v, w \in V$. Then $x - y = w - v \in V - V \subset U$ and so $x - y \in U$ which is a contradiction. Hence, $(V + x) \cap (V + y) = \emptyset$ and by Proposition 1.2.4 we know that $V + x \in \mathcal{F}(x)$ and $V + y \in \mathcal{F}(y)$. Hence, X is Hausdorff. \Box

We have already seen that a \mathbb{K} -algebra can be always embedded in a unital one, called unitization see Definition 1.1.3-4). In the rest of this section, we will discuss about which topologies on the unitization of a \mathbb{K} -algebra makes it into a TA. To start with, let us look at normed algebras.

Proposition 1.3.3. If A is a normed algebra, then there always exists a norm on its unitization A_1 making both A_1 into a normed algebra and the canonical embedding an isometry. Such a norm is called a unitization norm.

Proof.

Let $(A, \|\cdot\|)$ be a normed algebra and $A_1 = \mathbb{K} \times A$ its unitization. Define

$$||(k,a)||_1 := |k| + ||a||, \ \forall k \in \mathbb{K}, a \in A.$$

⁵ A topological space X is said to be (T1) if, given two distinct points of X, each lies in a neighborhood which does not contain the other point; or equivalently if, for any two distinct points, each of them lies in an open subset which does not contain the other point.

Then $||(1, o)||_1 = 1$ and it is straightforward that $|| \cdot ||_1$ is a norm on A_1 since $| \cdot |$ is a norm on \mathbb{K} and $|| \cdot ||$ is a norm on A. Also, for any $\lambda, k \in \mathbb{K}, a, b \in A$ we have:

$$\begin{aligned} \|(k,a)(\lambda,b)\|_{1} &= \|(k\lambda,ka+\lambda b+ab)\|_{1} = |k\lambda| + \|ka+\lambda b+ab\| \\ &\leq |k||\lambda|+k\|a\|+\lambda\|b\|+\|a\|\|b\| = |k|(|\lambda|+\|b\|) + \|a\|(|\lambda|+\|b\|) \\ &= (|k|+\|a\|)(|\lambda|+\|b\|) = \|(k,a)\|_{1}\|(\lambda,b)\|_{1}. \end{aligned}$$

This proves that $(A_1, \|\cdot\|_1)$ is a unital normed algebra. Moreover, the canonical embedding $e: A \to A_1, a \mapsto (0, a)$ is an isometry because $\|e(a)\|_1 = |0| + \|a\| = \|a\|$ for all $a \in A$. This in turn gives that e is continuous and so a topological embedding.

Remark 1.3.4. Note that $\|\cdot\|_1$ induces the product topology on A_1 given by $(\mathbb{K}, |\cdot|)$ and $(A, \|\cdot\|)$ but there might exist other unitization norms on A_1 not necessarily equivalent to $\|\cdot\|_1$ (see Sheet 1, Exercise 3).

The latter remark suggests the following generalization of Proposition 1.3.3 to any TA.

Proposition 1.3.5. Let A be a TA. Its unitization A_1 equipped with the corresponding product topology is a TA and A is topologically embedded in A_1 . Note that A_1 is Hausdorff if and only if A is Hausdorff.

Proof. Suppose (A, τ) is a TA. By Proposition 1.1.4, we know that the unitization A_1 of A is a \mathbb{K} -algebra. Moreover, since $(\mathbb{K}, |\cdot|)$ and (A, τ) are both TVS, we have that $A_1 := \mathbb{K} \times A$ endowed with the corresponding product topology τ_{prod} is also a TVS. Then the definition of multiplication in A_1 together with the fact that the multiplication in A is separately continuous imply that the multiplication in A_1 is separately continuous, too. Hence, (A_1, τ_{prod}) is a TA.

The canonical embedding e of A in A_1 is then a continuous monomorphism, since for any U neighbourhood of (0, o) in (A_1, τ_{prod}) there exist $\varepsilon > 0$ and a neighbourhood V of o in (A, τ) such that $B_{\varepsilon}(0) \times V \subseteq U$ and so $V = e^{-1}(B_{\varepsilon}(0) \times V) \subseteq e^{-1}(U)$. Hence, (A, τ) is topologically embedded in (A_1, τ_{prod}) .

Finally, recall that the cartesian product of topological spaces endowed with the corresponding product topology is Hausdorff iff each of them is Hausdorff. Then, as $(\mathbb{K}, |\cdot|)$ is Hausdorff, it is clear that (A_1, τ_{prod}) is Hausdorff iff (A, τ) is Hausdorff. \Box

⁶Alternative proof:

 $A \text{ Hausdorff} \stackrel{1.3.2}{\longleftrightarrow} \{o\} \text{ closed in } A \stackrel{\{0\}\text{ closed in } \mathbb{K}}{\longleftrightarrow} \{(0, o)\} \text{ closed in } A_1 \stackrel{1.3.2}{\longleftrightarrow} (A_1, \tau_{prod}) \text{ Hausdorff.}$

If A is a TA with continuous multiplication, then A_1 endowed with the corresponding product topology is also a TA with continuous multiplication. Moreover, from Remark 1.3.4, it is clear that the product topology is not the unique one making the unitization of a TA into a TA itself.

1.4 Subalgebras and quotients of a TA

In this section we are going to see some methods which allow us to construct new TAs from a given one. In particular, we will see under which conditions the TA structure is preserved under taking subalgebras and quotients.

Let us start with an immediate application of Theorem 1.2.9.

Proposition 1.4.1. Let X be a \mathbb{K} -algebra, (Y, ω) a TA (resp. TA with continuous multiplication) over \mathbb{K} and $\varphi : X \to Y$ a homomorphism. Denote by \mathcal{B}_{ω} a basis of neighbourhoods of the origin in (Y, ω) . Then the collection $\mathcal{B} := \{\varphi^{-1}(U) : U \in \mathcal{B}_{\omega}\}$ is a basis of neighbourhoods of the origin for a topology τ on X such that (X, τ) is a TA (resp. TA with continuous multiplication).

The topology τ constructed in the previous proposition is usually called *initial topology* or *inverse image topology* induced by φ .

Proof.

We first show that \mathcal{B} is a basis for a filter in X.

For any $B_1, B_2 \in \mathcal{B}$, we have $B_1 = \varphi^{-1}(U_1)$ and $B_2 = \varphi^{-1}(U_2)$ for some $U_1, U_2 \in \mathcal{B}_{\omega}$. Since \mathcal{B}_{ω} is a basis of the filter of neighbourhoods of the origin in (Y, ω) , there exists $U_3 \in \mathcal{B}_{\omega}$ such that $U_3 \subseteq U_1 \cap U_2$ and so $B_3 := \varphi^{-1}(U_3) \subseteq \varphi^{-1}(U_1) \cap \varphi^{-1}(U_2) = B_1 \cap B_2$ and clearly $B_3 \in \mathcal{B}$.

Now consider the filter \mathcal{F} generated by \mathcal{B} . For any $M \in \mathcal{F}$, there exists $U \in \mathcal{B}_{\omega}$ such that $\varphi^{-1}(U) \subseteq M$ and so we have the following:

- 1. $o_Y \in U$ and so $o_X \in \varphi^{-1}(o_Y) \in \varphi^{-1}(U) = M$.
- 2. by Theorem 1.2.6-2 applied to the TVS (Y, ω) , we have that there exists $V \in \mathcal{B}_{\omega}$ such that $V + V \subseteq U$. Hence, setting $N := \varphi^{-1}(V) \in \mathcal{F}$ we have $N + N \subseteq \varphi^{-1}(V + V) \subseteq \varphi^{-1}(U) = M$.
- 3. by Theorem 1.2.6-3 applied to the TVS (Y, ω) , we have that for any $\lambda \in \mathbb{K} \setminus \{0\}$ there exists $V \in \mathcal{B}_{\omega}$ such that $V \subseteq \lambda U$. Therefore, setting $N := \varphi^{-1}(V) \in \mathcal{B}$ we have $N \subseteq \varphi^{-1}(\lambda U) = \lambda \varphi^{-1}(U) \subseteq \lambda M$, and so $\lambda M \in \mathcal{F}$.
- 4. For any $x \in X$ there exists $y \in Y$ such that $x = \varphi^{-1}(y)$. As U is absorbing (by Theorem 1.2.6-4 applied to the TVS (Y, ω)), we have that there exists $\rho > 0$ such that $\lambda y \in U$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq \rho$.

This yields $\lambda x = \lambda \varphi^{-1}(y) = \varphi^{-1}(\lambda y) \in \varphi^{-1}(U) = M$ and hence, M is absorbing in X.

5. by Theorem 1.2.6-5 applied to the TVS (Y, ω) , we have that there exists $V \in \mathcal{B}_{\omega}$ balanced such that $V \subseteq U$. By the linearity of φ also $\varphi^{-1}(V)$ is balanced and so, setting $N := \varphi^{-1}(V)$ we have $N \subseteq \varphi^{-1}(U) = M$.

Therefore, we have showed that \mathcal{F} fulfills itself all the 5 properties of Theorem 1.2.6 and so it is a filter of neighbourhoods of the origin for a topology τ making (X, τ) a TVS.

Furthermore, for any $x \in X$ and any $B \in \mathcal{B}$ we have that there exist $y \in Y$ and $U \in \mathcal{B}_{\omega}$ such that $x = \varphi^{-1}(y)$ and $B = \varphi^{-1}(U)$. Then, as (Y, ω) is a TA, Theorem 1.2.9 guarantees that there exist $V_1, V_2 \in \mathcal{B}_{\omega}$ such that $yV_1 \subseteq U$ and $V_2y \subseteq U$. Setting $N_1 := \varphi^{-1}(V_1)$ and $N_2 := \varphi^{-1}(V_2)$, we obtain that $N_1, N_2 \in \mathcal{B}$ and $xN_1 = \varphi^{-1}(y)\varphi^{-1}(V_1) = \varphi^{-1}(yV_1) \subseteq \varphi^{-1}(U) = B$ and $xN_2 = \varphi^{-1}(y)\varphi^{-1}(V_2) = \varphi^{-1}(yV_2) \subseteq \varphi^{-1}(U) = B$. (Similarly, if (Y, ω) is a TA with continuous multiplication, then one can show that for any $B \in \mathcal{B}$ there exists $N \in \mathcal{B}$ such that $NN \subseteq B$.)

Hence, by Theorem 1.2.9 (resp. Theorem 1.2.10), (X, τ) is a TA (resp. TA with continuous multiplication).

Corollary 1.4.2. Let (A, ω) be a TA (resp. TA with continuous multiplication) and M a subalgebra of A. If we endow M with the relative topology τ_M induced by A, then (M, τ_M) is a TA (resp. TA with continuous multiplication).

Proof.

Consider the identity map $id: M \to A$ and let \mathcal{B}_{ω} a basis of neighbourhoods of the origin in (A, ω) Clearly, id is a homomorphism and the initial topology induced by id on M is nothing but the relative topology τ_M induced by Asince

 $\{id^{-1}(U): U \in \mathcal{B}_{\omega}\} = \{U \cap M: U \in \mathcal{B}_{\omega}\} = \tau_M.$

Hence, Proposition 1.4.1 ensures that (M, τ_M) is a TA (resp. TA with continuous multiplication).

With similar techniques to the ones used in Proposition 1.4.1 one can show:

Proposition 1.4.3. Let (X, ω) be a TA (resp. TA with continuous multiplication) over \mathbb{K} , Y a \mathbb{K} -algebra and $\varphi : X \to Y$ a surjective homomorphism. Denote by \mathcal{B}_{ω} a basis of neighbourhoods of the origin in (X, ω) . Then $\mathcal{B} := \{\varphi(U) : U \in \mathcal{B}_{\omega}\}$ is a basis of neighbourhoods of the origin for a topology τ on Y such that (Y, τ) is a TA (resp. TA with continuous multiplication).

Proof. (Sheet 2)
$$\Box$$

Using the latter result one can show that the quotient of a TA over an ideal endowed with the quotient topology is a TA (Sheet 2). However, in the following we are going to give a direct proof of this fact without making use of bases. Before doing that, let us briefly recall the notion of quotient topology.

Given a topological space (X, ω) and an equivalence relation \sim on X. The *quotient set* X/\sim is defined to be the set of all equivalence classes w.r.t. to \sim . The map $\phi: X \to X/\sim$ which assigns to each $x \in X$ its equivalence class $\phi(x)$ w.r.t. \sim is called the *canonical map* or *quotient map*. Note that ϕ is surjective. The *quotient topology* on X/\sim is the collection of all subsets U of X/\sim such that $\phi^{-1}(U) \in \omega$. Hence, the quotient map ϕ is continuous and actually the quotient topology on X/\sim is the finest topology on X/\sim such that ϕ is continuous.

Note that the quotient map ϕ is not necessarily open or closed.

Example 1.4.4. Consider \mathbb{R} with the standard topology given by the modulus and define the following equivalence relation on \mathbb{R} :

$$x \sim y \Leftrightarrow (x = y \lor \{x, y\} \subset \mathbb{Z}).$$

Let \mathbb{R}/\sim be the quotient set w.r.t \sim and $\phi : \mathbb{R} \to \mathbb{R}/\sim$ the correspondent quotient map. Let us consider the quotient topology on \mathbb{R}/\sim . Then ϕ is not an open map. In fact, if U is an open proper subset of \mathbb{R} containing an integer, then $\phi^{-1}(\phi(U)) = U \cup \mathbb{Z}$ which is not open in \mathbb{R} with the standard topology. Hence, $\phi(U)$ is not open in \mathbb{R}/\sim with the quotient topology.

For an example of not closed quotient map see e.g. [15, Example 2.3.3].

Let us consider now a \mathbb{K} -algebra A and an ideal I of A. We denote by A/I the quotient set A/\sim_I , where \sim_I is the equivalence relation on A defined by $x \sim_I y$ iff $x - y \in I$. The canonical (or quotient) map $\phi : A \to A/I$ which assigns to each $x \in A$ its equivalence class $\phi(x)$ w.r.t. the relation \sim_I is clearly surjective.

Using the fact that I is an ideal of the algebra A (see Definition 1.1.3-2), it is easy to check that:

- 1. if $x \sim_I y$, then $\forall \lambda \in \mathbb{K}$ we have $\lambda x \sim_I \lambda y$.
- 2. if $x \sim_I y$, then $\forall z \in A$ we have $x + z \sim_I y + z$.
- 3. if $x \sim_I y$, then $\forall z \in A$ we have $xz \sim_I yz$ and $zx \sim_I zy$.

These three properties guarantee that the following operations are well-defined on A/I:

- vector addition: $\forall \phi(x), \phi(y) \in A/I, \ \phi(x) + \phi(y) := \phi(x+y)$
- scalar multiplication: $\forall \lambda \in \mathbb{K}, \forall \phi(x) \in A/I, \lambda \phi(x) := \phi(\lambda x)$
- vector multiplication: $\forall \phi(x), \phi(y) \in A/I, \phi(x) \cdot \phi(y) := \phi(xy)$

A/I equipped with the three operations defined above is a \mathbb{K} -algebra which is often called *quotient algebra*. Then the quotient map ϕ is clearly a homomorphism. Moreover, if A is unital and I proper then also the quotient algebra A/I is unital. Indeed, as I is a proper ideal of A, the unit 1_A does not belong to I and so we have $\phi(1_A) \neq o$ and for all $x \in A$ we get $\phi(x)\phi(1_A) = \phi(x \cdot 1_A) = \phi(x) = \phi(1_A \cdot x) = \phi(1_A)\phi(x)$.

Suppose now that (A, ω) is a TA and I an ideal of A. Since A is in particular a topological space, we can endow it with the quotient topology w.r.t. the equivalence relation \sim_I . We already know that in this setting ϕ is a continuous homomorphism but actually the structure of TA on A guarantees also that it is open. Indeed, the following holds for any TVS and so for any TA:

Proposition 1.4.5. For a linear subspace M of a t.v.s. X, the quotient mapping $\phi : X \to X/M$ is open (i.e. carries open sets in X to open sets in X/M) when X/M is endowed with the quotient topology.

Proof.

Let V be open in X. Then we have

$$\phi^{-1}(\phi(V)) = V + M = \bigcup_{m \in M} (V + m).$$

Since X is a t.v.s, its topology is translation invariant and so V + m is open for any $m \in M$. Hence, $\phi^{-1}(\phi(V))$ is open in X as union of open sets. By definition, this means that $\phi(V)$ is open in X/M endowed with the quotient topology.

Theorem 1.4.6. Let (A, ω) be a TA (resp. TA with continuous multiplication) and I an ideal of A. Then the quotient algebra A/I endowed with the quotient topology is a TA (resp. TA with continuous multiplication).

Proof.

For convenience, in this proof we denote by a (resp. m) the vector addition (resp. vector multiplication) in A/I and just by + (resp. \cdot) the vector addition (resp. vector multiplication) in A. Let W be a neighbourhood of the origin o in A/I endowed with the quotient topology τ_Q . We first aim to prove that $a^{-1}(W)$ is a neighbourhood of (o, o) in $A/I \times A/I$.

By definition of τ_Q , $\phi^{-1}(W)$ is a neighbourhood of the origin in (A, ω) and so, by Theorem 1.2.6-2 (we can apply the theorem because (A, ω) is a TA and so a TVS), there exists V neighbourhood of the origin in (A, ω) s.t. $V + V \subseteq \phi^{-1}(W)$. Hence, by the linearity of ϕ , we get

$$a(\phi(V) \times \phi(V)) = \phi(V+V) \subseteq \phi(\phi^{-1}(W)) \subseteq W, \text{ i.e. } \phi(V) \times \phi(V) \subseteq a^{-1}(W).$$

Since ϕ is also an open map, $\phi(V)$ is a neighbourhood of the origin o in $(A/I, \tau_Q)$ and so $a^{-1}(W)$ is a neighbourhood of (o, o) in $A/I \times A/I$ endowed with the product topology given by τ_Q . A similar argument gives the continuity of the scalar multiplication. Hence, A/I endowed with the quotient topology is a TVS.

Furthermore, for any $\tilde{x} \in A/I$ and any W neighbourhood of the origin in $(A/I, \tau_Q)$, we know that $\tilde{x} = \phi(x)$ for some $x \in A$ and $\phi^{-1}(W)$ is a neighbourhood of the origin in (A, ω) . Since (A, ω) is a TA, the multiplication \cdot in A is separately continuous so there exist V_1, V_2 neighbourhoods of the origin in (A, ω) such that $x \cdot V_1 \subseteq \phi^{-1}(W)$ and $V_2 \cdot x \subseteq \phi^{-1}(W)$. Setting $N_1 := \phi(V_1)$ and $N_2 := \phi(V_2)$, we get $\mathfrak{m}(\tilde{x} \times N_1) = \mathfrak{m}(\phi(x) \times \phi(V_1)) = \phi(x \cdot V_1) \subseteq \phi(\phi^{-1}(W)) \subseteq W$ and similarly $\mathfrak{m}(N_2 \times \tilde{x}) \subseteq \phi(\phi^{-1}(W)) \subseteq W$. This yields that \mathfrak{m} is separately continuous as the quotient map is open and so N_1, N_2 are both neighbourhoods of the origin in $(A/I, \tau_Q)$.

Proposition 1.4.7. Let A be a TA and I an ideal of A. Consider A/I endowed with the quotient topology. Then the two following properties are equivalent:

a) I is closedb) A/I is Hausdorff

Proof.

In view of Proposition 1.3.2, (b) is equivalent to say that the complement of the origin in A/I is open w.r.t. the quotient topology. But the complement of the origin in A/I is exactly the image under the canonical map ϕ of the complement of I in A. Since ϕ is an open continuous map, the image under ϕ of the complement of I in X is open in A/I iff the complement of I in A is open, i.e. (a) holds.

Corollary 1.4.8. If A is a TA, then $A/\overline{\{o\}}$ endowed with the quotient topology is a Hausdorff TA. $A/\overline{\{o\}}$ is said to be the Hausdorff TA associated with A. When A is a Hausdorff TA, A and $A/\overline{\{o\}}$ are topologically isomorphic.

Proof.

First of all, let us observe that $\overline{\{o\}}$ is a closed ideal of A. Indeed, since A is a TA, the multiplication is separately continuous and so for all $x, y \in A$ we have $x\{o\} \subseteq \overline{\{x \cdot o\}} = \overline{\{o\}}$ and $\overline{\{o\}}y \subseteq \overline{\{o \cdot y\}} = \overline{\{o\}}$. Then, by Theorem 1.4.6 and Proposition 1.4.7, $A/\overline{\{o\}}$ is a Hausdorff TA. If in addition A is also Hausdorff, then Proposition 1.3.2 guarantees that $\overline{\{o\}} = \{o\}$ in A. Therefore, the quotient map $\phi : A \to A/\overline{\{o\}}$ is also injective because in this case $Ker(\phi) = \{o\}$.

Hence, ϕ is a topological isomorphism (i.e. bijective, continuous, open, linear) between A and $A/\overline{\{o\}}$ which is indeed $A/\{o\}$.

Let us finally focus on quotients of normed algebra. If $(A, \|\cdot\|)$ is a normed (resp. Banach) algebra and I a closed ideal of A, then Theorem 1.4.6 guarantees that A/I endowed with the quotient topology is a TA with continuous multiplication but, actually, the latter is also a normed (resp. Banach) algebra. Indeed, one can easily show that the quotient topology is generated by the so-called *quotient norm* defined by

$$q(\phi(x)) := \inf_{y \in I} \|x + y\|, \quad \forall x \in A$$

which has the nice property to be submultiplicative and so the following holds.

Proposition 1.4.9. If $(A, \|\cdot\|)$ is a normed (resp. Banach) algebra and I a closed ideal of A, then A/I equipped with the quotient norm is a normed (resp. Banach) algebra.

Proof. (Sheet 2)

Chapter 2

Locally multiplicative convex algebras

2.1 Neighbourhood definition of Imc algebras

In the study of locally multiplicative convex algebras a particular role will be played by multiplicative sets. Therefore, before starting the study of this class of topological algebras we are going to have a closer look to this concept.

Definition 2.1.1. A subset U of a \mathbb{K} -algebra A is said to be a multiplicative set or m-set if $U \cdot U \subseteq U$. We call m-convex (resp. m-balanced) a multiplicative convex (resp. balanced) subset of A and absolutely m-convex a multiplicative subset of A which is both balanced and convex.

The notions defined above are totally algebraic and so independent from the topological structure with which the algebra is endowed.

Example 2.1.2.

- Any ideal of an algebra is an m-set.
- Fixed an element $a \neq o$ of an algebra, the set $\{a^n : n \in \mathbb{N}\}$ is an m-set.
- Given a normed algebra $(A, \|\cdot\|)$ and an integer $n \in \mathbb{N}$, the open and the closed ball centered at origin with radius $\frac{1}{n}$ are both examples of absolutely m-convex sets in A.

The following proposition illustrates some operations under which the multiplicativity of a subset of an algebra is preserved.

Proposition 2.1.3. Let A be a \mathbb{K} -algebra and $U \subset A$ multiplicative, then

- a) The convex hull of U is an m-convex set in A.
- b) The balanced hull of U is an m-balanced set in A.
- c) The convex balanced hull of U is an absolutely m-convex set in A.
- d) Any direct or inverse image via a homomorphism is a m-set.

Proof. (Sheet 2)

Recall that

Definition 2.1.4. Let S be any subset of a vector space X over \mathbb{K} . The convex (resp. balanced) hull of S, denoted by $\operatorname{conv}(S)$ (resp. $\operatorname{bal}(S)$) is the smallest convex (resp. balanced) subset of X containing S, i.e. the intersection of all convex (resp. balanced) subsets of X containing S. Equivalently,

$$\operatorname{conv}(S) := \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in S, \lambda_i \in [0,1], \sum_{i=1}^{n} \lambda_i = 1, n \in \mathbb{N} \right\}$$

and the balanced hull of S, denoted by bal(S) as

$$\operatorname{bal}(S) := \bigcup_{\lambda \in \mathbb{K}, |\lambda| \le 1} \lambda S$$

The convex balanced hull of S, denoted by $\operatorname{conv}_b(S)$, is defined as the smallest convex and balanced subset of X containing S and it can be easily proved that $\operatorname{conv}_b(S) = \operatorname{conv}(\operatorname{bal}(S))$.

Let us come back now to topological algebras.

Proposition 2.1.5. In any topological algebra, the operation of closure preserves the multiplicativity of a subset as well as its m-convexity and absolute m-convexity.

Proof.

First of all let us show that the following property holds in any TA (A, τ) :

$$\forall V, W \subseteq A, \ \overline{V} \cdot \overline{W} \subseteq \overline{VW}.$$

$$(2.1)$$

where the closure in A is here clearly intended w.r.t. the topology τ . Let $x \in \overline{V}, y \in \overline{W}$ and $O \in \mathcal{F}(o)$ where $\mathcal{F}(o)$ denotes the filter of neighbourhoods of the origin in A. As A is in particular a TVS, Theorem 1.2.6-2 ensures that there exists $N \in \mathcal{F}(o)$ s.t. $N + N \subseteq O$. Then for each $a \in A$, by Theorem 1.2.9, there exist $N_1, N_2 \in \mathcal{F}(o)$ such that $N_1 a \subseteq N$ and $aN_2 \subseteq N$. Moreover, since $x \in \overline{V}$ and $y \in \overline{W}$, there exist $v \in V$ and $w \in W$ s.t. $v \in x + N_1$ and $w \in y + N_2$. Putting all together, we have that

$$vw \in (x+N_1)w = xw + N_1w \subseteq xw + N \subseteq x(y+N_2) + N$$
$$= xy + xN_2 + N \subseteq xy + N + N \subseteq xy + O.$$

Hence, $(xy + O) \cap VW \neq \emptyset$, which proves that $xy \in \overline{VW}$. Therefore, if U is an m-set in A then by (2.1) we get $\overline{U} \cdot \overline{U} \subseteq \overline{U} \cdot \overline{U} \subseteq \overline{U}$, which proves that \overline{U} is an m-set.

Suppose now that U is m-convex. The first part of the proof guarantees that \overline{U} is an m-set. Moreover, using that A is in particular a TVS, we have that for any $\lambda \in [0, 1]$ the mapping

$$\begin{array}{rccc} \varphi_{\lambda} : & A \times A & \to & A \\ & & (x,y) & \mapsto & \lambda x + (1-\lambda)y \end{array}$$

is continuous and so $\varphi_{\lambda}(\overline{U \times U}) \subseteq \overline{\varphi_{\lambda}(U \times U)}$. Since U is also convex, for any $\lambda \in [0, 1]$ we have that $\varphi_{\lambda}(U \times U) \subseteq U$ and so $\overline{\varphi_{\lambda}(U \times U)} \subseteq \overline{U}$. Putting all together, we can conclude that $\varphi_{\lambda}(\overline{U} \times \overline{U}) = \varphi_{\lambda}(\overline{U} \times \overline{U}) \subseteq \overline{U}$, i.e. \overline{U} is convex. Hence, \overline{U} is an m-convex set.

Finally, assume that U is absolutely m-convex. As U is in particular mconvex, by the previous part of the proof, we can conclude immediately that \overline{U} is an m-convex set. Furthermore, since U is balanced and A has the TVS structure, we can conclude that \overline{U} is also balanced. Indeed, in any TVS the closure of a balanced set is still balanced because the multiplication by scalar is continuous and so for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$ we have $\lambda \overline{U} \subseteq \overline{\lambda U} \subseteq U$. \Box

Definition 2.1.6. A closed absorbing absolutely convex multiplicative subset of a TA is called a m-barrel.

Proposition 2.1.7. Every multiplicative neighbourhood of the origin in a TA is contained in a neighbourhood of the origin which is an m-barrel.

Proof.

Let U be a multiplicative neighbourhood of the origin and define $T(U) := \overline{\operatorname{conv}_b(U)}$. Clearly, $U \subseteq T(U)$. Therefore, T(U) is a neighbourhood of the origin and so it is absorbing by Theorem 1.2.6-4). By Proposition 2.1.3-c), $\operatorname{conv}_b(U)$ is an absolutely m-convex set as U is an m-set. Hence, Proposition 2.1.5 ensures that T(U) is closed and absolutely m-convex, i.e. an m-barrel.

Note that the converse inclusion in Proposition 2.1.7 does not hold in general. Indeed, in any TA not every neighbourhood of the origin (not even every multiplicative one) contains another one which is a m-barrel. This means that not every TA has a basis of neighbourhoods consisting of m-barrels. However, this is true for any lmc TA.

Definition 2.1.8. A TA is said to be locally multiplicative convex (lmc) if it has a basis of neighbourhoods of the origin consisting of m-convex sets.

It is then easy to show that

Proposition 2.1.9. A locally multiplicative convex algebra is a TA with continuous multiplication.

Proof.

Let (A, τ) be an lmc algebra and let \mathcal{B} denote a basis of neighbourhoods of the origin in (A, τ) consisting of m-convex sets. Then (A, τ) is in particular a TVS and for any $U \in \mathcal{B}$ we have $U \cdot U \subset U$. Hence, both conditions of Theorem 1.2.10 are fulfilled by \mathcal{B} , which proves that (A, τ) is a TA with continuous multiplication.

Note that any lmc algebra is in particular a *locally convex TVS*, i.e. a TVS having a basis of neighbourhoods of the origin consisting of convex sets. Hence, in the study of this class of TAs we can make use of all the powerful results about locally convex TVS. To this aim let us recall that the class of locally convex TVS can be characterized in terms of absorbing absolutely convex neighbourhoods of the origin.

Theorem 2.1.10. If X is a lc TVS then there exists a basis \mathcal{B} of neighbourhoods of the origin consisting of absorbing absolutely convex subsets s.t.

a) $\forall U, V \in \mathcal{B}, \exists W \in \mathcal{B} \ s.t. \ W \subseteq U \cap V$

b) $\forall U \in \mathcal{B}, \forall \rho > 0, \exists W \in \mathcal{B} \ s.t. \ W \subseteq \rho U$

Conversely, if \mathcal{B} is a collection of absorbing absolutely convex subsets of a vector space X s.t. a) and b) hold, then there exists a unique topology compatible with the linear structure of X s.t. \mathcal{B} is a basis of neighbourhoods of the origin in X for this topology (which is necessarily locally convex).

Proof.

Let N be a neighbourhood of the origin in the lc TVS (X, τ) . The local convexity ensures that there exists W convex neighbourhood of the origin in (X, τ) s.t. $W \subseteq N$. Moreover, by Theorem 1.2.6-5), there exists U balanced neighbourhood of the origin in X s.t. $U \subseteq W$. Then, using that W is a convex set containing U, we get conv $(U) \subseteq W \subseteq N$. Now conv(U) is convex by definition, balanced because U is balanced and it is also a neighbourhood of the origin (and so an absorbing set) since $U \subseteq \text{conv}(U)$. Hence, the collection $\mathcal{B} :=$ $\{\text{conv}(U) : U \in \mathcal{B}_b\}$ is a basis of absorbing absolutely convex neighbourhoods of the origin in (X, τ) ; here \mathcal{B}_b denotes a basis of balanced neighbourhoods of the origin in (X, τ) . Observing that for any $U, W \in \mathcal{B}_b$ and any $\rho > 0$ we have $\text{conv}(U \cap W) \subseteq \text{conv}(U) \cap \text{conv}(W)$ and $\text{conv}(\rho U) \subseteq \rho \text{conv}(U)$, we see that \mathcal{B} fulfills both a) and b).

The converse direction is left as an exercise for the reader.

This theorem will be a handful tool in the proof of the following characterization of lmc algebras in terms of neighbourhood basis. **Theorem 2.1.11.** Let A be a \mathbb{K} -algebra. Then the following are equivalent: a) A is an lmc algebra

- b) A is a TVS having a basis of neighbourhoods consisting of m-barrels.
- c) There exists a basis for a filter on A consisting of absorbing absolutely *m*-convex subsets.

Proof.

 $a) \Rightarrow b)$ If A is an lmc algebra, then we have already observed that it is a lc TVS. Let $\mathcal{F}(o)$ be the filter of neighbourhoods of the origin in A and let $N \in \mathcal{F}(o)$. The TVS structure ensures that there exists $V \in \mathcal{F}(o)$ closed s.t. $V \subseteq N^1$ and the local convexity allows to apply Theorem 2.1.10 which guarantees that we can always find $M \in \mathcal{F}(o)$ absolutely convex s.t. $M \subseteq V$. Finally, since A is an lmc algebra, we know that there exists $C \in \mathcal{F}(o)$ m-convex s.t. $C \subseteq M$. Using the previous inclusions we have that

$$T(C) := \overline{\operatorname{conv}_b(C)} \subseteq \overline{M} \subseteq \overline{V} = V \subseteq N.$$

(Note that the first inclusion follows from the fact that M is a convex and balanced subset containing C.) Hence, the conclusion holds because T(C) is an m-barrel set as C is a multiplicative neighbourhood of the origin (see last part of proof of Proposition 2.1.7).

 $b) \Rightarrow c$) This is clear because every m-barrelled neighbourhood of the origin is an absorbing absolutely m-convex subsets of A.

c) $\Rightarrow a$) Suppose that \mathcal{M} is a basis for a filter on A consisting of absorbing absolutely convex m-subsets. Then it is easy to verify that the collection $\widetilde{\mathcal{M}} := \{\lambda U : U \in \mathcal{M}, 0 < \lambda \leq 1\}$ also consists of absorbing absolutely m-convex subsets of A. Moreover, for any $U, V \in \mathcal{M}$ we know that there exists $W \in \mathcal{M}$ s.t. $W \subseteq U \cap W$ and so for any $0 < \lambda, \mu \leq 1$ we have that $\delta W \subseteq \delta(U \cap V) = \delta U \cap \delta V \subseteq \lambda U \cap \mu V$ where $\delta := \min\{\lambda, \mu\}$. As $\delta W \in \widetilde{\mathcal{M}}$ we have that a) of Theorem 2.1.10. Also b) of this same theorem is satisfied because for any $\rho > 0, 0 < \lambda \leq 1$ and $U \in \mathcal{M}$ we easily get that there exists $M \in \widetilde{\mathcal{M}}$ s.t. $M \subseteq \rho(\lambda U)$ by choosing $M = \rho(\lambda U)$ when $0 < \rho \leq 1$ and $M = \lambda U$ when $\rho > 1$. Hence, \widetilde{M} fulfills all the assumptions of the second part of Theorem 2.1.10 and so it is a basis of neighbourhoods of the origin for a uniquely defined topology τ on A making (A, τ) a lc TVS. As every set in $\widetilde{\mathcal{M}}$ is m-convex, (A, τ) is in fact a lmc algebra.

 $^{^{1}\}mathrm{Every}$ TVS has basis of closed neighbourhoods of the origin. *Proof.*

Let $\mathcal{F}(o)$ be the filter of neighbourhoods of the origin in a TVS X and $N \in \mathcal{F}(o)$. Then Theorem 1.2.6 guarantees that there exists $V \in \mathcal{F}(o)$ balanced such that $V - V \subseteq N$. If $x \in \overline{V}$ then $(V + x) \cap V \neq \emptyset$ and so there exist $u, v \in V$ s.t. u + x = v, which gives $x = v - u \in V - V \subseteq N$. Hence, $\overline{V} \subseteq N$.

From the last part of the proof we can immediately see that

Corollary 2.1.12. If \mathcal{M} is a basis for a filter on a \mathbb{K} -algebra A consisting of absorbing absolutely convex m-subsets, then there exists a unique topology τ on A both having $\widetilde{\mathcal{M}} := \{\lambda U : U \in \mathcal{M}, 0 < \lambda \leq 1\}$ as a basis of neighbourhoods of the origin and making (A, τ) an lmc algebra.

Theorem 2.1.11 shows that in an lmc algebra every neighbourhood of the origin contains an m-barrel set. However, it is important to remark that not every m-barrel subset of a topological algebra, not even of an lmc algebra, is a neighbourhood of the origin (see Examples 2.2.19)! Topological algebras having this property are called *m-barrelled algebras*.

2.2 Seminorm characterization of Imc algebras

In this section we will investigate the intrinsic and very useful connection between lmc algebras and seminorms. Therefore, let us briefly recall this concept and focus in particular on submultiplicative seminorms.

Definition 2.2.1. Let X be a \mathbb{K} -vector space. A function $p: X \to \mathbb{R}$ is called a seminorm if it satisfies the following conditions:

1. p is subadditive: $\forall x, y \in X, p(x+y) \le p(x) + p(y)$.

2. p is positively homogeneous: $\forall x \in X, \forall \lambda \in \mathbb{K}, p(\lambda x) = |\lambda| p(x).$

A seminorm on a \mathbb{K} -algebra X is called submultiplicative if

 $\forall x, y \in X, \, p(xy) \le p(x)p(y).$

Definition 2.2.2. A seminorm p on a vector space X is a norm if p(x) = 0 implies x = o (i.e. if $p^{-1}(\{0\}) = \{o\}$).

The following properties are an easy consequence of Definition 2.2.1.

Proposition 2.2.3. Let p be a seminorm on a vector space X. Then:

- p is symmetric, i.e. $p(x) = p(-x), \forall x \in X$.
- p(o) = 0.
- $|p(x) p(y)| \le p(x y), \forall x, y \in X.$
- $p(x) \ge 0, \forall x \in X.$
- $\ker(p)$ is a linear subspace of X.

Examples 2.2.4.

a) Suppose $X = \mathbb{R}^n$ is equipped with the componentwise operations of addition, scalar and vector multiplication. Let M be a linear subspace of X. For any $x \in X$, set

$$q_M(x) := \inf_{m \in M} \|x - m\|,$$

where $\|\cdot\|$ is the Euclidean norm on \mathbb{R}^n , i.e. $q_M(x)$ is the distance from the point x to M in the usual sense. If $\dim(M) \ge 1$ then q_M is a submultiplicative seminorm but not a norm (M is exactly the kernel of q_M). When $M = \{o\}, p_M(\cdot)$ and $\|\cdot\|$ coincide.

b) Let $\mathcal{C}(\mathbb{R})$ be the vector space of all real valued continuous functions on the real line equipped with the pointwise operations of addition, multiplication and scalar multiplication. For any $a \in \mathbb{R}^+$, we define

$$p_a(f) := \sup_{-a \le t \le a} |f(t)|, \ \forall f \in \mathcal{C}(\mathbb{R}).$$

Then p_a is a submultiplicative seminorm but is never a norm because it might be that f(t) = 0 for all $t \in [-a, a]$ (and so that $p_a(f) = 0$) but $f \not\equiv 0$. c) Let $n \geq 2$ be an integer and consider the algebra $\mathbb{R}^{n \times n}$ of real square ma-

trices of order n. Then

$$q(A) := \max_{i,j=1,...,n} |A_{ij}|, \quad \forall A = (A_{ij})_{i,j=1}^n \in \mathbb{R}^{n \times n}$$

is a norm (so in particular a seminorm) but it is not submultiplicative because for example if A is the matrix with all entries equal to 1 then it is easy to check that $||A^2|| > ||A||$.

Seminorms on vector spaces are strongly related to a special kind of functionals, i.e. *Minkowski functionals*. Let us investigate more in details such a relation. Note that we are still in the realm of vector spaces with no topology!

Definition 2.2.5. Let X be a vector space and V a non-empty subset of X. We define the Minkowski functional (or gauge) of V to be the mapping:

$$p_V: X \to \mathbb{R}$$

$$x \mapsto p_V(x) := \inf\{\lambda > 0 : x \in \lambda V\}$$

(where $p_V(x) = \infty$ if the set $\{\lambda > 0 : x \in \lambda V\}$ is empty).

It is then natural to ask whether there exists a class of subsets for which the associated Minkowski functionals are actually seminorms, and in particular submultiplicative seminorms in the context of algebras. The answer is positive in both cases as established in the following lemma.

Notation 2.2.6. Let X be a vector space and p a seminorm on X. The sets

$$U_p = \{x \in X : p(x) < 1\}$$
 and $U_p = \{x \in X : p(x) \le 1\}$

are said to be, respectively, the open and the closed unit semiball of p.

Lemma 2.2.7. Let X be a \mathbb{K} -vector space (resp. \mathbb{K} -algebra).

- a) If V is a non-empty subset of X which is absorbing and absolutely convex (resp. absolutely m-convex), then the associated Minkowski functional p_V is a seminorm (resp. submultiplicative seminorm) and $\mathring{U}_{p_V} \subseteq V \subseteq U_{p_V}$.
- b) If q is a seminorm (resp. submultiplicative seminorm) on X then both \mathring{U}_q and U_q are absorbing absolutely convex sets [resp. absolutely m-convex] and for any absorbing absolutely convex (resp. absolutely m-convex) V such that $\mathring{U}_q \subseteq V \subseteq U_q$ we have $q = p_V$.

Proof.

a) Let V be a non-empty subset of X which is absorbing and absolutely convex and denote by p_V the associated Minkowski functional. We want to show that p_V is a seminorm.

- First of all, note that $p_V(x) < \infty$ for all $x \in X$ because V is absorbing. Indeed, for any $x \in X$ there exists $\rho_x > 0$ s.t. for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq \rho_x$ we have $\lambda x \in V$ and so the set $\{\lambda > 0 : x \in \lambda V\}$ is never empty, i.e. p_V has only finite nonnegative values. Moreover, since $o \in V$, we also have that $o \in \lambda V$ for any $\lambda \in \mathbb{K}$ and so $p_V(o) = \inf\{\lambda > 0 : o \in \lambda V\} = 0$.
- The balancedness of V implies that p_V is positively homogeneous. Since we have already showed that $p_V(o) = 0$ it remains to prove the positive homogeneity of p_V for non-zero scalars. Since V is balanced we have that for any $x \in X$ and for any $\xi, \lambda \in \mathbb{K}$ with $\xi \neq 0$ the following holds:

$$\xi x \in \lambda V$$
 if and only if $x \in \frac{\lambda}{|\xi|} V$. (2.2)

Indeed, V balanced guarantees that $\xi V = |\xi|V$ and so $x \in \frac{\lambda}{|\xi|}V$ is equivalent to $\xi x \in \lambda \frac{\xi}{|\xi|}V = \lambda V$. Using (2.2), we get that for any $x \in X$ and for any $\xi \in \mathbb{K}$ with $\xi \neq 0$:

$$p_{V}(\xi x) = \inf \{\lambda > 0 : \xi x \in \lambda V\}$$

=
$$\inf \left\{\lambda > 0 : x \in \frac{\lambda}{|\xi|}V\right\}$$

=
$$\inf \left\{|\xi|\frac{\lambda}{|\xi|} > 0 : x \in \frac{\lambda}{|\xi|}V\right\}$$

=
$$|\xi|\inf\{\mu > 0 : x \in \mu V\} = |\xi|p_{V}(x).$$

• The convexity of V ensures the subadditivity of p_V . Take $x, y \in X$. By the definition of Minkowski functional, for every $\varepsilon > 0$ there exist $\lambda, \mu > 0$ s.t.

$$\lambda < p_V(x) + \frac{\varepsilon}{2} \text{ and } x \in \lambda V$$

and

$$\mu < p_V(y) + \frac{\varepsilon}{2} \text{ and } y \in \mu V.$$

Then, by the convexity of V, we obtain that $\frac{\lambda}{\lambda+\mu}V + \frac{\mu}{\lambda+\mu}V \subseteq V$, i.e. $\lambda V + \mu V \subseteq (\lambda + \mu)V$, and therefore $x + y \in (\lambda + \mu)V$. Hence:

$$p_V(x+y) = \inf\{\delta > 0 : x+y \in \delta V\} \le \lambda + \mu < p_V(x) + p_V(y) + \varepsilon$$

which proves the subadditivity of p_V since ε is arbitrary.

We can then conclude that p_V is a seminorm. Furthermore, we have the following inclusions:

$$\check{U}_{p_V} \subseteq V \subseteq U_{p_V}.$$

In fact, if $x \in U_{p_V}$ then $p_V(x) < 1$ and so there exists $0 < \lambda < 1$ s.t. $x \in \lambda V$. Since V is balanced, for such λ we have $\lambda V \subseteq V$ and therefore $x \in V$. On the other hand, if $x \in V$ then clearly $1 \in \{\lambda > 0 : x \in \lambda V\}$ which gives $p_V(x) \leq 1$ and so $x \in U_{p_V}$.

If X is a \mathbb{K} -algebra and V an absorbing absolutely m-convex subset of X, then the previous part of the proof guarantees that p_V is a seminorm and $\mathring{U}_{p_V} \subseteq V \subseteq U_{p_V}$. Moreover, for any $a, b \in X$, the multiplicativity of V implies that $\{\lambda > 0 : a \in \lambda V\}\{\mu > 0 : b \in \mu V\} \subseteq \{\delta > 0 : ab \in \delta V\}$ and so

$$p_V(a)p_V(b) = \inf \left\{ \{\lambda > 0 : a \in \lambda V \} \{\mu > 0 : b \in \mu V \} \right\}$$

$$\geq \inf \{\delta > 0 : ab \in \delta V \} = p_V(ab).$$

Hence, p_V is a submultiplicative seminorm.

b) Let us take any seminorm q on X. Let us first show that \check{U}_q is absorbing and absolutely convex.

• \check{U}_q is absorbing.

Let x be any point in X. If q(x) = 0 then clearly $x \in \mathring{U}_q$. If q(x) > 0, we can take $0 < \rho < \frac{1}{q(x)}$ and then for any $\lambda \in \mathbb{K}$ s.t. $|\lambda| \leq \rho$ the positive homogeneity of q implies that $q(\lambda x) = |\lambda|q(x) \leq \rho q(x) < 1$, i.e. $\lambda x \in \mathring{U}_q$.

For any $x \in \mathring{U}_q$ and for any $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$, again by the positive homogeneity of q, we get: $q(\lambda x) = |\lambda|q(x) \leq q(x) < 1$ i.e. $\lambda x \in \mathring{U}_q$.

• U_q is convex.

For any $x, y \in \mathring{U}_q$ and any $t \in [0, 1]$, by both the properties of seminorm, we have that $q(tx + (1 - t)y) \leq tq(x) + (1 - t)q(y) < t + 1 - t = 1$ i.e. $tx + (1 - t)y \in \mathring{U}_q$. The proof above easily adapts to show that U_q is absorbing and absolutely convex. Also, it is easy to check that

$$p_{U_{q}}(x) = q(x) = p_{U_{q}}(x), \, \forall x \in X.$$
 (2.3)

Since for any absorbing absolutely convex subset V of X s.t. $\mathring{U}_q \subseteq V \subseteq U_q$ and for any $x \in X$ we have that

$$p_{U_q}(x) \le p_V(x) \le p_{\mathring{U}_q}(x),$$

by (2.3) we can conclude that $p_V(x) = q(x)$.

If X is a \mathbb{K} -algebra and q is submultiplicative, then the previous part of the proof of b) applies but in addition we get that both \mathring{U}_q and U_q are multiplicative sets. Indeed, for any $a, b \in \mathring{U}_q$ we have $q(ab) \leq q(a)q(b) < 1$, i.e. $ab \in \mathring{U}_q$ and similarly for U_q . \Box

In a nutshell this lemma says that: a real-valued functional on a \mathbb{K} -vector space X (resp. a \mathbb{K} -algebra) is a seminorm (resp-submultiplicative seminorm) if and only if it is the Minkowski functional of an absorbing absolutely convex (resp. absolutely m-convex) non-empty subset of X.

Let us collect some interesting properties of semiballs in a vector space, which we will repeatedly use in the following.

Proposition 2.2.8. Let X be a \mathbb{K} -vector space and p a seminorm on X. Then:

- a) $\forall r > 0, r \mathring{U}_p = \{x \in X : p(x) < r\} = \mathring{U}_{\frac{1}{2}p}.$
- b) $\forall x \in X, x + \mathring{U}_p = \{y \in X : p(y x) < 1\}.$
- c) If q is also a seminorm on X, then $p \leq q$ if and only if $\check{U}_q \subseteq \check{U}_p$.
- d) If n ∈ N and s₁,..., s_n are seminorms on X, then their maximum s defined as s(x) := max s_i(x), ∀x ∈ X is also seminorm on X and Ů_s = ∩ⁿ_{i=1} Ů_{s_i}. In particular, if X is a K-algebra and all s_i's are submultiplicative seminorms, then s(x) is also submultiplicative.

All the previous properties also hold for closed semballs.

Proof. (Sheet 3)

Let us start to put some topological structure on our space and so to consider continuous seminorms on it. The following result holds in any TVS and so in particular in any TA.
Proposition 2.2.9. Let X be a TVS and p a seminorm on X. Then the following conditions are equivalent:

- a) The open unit semiball U_p of p is an open neighbourhood of the origin and coincides with the interior of U_p .
- b) p is continuous at the origin.
- c) The closed unit semiball U_p of p is a closed neighbourhood of the origin and coincides with the closure of \mathring{U}_p .
- d) p is continuous at every point.

Proof.

 $a) \Rightarrow b$ Suppose that \mathring{U}_p is open in the topology on X. Then for any $\varepsilon > 0$ we have that $p^{-1}(] - \varepsilon, \varepsilon[) = \{x \in X : p(x) < \varepsilon\} = \varepsilon \mathring{U}_p$ is an open neighbourhood of the origin in X. This is enough to conclude that $p: X \to \mathbb{R}^+$ is continuous at the origin.

 $b) \Rightarrow c)$ Suppose that p is continuous at the origin, then $U_p = p^{-1}([0,1])$ is a closed neighbourhood of the origin. Also, by definition $\mathring{U}_p \subseteq U_p$ and so $\overline{\mathring{U}_p} \subseteq \overline{U_p} = U_p$. To show the converse inclusion, we consider $x \in X$ s.t. p(x) = 1 and take $(\lambda_n)_{n \in \mathbb{N}} \subset (0,1)$ s.t. $\lim_{n \to \infty} \lambda_n = 1$. Then $\lambda_n x \in \mathring{U}_p$ and $\lim_{n \to \infty} \lambda_n x = x$ since the scalar multiplication is continuous. Hence, $x \in \overline{\mathring{U}_p}$ which completes the proof of c).

 $c) \Rightarrow d$) Assume that c) holds and fix $x \in X$. Using Proposition 2.2.8 and Proposition 2.2.3, we get that for any $\varepsilon > 0$: $p^{-1}([-\varepsilon + p(x), p(x) + \varepsilon]) = \{y \in X : |p(y) - p(x)| \le \varepsilon\} \supseteq \{y \in X : p(y - x) \le \varepsilon\} = x + \varepsilon U_p$, which is a closed neighbourhood of x since X is a TVS and by the assumption c). Hence, p is continuous at x.

 $d) \Rightarrow a)$ If p is continuous on X then a) holds because $\mathring{U}_p = p^{-1}(]-1,1[)$ and the preimage of an open set under a continuous function is open. Also, by definition $\mathring{U}_p \subseteq U_p$ and so $\mathring{U}_p = \operatorname{int} \left(\mathring{U}_p \right) \subseteq \operatorname{int} (U_p)$. To show the converse inclusion, we consider $x \in \operatorname{int} (U_p)$. Then $p(x) \leq 1$ but, since $p(x) = p_{\mathring{U}_p}(x)$, we also have that for any $\varepsilon > 0$ there exists $\lambda > 0$ s.t. $x \in \lambda \mathring{U}_p$ and $\lambda < p(x) + \varepsilon$. This gives that $p(x) < \lambda < 1 + \varepsilon$ and so p(x) < 1, i.e. $x \in \mathring{U}_p$ which completes the proof of a).

Definition 2.2.10. Let X be a vector space and $\mathcal{P} := \{p_i\}_{i \in I}$ a family of seminorms on X. The coarsest topology $\tau_{\mathcal{P}}$ on X s.t. each p_i is continuous is said to be the topology induced or generated by the family of seminorms \mathcal{P} .

We are now ready to see the connection between submultiplicative seminorms and locally convex multiplicative algebras. **Theorem 2.2.11.** Let X be a \mathbb{K} -algebra and $\mathcal{P} := \{p_i\}_{i \in I}$ a family of submultiplicative seminorms. Then the topology induced by the family \mathcal{P} is the unique topology both making X into an lmc algebra and having as a basis of neighbourhoods of the origin the following collection:

$$\mathcal{B} := \Big\{ \{ x \in X : p_{i_1}(x) \le \varepsilon, \dots, p_{i_n}(x) \le \varepsilon \} : i_1, \dots, i_n \in I, n \in \mathbb{N}, 0 < \varepsilon \le 1 \Big\}.$$

Viceversa, the topology of an arbitrary lmc algebra is always induced by a family of submultiplicative seminorms (often called generating).

Proof.

Let us first observe that

$$\mathcal{B} = \left\{ \bigcap_{j=1}^{n} \varepsilon U_{p_{i_j}} : n \in \mathbb{N}, i_1, \dots, i_n \in I, 0 < \varepsilon \le 1 \right\}$$

and is a basis for a filter on X as it is closed under finite intersections. Moreover, by Proposition 2.2.8-a) and Lemma 2.2.7-b), we have that for any $i \in I$ the semiball εU_{p_i} is absorbing and absolutely m-convex. Therefore, any element in \mathcal{B} is an absorbing absolutely m-convex subset of X as finite intersection of sets having such properties. Hence, Corollary 2.1.12 guarantees that there exists a unique topology τ having \mathcal{B} as a basis of neighbourhoods of the origin and s.t. (X, τ) is an lmc algebra.

Since for any $i \in I$ we have $U_{p_i} \in \mathcal{B}$, U_{p_i} is a neighbourhood of the origin in (X, τ) , then by Proposition 2.2.9, the seminorm p_i is τ -continuous. Therefore, the topology $\tau_{\mathcal{P}}$ induced by the family \mathcal{P} is by definition coarser than τ . On the other hand, each p_i is also $\tau_{\mathcal{P}}$ -continuous and so U_{p_i} is a closed neighbourhood of the origin in $(X, \tau_{\mathcal{P}})$. Then \mathcal{B} consists of neighbourhoods of the origin in $(X, \tau_{\mathcal{P}})$ which implies that τ is coarser than $\tau_{\mathcal{P}}$. Hence, $\tau \equiv \tau_{\mathcal{P}}$.

Viceversa, let us assume that (X, τ) is an lmc algebra. Then by Theorem 2.1.11 there exists a basis \mathcal{N} of neighbourhoods of the origin in (X, τ) consisting of m-barrels. Consider now the family $\mathcal{S} := \{p_N : N \in \mathcal{N}\}$. By Lemma 2.2.7-a), we know that each p_N is a submultiplicative seminorm and that $\mathring{U}_{p_N} \subseteq N \subseteq U_{p_N}$. Now each p_N is τ -continuous because $U_{p_N} \supseteq N \in \mathcal{N}$ and hence, $\tau_{\mathcal{S}} \subseteq \tau$. Moreover, each p_N is clearly $\tau_{\mathcal{S}}$ -continuous and so, by Proposition 2.2.9, \mathring{U}_{p_N} is open in $(X, \tau_{\mathcal{S}})$. Since $\mathring{U}_{p_N} \subseteq N$, we have that \mathcal{N} consists of neighbourhoods of the origin in $(X, \tau_{\mathcal{S}})$, which implies $\tau \subseteq \tau_{\mathcal{S}}$. \Box

Historically the following more general result holds for locally convex tvs and the previous theorem could be also derived as a corollary of: **Theorem 2.2.12.** Let X be a vector space and $\mathcal{P} := \{p_i\}_{i \in I}$ a family of seminorms. Then the topology induced by the family \mathcal{P} is the unique topology both making X into a locally convex TVS and having as a basis of neighbourhoods of the origin the following collection:

$$\mathcal{B} := \Big\{ \{ x \in X : p_{i_1}(x) \le \varepsilon, \dots, p_{i_n}(x) \le \varepsilon \} : i_1, \dots, i_n \in I, n \in \mathbb{N}, 0 < \varepsilon \le 1 \Big\}.$$

Viceversa, the topology of an arbitrary locally convex TVS is always induced by a family of seminorms (often called generating).

Coming back to lmc algebras, Theorem 2.2.11 allows us to give another characterization of such a class, namely: A TA is lmc if and only if its topology is induced by a family of submultiplicative seminorms. This is very helpful in establishing whether a given topological algebras is lmc or not as we will see from the following examples.

Examples 2.2.13.

have

- 1. Normed algebras are clearly lmc algebras.
- 2. A seminormed algebra, i.e. a K-algebra endowed with the topology generated by a submultiplicative seminorm, is lmc.
- 3. The weak and the strong operator topologies on the space L(H) introduced in Example 1.2.17 both make L(H) into a locally convex algebra which is not lmc. Indeed, the weak operator topology τ_w is generated by the family of seminorms $\{p_{x,y} : x, y \in H\}$ where $p_{x,y}(T) := |\langle Tx, y \rangle|$, while the strong operator topology τ_s is generated by the family of seminorms $\{p_x : x \in H\}$ where $p_x(T) := ||Tx||$. If $(L(H), \tau_w)$ and $(L(H), \tau_s)$ were lmc algebras, then by Proposition 2.1.9 the multiplication should have been jointly continuous in both of them but this is not the case as we have already showed in Example 1.2.17.
- 4. Consider L^ω([0,1]) := ∩_{p≥1} L^p([0,1]), where for each p ≥ 1 we define L^p([0,1]) to be the space of all equivalence classes of functions f : [0,1] → ℝ such that ||f||_p := (∫₀¹ |f(t)|^pdt)^{1/p} < ∞ which agree almost everywhere. The set L^ω([0,1]) endowed with the pointwise operations is a real algebra since for any q, r ≥ 1 such that 1/p = 1/q + 1/r we

$$||fg||_p \le ||f||_q ||g||_r, \forall f, g \in L^{\omega}([0,1]).$$

The algebra $L^{\omega}([0,1])$ endowed with the topology induced by the family $\mathcal{P} := \{ \| \cdot \|_p : p \geq 1 \}$ of seminorms is a locally convex algebra. However, $(L^{\omega}([0,1]), \tau_{\mathcal{P}})$ is not an lmc algebra because any m-convex subset U is open in $(L^{\omega}([0,1]), \tau_{\mathcal{P}})$ if and only if $U = L^{\omega}([0,1])$ (Sheet 3).

Let us conclude this section with a further very useful property of lmc algebras.

Proposition 2.2.14. The topology of an lmc algebra can be always induced by a directed family of submultiplicative seminorms.

Definition 2.2.15. A family $Q := \{q_j\}_{j \in J}$ of seminorms on a vector space X is said to be directed (or fundamental or saturated) if

$$\forall n \in \mathbb{N}, j_1, \dots, j_n \in J, \exists j \in J, C > 0 \quad s.t. \quad Cq_j(x) \ge \max_{k=1,\dots,n} q_{j_k}(x), \forall x \in X.$$

$$(2.4)$$

To prove Proposition 2.2.14 we need to recall an important criterion to compare topologies induced by families of seminorms.

Theorem 2.2.16.

Let $\mathcal{P} = \{p_i\}_{i \in I}$ and $\mathcal{Q} = \{q_j\}_{j \in J}$ be two families of seminorms on a \mathbb{K} -vector space X inducing respectively the topologies $\tau_{\mathcal{P}}$ and $\tau_{\mathcal{Q}}$. Then $\tau_{\mathcal{P}}$ is finer than $\tau_{\mathcal{Q}}$ (i.e. $\tau_{\mathcal{Q}} \subseteq \tau_{\mathcal{P}}$) iff

$$\forall q \in \mathcal{Q} \ \exists n \in \mathbb{N}, \, i_1, \dots, i_n \in I, \, C > 0 \ s.t. \ Cq(x) \le \max_{k=1,\dots,n} p_{i_k}(x), \, \forall x \in X.$$

$$(2.5)$$

Proof.

Let us first recall that, by Theorem 2.2.12, we have that

$$\mathcal{B}_{\mathcal{P}} := \left\{ \bigcap_{k=1}^{n} \varepsilon \mathring{U}_{p_{i_k}} : i_1, \dots, i_n \in I, n \in \mathbb{N}, 0 < \varepsilon \le 1 \right\}$$

and

$$\mathcal{B}_{\mathcal{Q}} := \Big\{ \bigcap_{k=1}^{n} \varepsilon \mathring{U}_{q_{j_k}} : j_1, \dots, j_n \in J, n \in \mathbb{N}, 0 < \varepsilon \le 1 \Big\}.$$

are respectively bases of neighbourhoods of the origin for $\tau_{\mathcal{P}}$ and $\tau_{\mathcal{Q}}$.

By using Proposition 2.2.8, the condition (2.5) can be rewritten as

$$\forall q \in \mathcal{Q}, \exists n \in \mathbb{N}, i_1, \dots, i_n \in I, C > 0 \text{ s.t. } C \bigcap_{k=1}^n \mathring{U}_{p_{i_k}} \subseteq \mathring{U}_q.$$

which means that

$$\forall q \in \mathcal{Q}, \exists B_q \in \mathcal{B}_{\mathcal{P}} \text{ s.t. } B_q \subseteq U_q.$$
(2.6)

since $C \bigcap_{k=1}^{n} \mathring{U}_{p_{i_k}} \in \mathcal{B}_{\mathcal{P}}$.

Condition (2.6) means that for any $q \in Q$ the set \check{U}_q is a neighbourhood of the origin in $(X, \tau_{\mathcal{P}})$, which by Proposition 2.2.9 is equivalent to say that qis continuous w.r.t. $\tau_{\mathcal{P}}$. By definition of τ_Q , this gives that $\tau_Q \subseteq \tau_{\mathcal{P}}$. This theorem allows us to easily see that the topology induced by a family of seminorms on a vector space does not change if we close the family under taking the maximum of finitely many of its elements. Indeed, the following result holds.

Proposition 2.2.17. Let $\mathcal{P} := \{p_i\}_{i \in I}$ be a family of seminorms on a \mathbb{K} -vector space (resp. submultiplicative seminorms on a \mathbb{K} -algebra) X. Then we have that $\mathcal{Q} := \{\max_{i \in B} p_i : \emptyset \neq B \subseteq I \text{ with } B \text{ finite }\}$ is a directed family of seminorms (resp. submultiplicative seminorms) and $\tau_{\mathcal{P}} = \tau_{\mathcal{Q}}$, where $\tau_{\mathcal{P}}$ and $\tau_{\mathcal{Q}}$ denote the topology induced on X by \mathcal{P} and \mathcal{Q} , respectively.

Proof.

First of all let us note that, by Proposition 2.2.8-d), \mathcal{Q} is a family of seminorms. On the one hand, since $\mathcal{P} \subseteq \mathcal{Q}$, by definition of induced topology we have $\tau_{\mathcal{P}} \subseteq \tau_{\mathcal{Q}}$. On the other hand, for any $q \in \mathcal{Q}$ we have $q = \max_{i \in B} p_i$ for some $\emptyset \neq B \subseteq I$ finite. Then (2.5) is fulfilled for n = |B| (where |B| denotes the cardinality of the finite set B), i_1, \ldots, i_n being the n elements of B and for any $0 < C \leq 1$. Hence, by Theorem 2.2.16, $\tau_{\mathcal{Q}} \subseteq \tau_{\mathcal{P}}$. If X is a \mathbb{K} -algebra and \mathcal{P} consists of submultiplicative seminorms, then \mathcal{Q} consists of submultiplicative seminorms by the second part of Proposition 2.2.8-d).

We claim that \mathcal{Q} is directed.

Let $n \in \mathbb{N}$ and $q_1, \ldots, q_n \in \mathcal{Q}$. Then for each $j \in \{1, \ldots, n\}$ we have $q_j = \max_{i \in B_j} p_i$ for some non-empty finite subset B_j of I. Let us define $B := \bigcup_{j=1}^n B_j$ and $q := \max_{i \in B} p_i$. Then $q \in \mathcal{Q}$ and for any $C \ge 1$ we have that (2.4) is satisfied, because we get that for any $x \in X$

$$Cq(x) \ge \max_{i \in B} p_i(x) = \max_{j=1,\dots,n} \left(\max_{i \in B_j} p_i(x) \right) = \max_{j=1,\dots,n} q_j(x).$$

Hence, \mathcal{Q} is directed.

We are ready now to show Proposition 2.2.14.

Proof. of Proposition 2.2.14

Let (X, τ) be an lmc algebra. By Theorem 2.2.11, we have that there exists a family of submultiplicative seminorms $\mathcal{P} := \{p_i\}_{i \in I}$ on X s.t. $\tau = \tau_{\mathcal{P}}$. Let us define \mathcal{Q} as the collection obtained by forming the maximum of finitely many elements of \mathcal{P} , i.e. $\mathcal{Q} := \{\max_{i \in B} p_i : \emptyset \neq B \subseteq I \text{ with } B \text{ finite }\}$. By Proposition 2.2.17, Q is a directed family of submultiplicative seminorms and we have that $\tau_{\mathcal{P}} = \tau_{Q}$.

It is possible to show (Sheet 3) that a basis of neighbourhoods of the origin for the lmc topology $\tau_{\mathcal{Q}}$ induced by a directed family of submultiplicative seminorms \mathcal{Q} is given by:

$$\mathcal{B}_d := \{ r U_q : q \in \mathcal{Q}, 0 < r \le 1 \}.$$
(2.7)

Remark 2.2.18. The proof of Proposition 2.2.14 can be easily adapted to show that the topology of a lc tvs can be always induced by a directed family of seminorms τ_Q and that the corresponding (2.7) is basis of neighbourhoods of the origin for τ_Q .

Example 2.2.19. Let $C_b(\mathbb{R})$ the set of all real-valued bounded continuous functions on the real line endowed with the pointwise operations of addition, multiplication and scalar multiplication and endowed with the topology τ_Q induced by the family $Q := \{p_a : a > 0\}$, where $p_a(f) := \sup_{-a \le t \le a} |f(t)|, \forall f \in C_b(\mathbb{R})$. Since each p_a is a submultiplicative seminorm (see Example 2.2.4-d)), the algebra $(C_b(\mathbb{R}), \tau_Q)$ is lmc.

Note that Q is directed since for any $n \in \mathbb{N}$ and any positive real numbers a_1, \ldots, a_n we have that $\max_{i=1,\ldots,n} p_{a_i}(f) = \sup_{t \in [-b,b]} |f(t)| = p_b(f)$, where $b := \max_{i=1,\ldots,n} a_i$, and so (2.4) is fulfilled. Hence, \mathcal{B}_d as in (2.7) is a basis of neighbourhoods of the origin for the lmc topology τ_Q .

The algebra $(\mathcal{C}_b(\mathbb{R}), \tau_Q)$ is not m-barrelled, because for instance the set $M := \{f \in \mathcal{C}_b(\mathbb{R}) : \sup_{t \in \mathbb{R}} |f(t)| \leq 1\}$ is an m-barrel but not a neighbourhood of the origin in $(\mathcal{C}_b(\mathbb{R}), \tau_Q)$. Indeed, no elements of the basis \mathcal{B}_d of neighbourhoods of the origin is entirely contained in M, because for any a > 0 and any $0 < r \leq 1$ the set rU_{p_a} also contains continuous functions bounded by r on [-a, a] but bounded by C > 1 on the whole \mathbb{R} and so not belonging to M.

2.3 Hausdorff Imc algebras

In Section 1.3, we gave some characterization of Hausdorff TVS which can of course be applied to establish whether an lmc algebra is Hausdorff or not. However, in this section we aim to provide necessary and sufficient conditions bearing only on the family of seminorms generating an lmc topology for being a Hausdorff topology.

Definition 2.3.1.

A family of seminorms $\mathcal{P} := \{p_i\}_{i \in I}$ on a vector space X is said to be separating if

$$\forall x \in X \setminus \{o\}, \exists i \in I \ s.t. \ p_i(x) \neq 0.$$

$$(2.8)$$

Note that the separation condition (2.8) is equivalent to

$$p_i(x) = 0, \forall i \in I \Rightarrow x = o$$

which by using Proposition 2.2.8 can be rewritten as

$$\bigcap_{i \in I, c > 0} c \mathring{U}_{p_i} = \{o\},$$
(2.9)

since $p_i(x) = 0$ is equivalent to say that $p_i(x) < c$, for all c > 0.

i

It is clear that if any of the elements in a family of seminorms is actually a norm, then the the family is separating.

Lemma 2.3.2. Let $\tau_{\mathcal{P}}$ be the topology induced by a separating family of seminorms $\mathcal{P} := (p_i)_{i \in I}$ on a vector space X. Then $\tau_{\mathcal{P}}$ is a Hausdorff topology.

Proof. 2

Let $x, y \in X$ be such that $x \neq y$. Since \mathcal{P} is separating, we have that $\exists i \in I$ with $p_i(x-y) \neq 0$. Then $\exists \varepsilon > 0$ s.t. $p_i(x-y) = 2\varepsilon$. Take $V_x := x + \varepsilon \mathring{U}_{p_i}$ and $V_y := y + \varepsilon \mathring{U}_{p_i}$. Since Theorem 2.2.12 guarantees that $(X, \tau_{\mathcal{P}})$ is a TVS where the set $\varepsilon \mathring{U}_{p_i}$ is a neighbourhood of the origin, V_x and V_y are neighbourhoods of x and y, respectively. They are clearly disjoint. Indeed, if there would exist $u \in V_x \cap V_y$ then $p_i(x-y) = p_i(x-u+u-y) \leq p_i(x-u) + p_i(u-y) < 2\varepsilon$, which is a contradiction. \Box

Proposition 2.3.3.

- a) A locally convex TVS is Hausdorff if and only if its topology can be induced by a separating family of seminorms.
- b) An lmc algebra is Hausdorff if and only if its topology can be induced by a separating family of submultiplicative seminorms.

²<u>Alternative proof</u> By Theorem 2.2.12, we know that $(X, \tau_{\mathcal{P}})$ is a TVS and that $\mathcal{B}_{\mathcal{P}} := \left\{ \bigcap_{k=1}^{n} \varepsilon \mathring{U}_{p_{i_{k}}} : i_{1}, \dots, i_{n} \in I, n \in \mathbb{N}, 0 < \varepsilon \leq 1 \right\}$ is a basis of neighbourhoods of the origin. Then $\bigcap_{B \in \mathcal{B}_{\mathcal{P}}} B = \bigcap_{i \in I, \varepsilon > 0} \varepsilon \mathring{U}_{p_{i}} \stackrel{(2.9)}{=} \{o\}$ and so Proposition 1.3.2 gives that $(X, \tau_{\mathcal{P}})$ is Hausdorff.

Proof.

a) Let (X, τ) be a locally convex TVS. Then we know that τ is induced by a directed family \mathcal{P} of seminorms on X and that $\mathcal{B}_d := \{rU_p : p \in \mathcal{Q}, 0 < r \leq 1\}$ (see Remark 2.2.18).

Suppose that (X, τ) is also Hausdorff. Then Proposition 1.3.2 ensures that for any $x \in X$ with $x \neq o$ there exists a neighbourhood V of the origin in Xs.t. $x \notin V$. This implies that there exists at least $B \in \mathcal{B}_d$ s.t. $x \notin B^3$ i.e. there exist $p \in \mathcal{P}$ and $0 < r \leq 1$ s.t. $x \notin rU_p$. Hence, p(x) > r > 0 and so $p(x) \neq 0$, i.e. \mathcal{P} is separating.

Conversely, if τ is induced by a separating family of seminorms \mathcal{P} , i.e. $\tau = \tau_{\mathcal{P}}$, then Lemma 2.3.2 ensures that X is Hausdorff.

b) A Hausdorff lmc algebra (X, τ) is in particular a Hausdorff lc tvs, so by a) there exists a separating family \mathcal{P} of seminorms s.t. $\tau = \tau_{\mathcal{P}}$. Since (X, τ) is an lmc algebra, Theorem 2.2.11 ensures that there exists \mathcal{Q} family of submultiplicative seminorms s.t. $\tau = \tau_{\mathcal{Q}}$. Hence, we have got $\tau_{\mathcal{P}} = \tau_{\mathcal{Q}}$ which gives in turn that for any $p \in \mathcal{P}$ there exist $q_1, q_2 \in \mathcal{Q}$ and $C_1, C_2 > 0$ s.t. $C_1q_1(x) \leq p(x) \leq C_2q_2(x), \forall x \in X$. This gives in turn that if q(x) = 0 for all $q \in \mathcal{Q}$ then we have p(x) = 0 for all $p \in \mathcal{P}$ which implies x = 0 because \mathcal{P} is separating. This shows that \mathcal{Q} is a separating family of submultiplicative seminorms \mathcal{P} , i.e. $\tau = \tau_{\mathcal{P}}$, then Lemma 2.3.2 ensures that X is Hausdorff and Theorem 2.2.11 that it is an lmc algebra.

Examples 2.3.4.

- 1. Every normed algebra is a Hausdorff lmc algebra, since every submultiplicative norm is a submultiplicative seminorm satisfying the separation property. Therefore, every Banach algebra is a complete Hausdorff lmc algebra.
- 2. Every family of submultiplicative seminorms on a vector space containing a submultiplicative norm induces a Hausdorff llmc topology.
- 3. Given an open subset Ω of \mathbb{R}^d with the euclidean topology, the space $\mathcal{C}(\Omega)$ of real valued continuous functions on Ω with the so-called topology of uniform convergence on compact sets is a lmc algebra. This topology is defined by the family \mathcal{P} of all the submultiplicative seminorms on $\mathcal{C}(\Omega)$ given by

$$p_K(f) := \max_{x \in K} |f(x)|, \forall K \subset \Omega \ compact.$$

³Since \mathcal{B}_d is a basis of neighbourhoods of the origin, $\exists B \in \mathcal{B}_d$ s.t. $B \subseteq V$. If x would belong to all elements of the basis then in particular it would be $x \in B$ and so also $x \in V$, contradiction.

Moreover, $(\mathcal{C}(\Omega), \tau_{\mathcal{P}})$ is Hausdorff, because the family \mathcal{P} is clearly separating. In fact, if $p_K(f) = 0$, $\forall K$ compact subsets of Ω then in particular $p_{\{x\}}(f) = |f(x)| = 0$, $\forall x \in \Omega$, which implies $f \equiv 0$ on Ω .

2.4 The finest Imc topology

In the previous sections we have seen how to generate topologies on an algebra which makes it into an lmc algebra. Among all of them, there is the finest one (i.e. the one having the largest number of open sets).

Proposition 2.4.1. The finest lmc topology on an algebra X is the topology induced by the family of all submultiplicative seminorms on X.

Proof.

Let us denote by S the family of all submultiplicative seminorms on the vector space X. By Theorem 2.2.11, we know that the topology τ_S induced by Smakes X into an lmc algebra. We claim that τ_S is the finest lmc topology. In fact, if there was a finer lmc topology τ (i.e. $\tau_S \subseteq \tau$ with (X, τ) lmc algebra) then Theorem 2.2.11 would give that τ is also induced by a family \mathcal{P} of submultiplicative seminorms. But then $\mathcal{P} \subseteq S$ and so $\tau = \tau_{\mathcal{P}} \subseteq \tau_S$ by definition of induced topology. Hence, $\tau = \tau_S$.

An alternative way of describing the finest lmc topology on an algebra without using the seminorms is the following:

Proposition 2.4.2. The collection of all absorbing absolutely m-convex sets of an algebra X is a basis of neighbourhoods of the origin for the finest lmc topology on X.

Proof.

Let τ_{max} be the finest lmc topology on X and \mathcal{M} the collection of all absorbing absolutely m-convex sets of X. Since \mathcal{M} fulfills all the properties required in Corollary 2.1.12, there exists a unique topology τ which makes X into an lmc algebra having as basis of neighbourhoods of the origin \mathcal{M} . Hence, by definition of finest lmc topology, $\tau \subseteq \tau_{max}$. On the other hand, (X, τ_{max}) is itself an lmc algebra and so Theorem 2.2.11 ensures that has a basis \mathcal{B}_{max} of neighbourhoods of the origin consisting of absorbing absolutely m-convex subsets of X. Then clearly \mathcal{B}_{max} is contained in \mathcal{M} and, hence, $\tau_{max} \subseteq \tau$. \Box

This result can be proved also using Proposition 2.4.1 and the correspondence between Minkowski functionals and absorbing absolutely convex subsets of X introduced in the Section 2.2.

Corollary 2.4.3. Every \mathbb{K} -algebra endowed with the finest lmc topology is an m-barrelled algebra.

Proof.

Let U be an m-barrel of (X, τ_{max}) . Then U is closed absolutely m-convex and so it is a neighbourhood of the origin by the previous proposition. \Box

Using basically the same proofs, we could show the analogous results for the finest lc topology, namely:

The finest lc topology on a \mathbb{K} -vector space X is the topology induced by the family of all seminorms on X or equivalently the topology having the collection of all absorbing absolutely convex sets of X as a basis of neighbourhoods of the origin. Hence, every vector space endowed with the finest lc topology is a barrelled space.

Recall that

Definition 2.4.4. A closed absorbing absolutely convex subset of a TVS is called a barrel. A TVS in which every barrel is a neighbourhood of the origin is called barrelled space.

It is also important to remark that while the finest lc topology on a \mathbb{K} -vector space (and in particular on a \mathbb{K} -algebra) is always Hausdorff, the finest lmc topology on a \mathbb{K} -algebra does not have necessarily this property.

Proposition 2.4.5. Any \mathbb{K} -vector space endowed with the finest lc topology is a Hausdorff TVS.

Proof.

Let X be any non-empty \mathbb{K} -vector space and S the family of all seminorms on X. By Proposition 2.3.3-a), it is enough to show that S is separating. We will do that, by proving that there always exists a non-zero norm on X. In fact, let $\mathcal{B} = (b_i)_{i \in I}$ be an algebraic basis of X then for any $x \in X$ there exist a finite subset J of I and $\lambda_j \in \mathbb{K}$ for all $j \in J$ s.t. $x = \sum_{j \in J} \lambda_j b_j$ and so we can define $||x|| := \max_{j \in J} |\lambda_j|$. Then it is easy to check that $|| \cdot ||$ is a norm on X and so $|| \cdot || \in S$.

Note that if X is a \mathbb{K} -algebra, then the previous proof does not guaranteed the existence of a non-zero norm on X because, depending on the multiplication in X, the norm $\|\cdot\|$ might be or not submultiplicative. In fact, there exist algebras on which no submultiplicative norm can be defined. For instance, if the algebra $\mathcal{C}(Y)$ of all complex valued continuous functions on a topological space Y contains an unbounded function then it does not admit a submultiplicative norm (see Sheet 4). Actually, there exist algebras on which no non-zero submultiplicative seminorms can be defined, e.g. the algebra of all linear operator on an infinite dimensional complex vector space (see [20, Theorem 3]). The finest lmc topology on such algebras is the trivial topology which is obviously not Hausdorff.

We conclude this section with a nice further property of the finest lmc topology involving characters of an algebra.

Definition 2.4.6. Let A be a \mathbb{K} -algebra. A character of A is a non-zero homomorphism of A into \mathbb{K} . The set of all characters is denoted by $\mathcal{X}(A)$.

Proposition 2.4.7. Every character on a \mathbb{K} -algebra A is continuous w.r.t. the finest lmc topology on A.

Proof. Let $\alpha : A \to \mathbb{K}$ be a character on A. For any $\varepsilon > 0$, we denote by $B_{\varepsilon}(0)$ the open ball in \mathbb{K} of radius ε and center $0 \in \mathbb{K}$, i.e. $B_{\varepsilon}(0) := \{k \in \mathbb{K} : |k| < \varepsilon\}$. Set $p(a) := |\alpha(a)|$ for all $a \in A$. Then p is a submultiplicative seminorm on A since for any $a, b \in A$ and $\lambda \in \mathbb{K} \setminus \{0\}$ we have that:

- $p(a+b) = |\alpha(a+b)| = |\alpha(a) + \alpha(b)| \le |\alpha(a)| + |\alpha(b)| = p(a) + p(b)$
- $p(\lambda a) = |\alpha(\lambda a)| = |\lambda\alpha(a)| = |\lambda||\alpha(a)| = |\lambda|p(a)$
- $p(ab) = |\alpha(ab)| = |\alpha(a)\alpha(b)| = |\alpha(a)||\alpha(b)| = p(a)p(b).$

Then $\alpha^{-1}(B_{\varepsilon}(0)) = \{a \in A : |\alpha(a)| < \varepsilon\} = \varepsilon U_p$, which is an absorbing absolutely m-convex subset of X and so, by Proposition 2.4.2, it is a neighbourhood of the origin in the finest lmc topology on X. Hence α is continuous at the origin and so continuous everywhere in A.

With a proof similar to the previous one, we can deduce that

Proposition 2.4.8. Every linear functional on a \mathbb{K} -vector space X is continuous w.r.t. the finest lc topology on X.

2.5 Topological algebras admitting lmc topologies

In this section we will look for sufficient conditions on a TA to be an lmc algebra. More precisely, we would like to find out under which conditions a locally convex algebra (i.e. a TA which is a locally convex TVS) is in fact an lmc algebra. The main result in this direction was proved by Michael in 1952 (see [21, Proposition 4.3]) and it is actually a generalization of a well-known theorem by Gel'fand within the theory of Banach algebras (see [9]).

Theorem 2.5.1 (Michael's Theorem). Let A be a lc algebra. If a) A is m-barrelled, and b) there exists a basis \mathcal{M} of neighbourhoods of the origin in A such that

$$\forall a \in A, \forall U \in \mathcal{M}, \exists \lambda > 0 : aU \subseteq \lambda U, \tag{2.10}$$

then A is an lmc algebra.

Proof.

Let us first give the main proof structure and then proceed to show the more technical details.

Claim 1 W.l.o.g. we can assume that \mathcal{M} consists of barrels. Consider the unitization A_1 of A equipped with the product topology (see Definition 1.1.3 and Section 1.3). Denote by \cdot the multiplication in A_1 and by $\overline{B}_1(0) := \{k \in \mathbb{K} : |k| \leq 1\}$. Then the family $\{\overline{B}_1(0) \times U : U \in \mathcal{M}\}$ is a basis of neighbourhoods of the origin (0, o) in A_1 and the following holds.

Claim 2 For any $U \in \mathcal{M}$, $V(U) := \{x \in A : (0,x) \cdot (\overline{B}_1(0) \times U) \subseteq (\overline{B}_1(0) \times U)\}$ is an m-barrel subset of A.

Then the assumption a) ensures that each V(U) is a neighbourhood of the origin in A. Moreover, for any $U \in \mathcal{M}$, $(1, o) \in (\overline{B}_1(0) \times U)$ and so

$$\forall x \in V(U), (0, x) = (0, x) \cdot (1, o) \in (\overline{B}_1(0) \times U),$$

which provides that $V(U) \subseteq U$. Hence, $\{V(U) : U \in \mathcal{M}\}$ is a basis of neighbourhoods of the origin in A consisting of m-barrels and so, by Theorem 2.1.11, A is an lmc algebra.

Proof. Claim 1

If \mathcal{M} is not already consisting of all barrels, then we can always replace it by $\widetilde{M} := \{\overline{\operatorname{conv}_{\mathrm{b}}(U)} : U \in \mathcal{M}\}$, because \widetilde{M} is a basis of neighbourhoods of the origin in A fulfilling (2.10).

In fact, since A is a lc TVS, then there exists a basis \mathcal{N} of neighbourhoods of the origin in A consisting of barrels. Then, since also \mathcal{M} is a basis of neighbourhoods of the origin in A, we have that:

$$\forall V \in \mathcal{N}, \exists U \in \mathcal{M} : U \subseteq V.$$

As $\overline{\operatorname{conv}_{\mathrm{b}}(U)}$ is the smallest closed convex balanced subset of A containing U and V has all such properties, we get that $\overline{\operatorname{conv}_{\mathrm{b}}(U)} \subseteq V$. Hence, \widetilde{M} is a basis of neighbourhoods of the origin in A.

Moreover, let $a \in A$ and $U \in \mathcal{M}$. By assumption b), we know that there exists $\lambda > 0$ such that $aU \subseteq \lambda U$. Now recalling that $\operatorname{conv}_{\mathrm{b}}(U) = \operatorname{conv}(\operatorname{bal}(U))$, we can write any $x \in \operatorname{conv}_{\mathrm{b}}(U)$ as $x = \sum_{i=1}^{n} \mu_i \delta_i u_i$ for some $n \in \mathbb{N}$, $u_i \in U$,

 $\mu_i \in [0,1]$ with $\sum_{i=1}^n \mu_i = 1$, and $\delta_i \in \mathbb{K}$ with $|\delta_i| \leq 1$. Then for each $i \in \{1, \ldots, n\}$ there exist $\tilde{u}_i \in U$ such that:

$$ax = \sum_{i=1}^{n} \mu_i \delta_i au_i = \sum_{i=1}^{n} \mu_i \delta_i \lambda \tilde{u}_i = \lambda \sum_{i=1}^{n} \mu_i (\delta_i \tilde{u}_i)$$

and so $ax \in \lambda \cdot \operatorname{conv_b}(U)$. Hence, $a \cdot \operatorname{conv_b}(U) \subseteq \lambda \cdot \operatorname{conv_b}(U)$. This together with the separate continuity of the multiplication in A and the fact that the scalar multiplication is a homeomorphism imply that

$$a \cdot \overline{\operatorname{conv}_{\mathrm{b}}(U)} \subseteq \overline{a \cdot \operatorname{conv}_{\mathrm{b}}(U)} \subseteq \overline{\lambda \cdot \operatorname{conv}_{\mathrm{b}}(U)} = \lambda \cdot \overline{\operatorname{conv}_{\mathrm{b}}(U)}.$$

This shows that \widetilde{M} fulfills (2.10).

Proof. Claim 2 Let $U \in \mathcal{M}$ and $V(U) := \{x \in A : (0, x) \cdot (\overline{B}_1(0) \times U) \subseteq (\overline{B}_1(0) \times U)\}$. Then: • V(U) is multiplicative.

For any $a, b \in V(U)$ we have

 $\begin{array}{l} (0,ab)\cdot(\overline{B}_1(0)\times U)=(0,a)\cdot(0,b)\cdot(\overline{B}_1(0)\times U)\subseteq(0,a)\cdot(\overline{B}_1(0)\times U)\subseteq(\overline{B}_1(0)\times U),\\ \text{i.e. } ab\in V(U). \end{array}$

• V(U) is closed.

Let us show that $A \setminus V(U)$ is open, i.e. that for any $x \in A \setminus V(U)$ there exists $N \in \mathcal{M}$ such that $x + N \subseteq A \setminus V(U)$. If $x \in A \setminus V(U)$, then there exist $t \in \overline{B}_1(0)$ and $u \in U$ such that $(0, x) \cdot (t, u) \notin (\overline{B}_1(0) \times U)$, i.e. $tx + ux \in A \setminus U$. As U is closed, $A \setminus U$ is open and so there exists $W \in \mathcal{M}$ s.t.

$$tx + ux + W \subseteq A \setminus U. \tag{2.11}$$

Take $N \in \mathcal{M}$ s.t. $uN \subseteq \frac{1}{2}W$ and $N \subseteq \frac{1}{2}W$ (this exists because left multiplication is continuous and \mathcal{M} is basis of neighbourhoods of the origin). Then $x + N \subseteq A \setminus U$, because otherwise there would exists $n \in N$ such that $x + n \in V(U)$ and so $(0, n+x) \cdot (t, u) \in (\overline{B}_1(0) \times U)$ that is $nt+nu+xt+xu \in U$, which in turns implies $xt+xu \in U-tN-uN \subseteq U-N-\frac{1}{2}W \subseteq U-\frac{1}{2}W-\frac{1}{2}W \subseteq U-W$, i.e. $xt+xu+W \subseteq U$ which contradicts (2.11).

• V(U) is absorbing.

Let $a \in A$. Then (2.10) ensures that there exists $\lambda > 0$ s.t. $aU \subseteq \lambda U$. Also, since U is absorbing, there exists $\mu > 0$ such that $a \in \mu U$. Take $\rho := \frac{1}{\lambda + \mu}$. Then for all $k \in \mathbb{K}$ with $|k| \leq \rho$ and for any $(t, u) \in (\overline{B}_1(0) \times U)$ we get that $kta + kau \in kt\mu U + k\lambda U \subseteq k\mu U + k\lambda U = k(\mu + \lambda)U \subseteq U$ where in both inclusions we have used that U is balanced together with $|t| \leq 1$ in the first and $|k(\mu + \lambda)| \leq \rho |\mu + \lambda| = 1$ in the second. Hence, we have obtained that $(0, ka) \cdot (t, u) = (0, kta + kau) \in (\overline{B}_1(0) \times U)$ which gives that $ka \in V(U)$.

• V(U) is balanced.

Let $a \in V(U)$ and $k \in \mathbb{K}$ with $|k| \leq 1$. Then

 $(0,ka) \cdot (\overline{B}_1(0) \times U) = k(0,a) \cdot (\overline{B}_1(0) \times U) \subseteq (k\overline{B}_1(0) \times kU) \subseteq (\overline{B}_1(0) \times U)$

where in the last inclusion we used that both $\overline{B}_1(0)$ and U are balanced. • V(U) is convex.

Let $a, b \in V(U)$ and $\mu \in [0, 1]$. Then for any $(t, u) \in (\overline{B}_1(0) \times U)$ we know that $(0, a) \cdot (t, u) \in (\overline{B}_1(0) \times U)$ and $(0, b) \cdot (t, u) \in (\overline{B}_1(0) \times U)$, which give in turn that $at + au \in U$ and $bt + bu \in U$. Therefore, the convexity of U implies that $\mu(at + au) + (1 - \mu)(bt + bu) \in U$ and so we obtain

$$(0, \mu a + (1 - \mu)b) \cdot (t, u) = (0, \mu(at + au) + (1 - \mu)(bt + bu)) \in (B_1(0) \times U),$$

i.e. $\mu a + (1 - \mu)b \in V(U).$

Let us present now a stronger version of Michael's theorem, which has however the advantage of providing a less technical and so more manageable sufficient condition for a topology to be lmc. This more convenient condition actually identifies an entire class of TA: the so-called *A-convex algebras* introduced by Cochran, Keown and Williams in the early seventies [5].

Definition 2.5.2. A \mathbb{K} -algebra X is called A-convex if it is endowed with the topology induced by an absorbing family of seminorms on X.

Definition 2.5.3. A seminorm p on a \mathbb{K} -algebra X is called:

- left absorbing if $\forall a \in X, \exists \lambda > 0 \ s.t. \ p(ax) \leq \lambda p(x), \forall x \in X.$
- right absorbing if $\forall a \in X, \exists \lambda > 0 \ s.t. \ p(xa) \leq \lambda p(x), \forall x \in X.$
- absorbing if it is both left and right absorbing.

Proposition 2.5.4. Every A-convex algebra is a lc algebra.

Proof.

Let (X, τ) be an A-convex algebra. Then by definition $\tau = \tau_{\mathcal{P}}$ where \mathcal{P} is a family of absorbing seminorm. Hence, by Theorem 2.2.12, (X, τ) is an lc TVS. It remains to show that it is a TA. Let $a \in X$ and consider the left multiplication $\ell_a : X \to X, x \mapsto ax$. Since any $p \in \mathcal{P}$ is left absorbing, we have that there exists $\lambda > 0$ such that $p(ax) \leq \lambda p(x)$ for all $x \in X$ and so that $\frac{1}{\lambda}U_p \subseteq \ell_a^{-1}(U_p)$. Hence, ℓ_a is τ -continuous. Similarly, one can prove the continuity of the right multiplication. We can then conclude that (X, τ) is an lc algebra. Note that not every lc algebra is A-convex (see Sheet 4) but every lmc algebra is A-convex as the submultiplicativity of the generating seminorms implies that they are absorbing. Let us focus now on the inverse question of establishing when an A-convex algebra is lmc.

Theorem 2.5.5. Every m-barrelled A-convex algebra is an lmc algebra.

Proof.

Let (X, τ) be an m-barrelled A-convex algebra. By the previous proposition, we have that (X, τ) is an lc algebra. Denote by $\mathcal{P} := \{p_i : i \in I\}$ a family of absorbing seminorm generating τ . Then, by Proposition 2.2.17, $\mathcal{Q} := \{\max_{i \in B} p_i : \emptyset \neq B \subseteq I \text{ with } B \text{ finite }\}$ is a directed family of seminorms such that $\tau = \tau_{\mathcal{Q}}$. Also, each $q \in \mathcal{Q}$ is absorbing. Indeed, $q = \max_{i \in B} p_i$ for some $\emptyset \neq B \subseteq I$ with B finite and so for any $i \in B$ and any $a \in X$ we have that there exists $\lambda_i > 0$ such that $p_i(ax) \leq \lambda_i p(x)$ for all $x \in X$. Hence, for any $a \in X$ we get

$$q(ax) = \max_{i \in B} p_i(ax) \le \max_{i \in B} \lambda_i p_i(x) \le \lambda \max_{i \in B} p_i(x) = q(x), \forall x \in X,$$

where $\lambda := \max_{i \in B} \lambda_i$. Then $\mathcal{M} := \{\varepsilon U_q : q \in \mathcal{Q}, 0 < \varepsilon \leq 1\}$ is a basis of neighbourhoods of the origin for (X, τ) and for each $a \in A, q \in \mathcal{Q}$ and $0 < \varepsilon \leq 1$ we have that if $x \in a\varepsilon U_q$ then $x = a\varepsilon y$ for some $y \in U_q$ and so $q(x) = q(a\varepsilon y) \leq \varepsilon q(ay) \leq \lambda \varepsilon q(y) \leq \lambda \varepsilon$ i.e. $x \in \lambda \varepsilon U_q$. Hence, we proved that $\forall a \in A, \forall q \in \mathcal{Q}, \forall 0 < \varepsilon \leq 1, a\varepsilon U_q \subseteq \lambda \varepsilon U_q$ which means that \mathcal{M} fulfills condition b) in Theorem 2.5.1. Then (X, τ) satisfies all the assumptions of Theorem 2.5.1 which guarantees that it is an Imc algebra. \Box

To conclude this section let us just restate the result by Gel'fand mentioned in the beginning in one of the many formulation which reveals the analogy with Michael's theorem.

Theorem 2.5.6. If X is a \mathbb{K} -algebra endowed with a norm which makes it into a Banach space and a TA, then there exists an equivalent norm on X which makes it into a Banach algebra.

Chapter 3

Further special classes of topological algebras

3.1 Metrizable and Fréchet algebras

Definition 3.1.1. A metrizable algebra X is a TA which is in particular a metrizable TVS, i.e. a TVS whose topology is induced by a metric.

We recall that a metric d on a set X is a mapping $d: X \times X \to \mathbb{R}^+$ with the following properties:

- 1. d(x, y) = 0 if and only if x = y (identity of indiscernibles);
- 2. d(x, y) = d(y, x) for all $x, y \in X$ (symmetry);
- 3. $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$ (triangular inequality).

Saying that the topology of a TVS (X, τ) is induced by a metric d means that for any $x \in X$ the sets of all open (or equivalently closed) balls:

$$B_r(x) := \{ y \in X : d(x, y) < r \}, \quad \forall r > 0$$

forms a basis of neighbourhoods of x for τ .

There exists a completely general characterization of metrizable TVS.

Theorem 3.1.2. A TVS X is metrizable if and only if X is Hausdorff and has a countable basis of neighbourhoods of the origin.

Note that one direction is quite straightforward. Indeed, suppose that X is a metrizable TVS and that d is a metric defining the topology of X, then the collection of all $B_{\frac{1}{n}}(o)$ with $n \in \mathbb{N}$ is a countable basis of neighbourhoods of the origin o in X. Moreover, the intersection of all these balls is just the singleton $\{o\}$, which proves that the TVS X is also Hausdorff (see Proposition 1.3.2) The other direction requires more work and we are not going to prove it in full generality as it would go beyond the aim of this course (see e.g. [23, Chapter I, Section 6.1] or [16, proof of Theorem 1.1] for a proof for locally convex TVS). However, we are going to use Theorem 3.1.2 to give a characterization of all metrizable lmc algebras.

Theorem 3.1.3. Let A be a \mathbb{K} -algebra. Then the following are equivalent: a) A is a metrizable lmc algebra

- b) A is a TVS having a decreasing sequence of m-barrels with trivial intersection as a basis of neighbourhoods of the origin.
- c) A is a TVS whose topology is generated by an increasing sequence of submultiplicative seminorms which form a separating family.

The obvious analogous statement is true for metrizable lc algebras.

Proof.

a) \Rightarrow b) Suppose that (A, τ) is a metrizable lmc algebra. Then in particular (A, τ) is a metrizable TVS and so by Theorem 3.1.2 it is Hausdorff and has a countable basis $\{U_n : n \in \mathbb{N}\}$ of neighbourhoods of the origin. As (A, τ) is an lmc algebra, by Theorem 2.1.11, we can assume w.l.o.g. that each U_n is an m-barrel. Now for each $n \in \mathbb{N}$ set $V_n := U_1 \cap \cdots \cap U_n$. Then one can easily verify that each V_n is still an m-barrel and clearly $V_{n+1} \subseteq V_n$. Hence, the decreasing sequence $\{V_n : n \in \mathbb{N}\}$ is a basis of neighbourhoods of the origin in (A, τ) consisting of m-barrels. The Hausdorfness of (A, τ) implies, by Proposition 1.3.2, that $\bigcap_{n \in \mathbb{N}} V_n = \{o\}$.

b) \Rightarrow c) Suppose that (A, τ) is a TVS and that $\{V_n : n \in \mathbb{N}\}$ is a basis of neighbourhoods of the origin such that, for any $n \in \mathbb{N}$, V_n is an m-barrel, $V_{n+1} \subseteq V_n$ and $\bigcap_{n \in \mathbb{N}} V_n = \{o\}$. Then Theorem 2.1.11 guarantees that (A, τ) is an lmc algebra and the family $S := \{p_{V_n} : n \in \mathbb{N}\}$ is a family of submultiplicative seminorms generating τ (see proof of Theorem 2.2.12). Actually, S is an increasing sequence, since $V_{n+1} \subseteq V_n$ implies that $p_{V_n} \leq p_{V_{n+1}}$. Moreover, we have that $\{o\} \subseteq \bigcap_{n \in \mathbb{N}} \mathring{U}_{p_{V_n}} \subseteq \bigcap_{n \in \mathbb{N}} V_n = \{o\}$ and so $\bigcap_{n \in \mathbb{N}, c > 0} c \mathring{U}_{p_{V_n}} = \{o\}$, i.e. S is separating (c.f. (2.9)).

c) \Rightarrow a) Suppose that (A, τ) is a TVS and that $\mathcal{P} := \{p_n : n \in \mathbb{N}\}$ is a separating increasing sequence of submultiplicative seminorm generating τ . By Theorem 2.2.12 and Proposition 2.3.3, (A, τ) is a Hausdorff lmc algebra. W.l.o.g. we can assume that \mathcal{P} is directed and so, by using Exercise 3 in Sheet 3, we have that $\{\frac{1}{n}U_{p_n} : n \in \mathbb{N}\}$ is a countable basis of neighbourhoods of the origin. Then Theorem 3.1.2 ensures that (A, τ) is also a metrizable TVS and, hence, a metrizable lmc algebra.

A special class of metrizable algebras are the so-called *Fréchet algebras*.

Definition 3.1.4. A Fréchet algebra is a TA which is in particular a Fréchet TVS, i.e. a complete metrizable lc TVS.

It is clear that every Fréchet algebra is a Hausdorff complete lc algebra whose topology is induced by an increasing family of seminorms, but these are not necessarily submultiplicative. If this is the case, we speak of *Fréchet lmc algebras*.

Definition 3.1.5. A Fréchet lmc algebra is a complete metrizable lmc algebra.

As completeness is fundamental to understand the structure of a Fréchet algebra, let us recall here some of the most important properties of complete TVS (for a more detailed exposition about complete TVS see e.g. [15, Section 2.5] or [24, Part I, Section 5]).

Definition 3.1.6.

A TVS X is said to be complete if every Cauchy filter on X converges to a point x of A.

It is important to distinguish between completeness and sequentially completeness.

Definition 3.1.7.

A TVS X is said to be sequentially complete if any Cauchy sequence in X converges to a point in A.

Clearly, a TA is complete (resp. sequentially complete) if it is in particular a complete (resp. sequentially complete) TVS. Remind that

Definition 3.1.8. A filter \mathcal{F} on a TVS (X, τ) is said to be a Cauchy filter if

$$\forall U \in \mathcal{F}(o) \text{ in } X, \exists M \in \mathcal{F} : M - M \subset U, \tag{3.1}$$

where $\mathcal{F}(o)$ denotes the filter of neighbourhoods of the origin o in (X, τ) .

Definition 3.1.9. A sequence $S := \{x_n\}_{n \in \mathbb{N}}$ of points in a TVS (X, τ) is said to be a Cauchy sequence if

$$\forall U \in \mathcal{F}(o) \text{ in } X, \exists N \in \mathbb{N} : x_m - x_n \in U, \forall m, n \ge N,$$
(3.2)

where $\mathcal{F}(o)$ denotes the filter of neighbourhoods of the origin o in (X, τ) .

Proposition 3.1.10.

The filter associated with a Cauchy sequence in a TVS X is a Cauchy filter.

Proof.

Let S be a Cauchy sequence. Then, recalling that the collection $\mathcal{B} := \{S_m : m \in \mathbb{N}\}$ with $S_m := \{x_n \in S : n \geq m\}$ is a basis of the filter \mathcal{F}_S associated with S, we can easily rewrite (3.2) as

$$\forall U \in \mathcal{F}(o) \text{ in } X, \exists N \in \mathbb{N} : S_N - S_N \subset U.$$

This immediately gives that \mathcal{F}_S fulfills (3.2) and so that it is a Cauchy filter.

It is then not hard to prove that

Proposition 3.1.11.

If a TVS X is completen then A is sequentially complete.

Proof.

Let $S := \{x_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence of points in A. Then Proposition 3.1.10 guarantees that the filter \mathcal{F}_S associated to S is a Cauchy filter in A. By the completeness of A we get that there exists $x \in A$ such that \mathcal{F}_S converges to x. This is equivalent to say that the sequence S is convergent to $x \in A$ (see [15, Proposition 1.1.29]). Hence, A is sequentially complete. \Box

The converse is false in general (see [15, Example 2.5.9]). However, the two notions coincide in metrizable TVS, and so we have that

Proposition 3.1.12. A metrizable lc algebra is a Fréchet algebra if and only if it is sequentially complete.

Another important property of Fréchet algebras is that they are Baire spaces, i.e. topological spaces in which the union of any countable family of closed sets, none of which has interior points, has no interior points itself (or, equivalently, the intersection of any countable family of everywhere dense open sets is an everywhere dense set). This is actually a consequence of the following more general result:

Proposition 3.1.13. A complete metrizable TVS X is a Baire space.

Proof. (see [16, Proposition 1.1.8])

Example 3.1.14. An example of Baire space is \mathbb{R} with the euclidean topology. Instead \mathbb{Q} with the subset topology given by the euclidean topology on \mathbb{R} is not a Baire space. Indeed, for any $q \in \mathbb{Q}$ the subset $\{q\}$ is closed and has empty interior in \mathbb{Q} , but $\cup_{q \in \mathbb{Q}} \{q\} = \mathbb{Q}$ which has interior points in \mathbb{Q} (actually its interior is the whole \mathbb{Q}).

Corollary 3.1.15. Every Fréchet TVS is barrelled. In particular, every Fréchet algebra is m-barrelled.

Proof.

Let (X, τ) be a Fréchet TVS and V a barrelled subset of X. Then V is absorbing and closed, so $X = \bigcup_{n \in \mathbb{N}} nV$ is a countable union of closed sets. Hence, as Proposition 3.1.13 ensures that (X, τ) is a Baire space, we have that there exists $k \in \mathbb{N}$ such that $(\mathring{kV}) \neq \emptyset$. This implies that there exists $x \in (\mathring{kV})$, i.e. there exists a neighbourhood N of the origin in (X, τ) such that $x + N \subseteq V$. As (X, τ) is in particular an lc TVS, we can assume that Nis absolutely convex. Then we get

$$N = \frac{1}{2}N - \frac{1}{2}N = \frac{1}{2}(x+N) + \frac{1}{2}(-x-N) \subseteq \frac{1}{2}V + \frac{1}{2}(-V) = V,$$

where in the last equality we used that V is a barrel and so absolutely convex. Hence, we can conclude that V is a neighbourhood of the origin and so (X, τ) is barrelled.

If (X, τ) is a Fréchet algebra, then it is in particular a Fréchet TVS and so the previous part of the proof guarantees that every m-barrelled subset of X is a neighbourhood of the origin, i.e. (X, τ) is an m-barrelled algebra.

This result together with Theorem 2.5.1 (resp. Theorem 2.5.5) clearly provides that every Fréchet algebra having a basis of neighbourhoods of the origin which satifies (2.10) (resp. every A-convex Fréchet algebra) is lmc. Proposition 3.1.13 plays also a fundamental role in proving the following general property of complete metrizable algebras and so of Fréchet algebras.

Proposition 3.1.16. Every complete metrizable algebra is a TA with continuous multiplication.

Proof.

Let A be a complete metrizable algebra. The metrizability provides the existence of a countable basis $\mathcal{B} := \{W_n : n \in \mathbb{N}\}$ of neighbourhoods of the origin. We aim to show that for any $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $W_m W_m \subseteq W_n$.

Fixed $n \in \mathbb{N}$, as we are in a TVS, there always exists a closed neighbourhood V of the origin such that $V - V \subseteq W_n$. As for any $b \in A$ the right multiplication $r_b : A \to A, a \mapsto ab$ is continuous we have that

 $r_b^{-1}(V) := \{a \in A : ab \in V\}$ is closed. For any $k \in \mathbb{N}$, set $U_k := \bigcap_{b \in W_k} r_b^{-1}(V)$. Then each U_k is closed and $A = \bigcup_{k \in \mathbb{N}} U_k$.

Since A is a Baire space by Proposition 3.1.13, we have that there exists $h \in \mathbb{N}$ such that $\mathring{U}_h \neq \emptyset$. Therefore, there exists $x \in \mathring{U}_h$, i.e. there exists $j \in \mathbb{N}$ such that $x + W_j \subseteq U_h$. This in turn provides that

$$U_h - U_h \supseteq x + W_j - x - W_j = W_j - W_j \supseteq W_j$$

Since \mathcal{B} is a basis for the filter of neighbourhoods of the origin, we can find $m \in \mathbb{N}$ such that $W_m \subseteq W_i \cap W_h$ and therefore

$$W_m W_m \subseteq W_j W_h \subseteq (U_h - U_h) W_h = U_h W_h - U_h W_h \subseteq V - V \subseteq W_n,$$

where in the last inclusion we have just used the definition of U_h . Hence, the multiplication in A is jointly continuous.

Example 3.1.17.

1) Let $C^{\infty}([0,1])$ be the space of all real valued infinitely differentiable functions on [0,1] equipped with pointwise operations. We endow the algebra $C^{\infty}([0,1])$ with the topology $\tau_{\mathcal{P}}$ generated by $\mathcal{P} := \{p_n : n \in \mathbb{N}_0\}$ with $p_n(f) := \sup_{x \in [0,1]} |(D^{(n)}f)(x)|$ for any $f \in C^{\infty}([0,1])$ (here $D^{(n)}f$ denotes the n-th derivative of f). \mathcal{P} is a countable separating family of seminorms so that $(C^{\infty}([0,1]), \tau_{\mathcal{P}})$ is a metrizable lc algebra but the seminorms in \mathcal{P} are not submultiplicative since if for example we take f(t) := t then $p_1(f^2) = 2 > 1 = p_1(f)p_1(f)$. However, we are going to show that $\tau_{\mathcal{P}}$ can be in fact generated by a countable separating family of submultiplicative seminorms and so it is actually an lmc algebra. First, let us consider the family $\mathcal{R} := \{r_n := \max_{j=0,...,n} p_j : n \in \mathbb{N}_0\}$. As each $p_n \leq r_n$, we have that $\tau_{\mathcal{P}} = \tau_{\mathcal{R}}$ and also for all $n \in \mathbb{N}_0$, $f, g \in C^{\infty}([0,1])$ the following holds:

$$\begin{aligned} r_n(fg) &= \max_{j=0,\dots,n} p_j(fg) = \max_{j=0,\dots,n} \sup_{x\in[0,1]} |(D^{(j)}fg)(x)| \\ &\leq \max_{j=0,\dots,n} \sum_{k=0}^j \binom{j}{k} \sup_{x\in[0,1]} |(D^{(j-k)}f)(x)| \sup_{x\in[0,1]} |(D^{(k)}g)(x)| \\ &\leq \max_{j=0,\dots,n} \sum_{k=0}^j \binom{j}{k} p_{j-k}(f) p_k(g) \\ &\leq \left(\max_{j=0,\dots,n} \sum_{k=0}^j \binom{j}{k} \right) r_n(f) r_n(g) = 2^n r_n(f) r_n(g). \end{aligned}$$

¹Clearly, each $U_k \subset A$ and so $\bigcup_{k \in \mathbb{N}} U_k \subseteq A$. Conversely, if $x \in A$, then the continuity of the left multiplication implies that there exists $j \in \mathbb{N}$ such that $xW_j \subseteq V$ and so $x \in r_b^{-1}(V)$ for all $b \in W_j$, i.e. $x \in \bigcup_{k \in \mathbb{N}} U_k$.

Therefore, setting $q_n(f) := 2^n r_n(f)$ for any $n \in \mathbb{N}_0$ and $f \in \mathcal{C}^{\infty}([0,1])$, we have that the family $\mathcal{Q} := \{q_n : n \in \mathbb{N}_0\}$ is a countable family of submultiplicative seminorms such that $\tau_{\mathcal{Q}} = \tau_{\mathcal{R}} = \tau_{\mathcal{P}}$. Hence, $(\mathcal{C}^{\infty}([0,1]), \tau_{\mathcal{P}})$ is a metrizable lmc algebra. Actually, it is also complete and so a Fréchet lmc algebra. Indeed, as it is metrizable we said it is enough to show sequentially complete. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathcal{C}^{\infty}([0,1]), \tau_{\mathcal{P}})$, i.e. $\forall U \in \tau_{\mathcal{P}} \exists N \in \mathbb{N} \text{ s.t. } f_n - f_m \in U \forall n, m \geq N$. Thus,

$$\forall \varepsilon > 0 \forall k \in \mathbb{N}_0 \exists N \in \mathbb{N} : p_k(f_n - f_m) \le \varepsilon \quad \forall n, m \ge N$$
(3.3)

which yields

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : \sup_{x \in [0,1]} |f_n(x) - f_m(x)| \le \varepsilon \quad \forall n, m \ge N$$
(3.4)

so that $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in \mathbb{R} for all $x \in [0,1]$. Since \mathbb{R} is complete for all $x \in [0,1]$ there exists $y_x \in \mathbb{R}$ s.t. $f_n(x) \to y_x$ as $n \to \infty$. Set $f(x) := y_x$ for all $x \in [0,1]$, then $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f. The latter combined with (3.4) yields that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to fwhich implies that $f \in \mathcal{C}([0,1])$ by [16, Lemma 1.2.2]. By (3.4) for k = 1, we get $((D^{(1)}f_n)(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in \mathbb{R} for all $x \in [0,1]$ and reasoning as above $(D^{(1)}f_n)_{n\in\mathbb{N}}$ uniformly converges to some g on [0,1]. By [16, Lemma 1.2.3], $g = D^{(1)}f$ and so $f \in \mathcal{C}^1([0,1])$. Proceeding by induction, we can show that $(D^{(j)}f_n)_{n\in\mathbb{N}}$ converges uniformly to $D^{(j)}f$ on [0,1] and $f \in \mathcal{C}^j([0,1])$ for all $j \in \mathbb{N}_0$, i.e.

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : \sup_{x \in [0,1]} \left| D^{(j)} f_n(x) - D^{(j)} f(x) \right| \le \varepsilon \quad \forall n, m \ge N.$$

Therefore, $(f_n)_{n \in \mathbb{N}}$ converges to $f \in \mathcal{C}^{\infty}([0,1])$ in $\tau_{\mathcal{P}}$. Hence, completeness is proven.

Note that we could have first proved completeness and then used Corollary 3.1.15 to show that $(\mathcal{C}^{\infty}([0,1]), \tau_{\mathcal{P}})$ is m-barrelled. Then, observing that $\tau_{\mathcal{P}} = \tau_{\mathcal{R}}$ and that the seminorms in \mathcal{R} are all absorbing, we could have applied Theorem 2.5.5 and concluded that $(\mathcal{C}^{\infty}([0,1]), \tau_{\mathcal{P}})$ is an lmc algebra.

2) Let $\mathbb{K}^{\mathbb{N}} = \{\underline{a} = (a_n)_{n \in \mathbb{N}} : a_n \in \mathbb{K}\}$ be the space of all \mathbb{K} -valued sequences endowed $\mathbb{K}^{\mathbb{N}}$ with the topology $\tau_{\mathcal{P}}$ generated by $\mathcal{P} := \{p_n : n \in \mathbb{N}\}$ with $p_n(\underline{a}) := \max_{k \leq n} |a_k|$ for any $\underline{a} \in \mathbb{K}^{\mathbb{N}}$ $(n \in \mathbb{N})$. Since \mathcal{P} is an increasing family of submultiplicative seminorms and separating, $(\mathbb{K}^{\mathbb{N}}, \tau_{\mathcal{P}})$ is a metrizable lmc algebra by Theorem 3.1.3. Indeed, if $\underline{a}, \underline{b} \in \mathbb{K}^{\mathbb{N}}$, then

$$p_n(\underline{ab}) = \max_{k \le n} |a_k b_k| \le \max_{k \le n} |a_k| \max_{k \le n} |b_k| = p_n(\underline{a}) p_n(\underline{b})$$

for all $n \in \mathbb{N}$. Further, if $p_n(\underline{a}) = 0$ for all $n \in \mathbb{N}$, then

$$\max_{k \le n} |a_k| = 0, \forall n \in \mathbb{N} \Rightarrow |a_k| = 0, \forall k \in \mathbb{N} \Rightarrow a \equiv 0.$$

Moreover, $(\mathbb{K}^{\mathbb{N}}, \tau_{\mathcal{P}})$ is sequentially complete and so complete (prove it yourself). Hence, it is a Fréchet lmc algebra.

3) The Arens-algebra L^ω([0,1]) := ∩_{p≥1} L^p([0,1]) endowed with the topology τ_P generated by P := { || · ||_p : p ∈ ℕ} is a Fréchet lc algebra which is not lmc. We have already showed that it is an lc algebra but not lmc. Metrizability comes from the fact that the family of seminorms is countable and increasing (Hölder-inequality). Proving completeness is more complicated which we will maybe see it later on.

3.2 Locally bounded algebras

The TAs we are going to study in this section were first introduced by W. Zelazko in the 1960's and provide non-trivial examples of TAs whose underlying space is not necessarily locally convex (so they are neither necessarily lc algebras nor lmc algebras) but they still share several nice properties of Banach and/or lmc algebras.

Definition 3.2.1. A TA is locally bounded (lb) if there exists a neighbourhood of the origin which is bounded. Equivalently, a locally bounded algebra is a TA which is in particular a locally bounded TVS (i.e. the space has a bounded neighbourhood of the origin).

Recall that:

Definition 3.2.2. A subset B of a TVS X is bounded if for any neighbourhood U of the origin in X there exists $\lambda > 0$ s.t. $B \subseteq \lambda U$ (i.e. B can be swallowed by any neighbourhood of the origin).

This generalizes the concept of boundedness we are used to in the theory of normed and metric spaces, where a subset is bounded whenever we can find a ball large enough to contain it. **Example 3.2.3.** The subset $Q := [0,1]^2$ is bounded in $(\mathbb{R}^2, \|\cdot\|)$ as for any $\varepsilon > 0$ there exists $\lambda > 0$ s.t. $Q \subseteq \lambda B_{\varepsilon}(o)$ namely, if $\varepsilon \ge \sqrt{2}$ take $\lambda = 1$, otherwise take $\lambda = \frac{\sqrt{2}}{\varepsilon}$.

Proposition 3.2.4. Every Hausdorff locally bounded algebra is metrizable.

Proof.

Let (A, τ) be a Hausdorff locally bounded algebra and $\mathcal{F}(o)$ its filter of neighbourhoods of the origin. Then there exists $U \in \mathcal{F}(o)$ bounded. W.l.o.g. we can assume that U is balanced. Indeed, if this is not the case, then we can replace it by some $V \in \mathcal{F}(o)$ balanced s.t. $V \subseteq U$. Then the boundedness of U provides that $\forall N \in \mathcal{F}(o) \exists \lambda > 0$ s.t. $U \subseteq \lambda N$ and so $V \subseteq \lambda N$, i.e. V is bounded and balanced.

The collection $\{\frac{1}{n}U : n \in \mathbb{N}\}$ is a countable basis of neighbourhoods of the origin o. In fact, for any $N \in \mathcal{F}(o)$ there exists $\lambda > 0$ s.t. $U \subseteq \lambda N$, i.e. $\frac{1}{\lambda}U \subseteq N$, and so $\frac{1}{n}U \subseteq \frac{1}{\lambda}U$ for all $n \geq \lambda$ as U is balanced. Hence, we obtain that for any $N \in \mathcal{F}(o)$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n}U \subseteq N$. Then we can apply Theorem 3.1.2 which gives that (A, τ) is a metrizable algebra.

The converse is not true in general as for example the countable product of 1–dimensional metrizable TVS is metrizable but not locally bounded.

Corollary 3.2.5. Every complete Hausdorff lb algebra has continuous multiplication.

Proof. Since local boundedness and Hausdorffness imply metrizability, Proposition 3.1.16 ensures that the multiplication is continuous.

The concept of lb TVS and so of lb TA can be characterized through extensions of the notion of norm, which will allow us to see how some results can be extended from Banach algebras to complete lb algebras.

Definition 3.2.6. Let X be a \mathbb{K} -vector space. A map $\|\cdot\|: X \to \mathbb{R}^+$ is said to be a quasi-norm if

1. $\forall x \in X : ||x|| = 0 \iff x = 0,$ 2. $\forall x \in X \forall \lambda \in \mathbb{K} : ||\lambda x|| = |\lambda| ||x||,$ 3. $\exists k \ge 1 : ||x + y|| \le k(||x|| + ||y||), \forall x, y \in X.$ If k = 1 this coincides with the notion of norm.

Example 3.2.7.

Let $0 and consider the space <math>L^p([0,1])$ with $\|\cdot\|_p : L^p([0,1]) \to \mathbb{R}^+$ defined by $\|f\|_p := (\int_0^1 |f(x)|^p dx)^{\frac{1}{p}}$ for all $f \in L^p([0,1])$. Then the Minkowski inequality does not hold but we still have that $||f+g||_p \leq 2^{\frac{1-p}{p}}(||f||+||g||)$ for all $f, g \in L^p([0,1])$ and so that $||\cdot||_p$ is a quasi-norm.

Proposition 3.2.8. Let (X, τ) be a Hausdorff TVS. Then (X, τ) is lb if and only if τ is induced by a quasi-norm on X.

Proof.

Assume that (X, τ) is lb and $\mathcal{F}(o)$ is its filter of neighbourhoods of the origin. Then there exists balanced and bounded $U \in \mathcal{F}(o)$ and $\mathcal{B} := \{\alpha U : \alpha > 0\}$ is a basis of neighbourhoods of the origin in (X, τ) because for any $N \in \mathcal{F}(o)$ there exists $\lambda > 0$ s.t. $U \subseteq \lambda N \Rightarrow \mathcal{B} \ni \frac{1}{\lambda}U \subseteq N$. Consider the Minkowski functional $p_U(x) := \inf\{\alpha > 0 : x \in \alpha U\}$. In the proof of Lemma 2.2.7 we have already seen that if U is absorbing and balanced, then $0 \leq p_U(x) < \infty$ and $p_U(\lambda x) = |\lambda| p_U(x)$ for all $x \in X$ and all $\lambda \in \mathbb{K}$. If $p_U(x) = 0$, then $x \in \alpha U$ for all $\alpha > 0$ and so $x \in \bigcap_{\alpha > 0} \alpha U = \{o\}$, i.e. x = o. Since X is a TVS, $\exists V \in \mathcal{F}(o)$ s.t. $V + V \subseteq U$ and also $\exists \alpha > 0$ s.t. $\alpha U \subseteq V$ as \mathcal{B} is a basis of neighbourhoods. Therefore, $\alpha U + \alpha U \subseteq V + V \subseteq U$ and taking $k \geq \max\{1, \frac{1}{\alpha}\}$, we obtain $U + U \subseteq \frac{1}{\alpha}U \subseteq kU$ as U is balanced.

Let $x, y \in X$ and $\rho > p_U(x), \delta > p_U(y)$, then $x \in \rho U, y \in \delta U$ since U is balanced, and so $\frac{x}{\rho}, \frac{y}{\delta} \in U$. Thus,

$$\frac{x+y}{\rho+\delta} = \frac{\rho}{\rho+\delta}\frac{x}{\rho} + \frac{\delta}{\rho+\delta}\frac{y}{\delta} \in U + U \subseteq kU.$$

and we obtain $x + y \in k(\rho + \delta)U$ which implies $p_U(x + y) \leq k(\rho + \delta)$. As $\rho > p_U(x)$ and $\delta > p_U(y)$ were chosen arbitrarily, we conclude $p_U(x + y) \leq k(p_U(x) + p_U(y))$. Hence, p_U is a quasi-norm.

Let $B_1^{p_U} := \{x \in X : p_U(x) \leq 1\}$. Then we have $U \subseteq B_1^{p_U} \subseteq (1 + \varepsilon)U$ for all $\varepsilon > 0$. Indeed, if $x \in U$, then $p_U(x) \leq 1$ and so $x \in B_1^{p_U}$. If $x \in B_1^{p_U}$, then $p_U(x) \leq 1$ and so $\forall \varepsilon > 0 \exists \alpha$ with $\alpha \leq 1 + \varepsilon$ s.t. $x \in \alpha U$. This gives that $x \in (1 + \varepsilon)U$ as U is balanced and so $\alpha U \subseteq (1 + \varepsilon)U$. Since $\{\varepsilon B_1^{p_U} : \varepsilon > 0\}$ is a basis of τ_{p_U} , this implies $\tau = \tau_{p_U}$.

Conversely, assume that $\tau = \tau_q$ for a quasi-norm q on X and $\mathcal{F}^q(o)$ its filter of neighbourhoods of the origin. The collection $\mathcal{B} := \{\varepsilon B_1^q : \varepsilon > 0\}$ is a basis of neighbourhoods of the origin in (X, τ) (by Theorem 1.2.6). Let us just show that $\forall N \in \mathcal{F}^q(o) \exists V \in \mathcal{F}^q(o)$ s.t. $V + V \subseteq N$. Indeed, $\frac{1}{2k}B_1^q + \frac{1}{2k}B_1^q \subseteq B_1^q$ because if $x, y \in B_1^q$, then

$$q\left(\frac{x+y}{2k}\right) = \frac{1}{2k}q(x+y) \le \frac{k(q(x)+q(y))}{2k} \le \frac{2k}{2k} = 1$$

and so $\frac{x+y}{2k} \in B_1^q$. Then for all $N \in \mathcal{F}^q(o)$ there is some $\varepsilon > 0$ s.t. $\varepsilon B_1^q \subseteq N$ and so $\frac{\varepsilon}{2k}B_1^q + \frac{\varepsilon}{2k}B_1^q \subseteq \varepsilon B_1^q \subseteq N$. Since \mathcal{B} is a basis for τ_q , for any $N \in \mathcal{F}^q(o)$ there exists $\varepsilon > 0$ s.t. $\varepsilon B_1^q \subseteq N$, which implies $B_1^q \subseteq \frac{1}{\varepsilon}N$. Therefore, B_1^q is bounded and so τ_q is a lb TVS. \Box

Using the previous proposition and equipping the space in Example 3.2.7 with pointwise multiplication, we get an example of lb but not lc algebra (see Sheet 5). An example of lc but not lb algebra is given by the following.

Example 3.2.9. Let K be any compact subset of $(\mathbb{R}, \|\cdot\|)$ and let us consider the algebra $\mathcal{C}^{\infty}(K)$ of all real valued infinitely differentiable functions on K equipped with pointwise operations. Using the same technique as in Example 3.1.17, we can show that $\mathcal{C}^{\infty}(K)$ endowed with the topology τ_K , generated by the family $\{r_n : n \in \mathbb{N}_0\}$ where $r_n(f) := \sup_{j=0,\dots,n} \sup_{K \in K} |(D^{(j)}f)(x)|$ for any $f \in \mathcal{C}^{\infty}(K)$, is a Fréchet lmc algebra, i.e. an lc metrizable and complete algebra.

Denote now by $\mathcal{C}^{\infty}(\mathbb{R})$ the space of all real valued infinitely differentiable functions on \mathbb{R} and by $\mathcal{C}^{\infty}_{c}(K)$ its subset consisting of all the functions $f \in \mathcal{C}^{\infty}(\mathbb{R})$ whose support lies in K, i.e.

$$\mathcal{C}_{c}^{\infty}(K) := \{ f \in \mathcal{C}^{\infty}(\mathbb{R}) : supp(f) \subseteq K \},\$$

where supp(f) denotes the support of the function f, that is the closure in $(\mathbb{R}, \|\cdot\|)$ of the subset $\{x \in \mathbb{R} : f(x) \neq 0\}$. Then it is easy to see that $\mathcal{C}_c^{\infty}(K) = \mathcal{C}^{\infty}(K)$ and this is a linear subspace of $\mathcal{C}^{\infty}(\mathbb{R})$. Indeed, for any $f, g \in \mathcal{C}_c^{\infty}(K)$ and any $\lambda \in \mathbb{R}$, we clearly have $f + g \in \mathcal{C}^{\infty}(\mathbb{R})$ and $\lambda f \in \mathcal{C}^{\infty}(\mathbb{R})$ but also $supp(f + g) \subseteq supp(f) \cup supp(g) \subseteq K$ and $supp(\lambda f) = supp(f) \subseteq K$, which gives $f + g, \lambda f \in \mathcal{C}_c^{\infty}(K)$.

Let $\mathcal{C}_c^{\infty}(\mathbb{R})$ be the union of the subspaces $\mathcal{C}_c^{\infty}(K)$ as K varies in all possible ways over the family of compact subsets of \mathbb{R} , i.e. $\mathcal{C}_c^{\infty}(\mathbb{R})$ consists of all the functions belonging to $\mathcal{C}^{\infty}(\mathbb{R})$ having compact support (this is what is actually encoded in the subscript "c"). In particular, the space $\mathcal{C}_c^{\infty}(\mathbb{R})$ is usually called space of test functions and plays an essential role in the theory of distributions.

Consider a sequence $(K_j)_{j\in\mathbb{N}}$ of compact subsets of \mathbb{R} s.t. $K_j \subseteq K_{j+1}, \forall j \in \mathbb{N}$ and $\bigcup_{j=1}^{\infty} K_j = \mathbb{R}$. Then $\mathcal{C}_c^{\infty}(\mathbb{R}) = \bigcup_{j=1}^{\infty} \mathcal{C}_c^{\infty}(K_j)$, as an arbitrary compact subset K of \mathbb{R} is contained in K_j for some sufficiently large j, and we have that $\mathcal{C}_c^{\infty}(K_j) \subseteq \mathcal{C}_c^{\infty}(K_{j+1})$. For any $j \in \mathbb{N}$, we endow $\mathcal{C}_c^{\infty}(K_j)$ with the topology $\tau_j := \tau_{K_j}$ defined as above. Then $(\mathcal{C}_c^{\infty}(K_j), \tau_{K_j})$ is a Fréchet lmc algebra and $\tau_{j+1} \upharpoonright_{\mathcal{C}_c^{\infty}(K_j)} = \tau_j$. Denote by τ_{ind} the finest lc topology on $\mathcal{C}_c^{\infty}(\mathbb{R})$ such that all the inclusions $\mathcal{C}_c^{\infty}(K_j) \subseteq \mathcal{C}_c^{\infty}(\mathbb{R})$ are continuous (τ_{ind} does not depend on

the choice of the sequence of compact sets K_j 's provided they fill \mathbb{R}). Then it is possible to show that $(\mathcal{C}_c^{\infty}(\mathbb{R}), \tau_{ind})$ is a complete lc algebra but not Baire. Hence, Proposition 3.1.13 provides that $(\mathcal{C}_c^{\infty}(\mathbb{R}), \tau_{ind})$ is not metrizable and so not lb by Proposition 3.2.4.

Definition 3.2.10. Let X be a K-vector space and $0 < \alpha \leq 1$. A map $q: X \to \mathbb{R}^+$ is an α -norm if 1. $\forall x \in X: q(x) = 0 \iff x = 0$, 2. $\forall x \in X \forall \lambda \in \mathbb{K}: q(\lambda x) = |\lambda|^{\alpha} q(x)$, 3. $\forall x, y \in X: q(x + y) \leq q(x) + q(y)$. If $\alpha = 1$, this coincides with the notion of norm.

Definition 3.2.11. A TVS (X, τ) is α -normable if τ can be induced by an α -norm for some $0 < \alpha \leq 1$.

In order to understand how α -norms relates to lb spaces we need to introduce a generalization of the concept of convexity.

Definition 3.2.12. Let $0 < \alpha \leq 1$ and X a \mathbb{K} -vector space.

- A subset V of X is α -convex if for any $x, y \in V$ we have $tx + sy \in V$ for all t, s > 0 such that $t^{\alpha} + s^{\alpha} = 1$.
- A subset V of X is absolutely α−convex if for any x, y ∈ V we have tx + sy ∈ V for all t, s ∈ K such that |t|^α + |s|^α ≤ 1.
- For any W ⊆ X, Γ_α(W) denotes the smallest absolutely α-convex subset of X containing W, i.e.

$$\Gamma_{\alpha}(W) := \left\{ \sum_{i=1}^{n} \lambda_{i} w_{i} : n \in \mathbb{N}, w_{i} \in W, \lambda_{i} \in \mathbb{K} \ s.t. \sum_{i=1}^{n} |\lambda_{i}|^{\alpha} \leq 1 \right\}.$$

Proposition 3.2.13. Let (X, τ) be a TVS and $0 < \alpha \leq 1$. Then (X, τ) is α -normable if and only if there exists an α -convex, bounded neighbourhood of the origin.

Proof.

Suppose that τ is induced by an α -norm q, i.e. the collection of all $B_r^q := \{x \in X : q(x) \leq r\}$ for all r > 0 is a basis of neighbourhoods of the origin for τ . Then for any $x, y \in B_1^q$ and any $t, s \in \mathbb{K}$ such that $|t|^{\alpha} + |s|^{\alpha} \leq 1$ we have that

$$q(tx + sy) \le |t|^{\alpha}q(x) + |s|^{\alpha}q(y) \le |t|^{\alpha} + |s|^{\alpha} \le 1,$$

i.e. B_1^q is absolutely α -convex. Also, the definition of α -norm easily implies that

$$\forall \rho > 0, \forall x \in B_1^q, \ q(\rho^{\frac{1}{\alpha}}) = \rho q(x) \le \rho$$

and so that $B_1^q \subseteq \rho^{-\frac{1}{\alpha}} B_{\rho}^q$. Hence, B_1^q is a bounded absolutely α -convex neighbourhood of the origin.

Conversely, suppose that V is an α -convex bounded neighbourhood of the origin in (X, τ) .

<u>Claim 1</u>: W.l.o.g. we can always assume that V is absolutely α -convex. Then, as we showed in the proof of Proposition 3.2.8, the Minkowski functional p_V of V is a quasi-norm generating τ . Hence, defining $q(x) := p_V(x)^{\alpha}, \forall x \in X$ we can prove that

<u>Claim 2</u>: q is an α -norm.

Now $V \subseteq B_1^q$ because for any $x \in V$ we have that $q(x) \leq 1$. Also, for any $x \in B_1^q$ we have that $p_V(x) \leq 1$ and so for any $\varepsilon > 0$ there exists $\rho > 0$ s.t. $x \in \rho V$ and $\rho < p_V(x) + \varepsilon \leq 1 + \varepsilon$. Then $x \in \rho V \subseteq (1 + \varepsilon)V$ as V is balanced. Then we have just showed that

$$\forall \varepsilon > 0, V \subseteq B_1^q \subseteq (1+\varepsilon)V,$$

which in turn provides that τ is generated by q.

Let us now complete the proof by showing both claims.

Proof. of Claim 1

By assumption V is α -convex bounded neighbourhood of the origin in (X, τ) . If V is also balanced, then there is nothing to prove as V is already absolutely α -convex. If V is not balanced, then we can replace it with $\Gamma_{\alpha}(W)$ for some W balanced neighbourhood of the origin in X such that $W \subseteq V$ (the existence of such a W is given by Theorem 1.2.6 as (X, τ) is a TVS). In fact, we can show that $\Gamma_{\alpha}(W) \subseteq V$, which provides in turn that $\Gamma_{\alpha}(W)$ is both bounded and absolutely α -convex.

Let $z \in \Gamma_{\alpha}(W)$. Then $z = \sum_{i=1}^{n} \lambda_{i} w_{i}$ for some $n \in \mathbb{N}, w_{i} \in W$, and some $\lambda_{i} \in \mathbb{K}$ s.t. $\sum_{i=1}^{n} |\lambda_{i}|^{\alpha} \leq 1$. Take $\rho > 0$ such that $\rho^{\alpha} = \sum_{i=1}^{n} |\lambda_{i}|^{\alpha}$ and for each $\in \{1, \ldots, n\}$ set $\varepsilon_{i} := \frac{\lambda_{i}}{|\lambda_{i}|} \rho$. Then

$$z = \sum_{i=1}^{n} \lambda_i w_i = \sum_{i=1}^{n} \frac{|\lambda_i|}{\rho} \varepsilon_i w_i.$$
(3.5)

As $\rho^{\alpha} \leq 1$, we have $\rho \leq 1$ and so $|\varepsilon_i| \leq 1$. Then by the balancedness of W, for each $i \in \{1, \ldots, n\}$, we get that $\varepsilon_i w_i \in W \subset V$. Since $\sum_{i=1}^n \left(\frac{|\lambda_i|}{\rho}\right)^{\alpha} = 1$ and V is α -convex, (3.5) provides that $z \in V$.

Proof. of Claim 2

Since p_V is a quasi-norm on X, we have that $\forall x \in X, p_V(x) \ge 0$, which clearly

implies that $\forall x \in X$, $q(x) = p_V(x)^{\alpha} \ge 0$. Moreover, we have that x = 0 if and only if $p_V(x) = 0$, which is equivalent to q(x) = 0. The positive homogeneity of p_V gives in turn that

$$\forall x \in X, \ \forall \lambda \in \mathbb{K}, \ q(\lambda x) = p_V(\lambda x)^\alpha = |\lambda|^\alpha p_V(x)^\alpha = |\lambda|^\alpha q(x).$$
(3.6)

To show the triangular inequality for q, let us fix $x, y \in X$ and choose $\rho, \sigma \in \mathbb{R}^+$ such that $\rho > p_V(x)$ and $\sigma > p_V(y)$. Then there exist $\lambda, \mu > 0$ such that $x \in \lambda V, \lambda < \rho$ and $y \in \mu V, \mu < \sigma$. These together with the balancedness of V imply that $x \in \rho V$ and $y \in \sigma V$. Hence, we have $\frac{x}{\rho}, \frac{y}{\sigma} \in V$ and so, by the α -convexity of V we can conclude that

$$\frac{x+y}{(\rho^{\alpha}+\sigma^{\alpha})^{\frac{1}{\alpha}}} = \frac{\rho}{(\rho^{\alpha}+\sigma^{\alpha})^{\frac{1}{\alpha}}} \cdot \frac{x}{\rho} + \frac{\sigma}{(\rho^{\alpha}+\sigma^{\alpha})^{\frac{1}{\alpha}}} \cdot \frac{y}{\sigma} \in V.$$

Then $p_V\left(\frac{x+y}{(\rho^{\alpha}+\sigma^{\alpha})^{\frac{1}{\alpha}}}\right) \leq 1$ and so $q\left(\frac{x+y}{(\rho^{\alpha}+\sigma^{\alpha})^{\frac{1}{\alpha}}}\right) \leq 1$. Then, by using (3.6), we get that $\left(\frac{1}{(\rho^{\alpha}+\sigma^{\alpha})^{\frac{1}{\alpha}}}\right)^{\alpha} \cdot q(x+y) \leq 1$, that is $q(x+y) \leq \rho^{\alpha}+\sigma^{\alpha}$. Since this holds for all $\rho, \sigma \in \mathbb{R}^+$ such that $\rho > p_V(x)$ and $\sigma > p_V(y)$, we obtain that $q(x+y) \leq p_V(x)^{\alpha} + p_V(y)^{\alpha}$.

Corollary 3.2.14. Every α -normable TVS is lb.

The converse also holds and in proving it the following notion turns out to be very useful.

Definition 3.2.15. If (X, τ) is an lb TVS, then for any balanced, bounded, neighbourhood U of the origin in X we define

$$C(U) := \inf\{\lambda : U + U \subseteq \lambda U\}.$$

The concavity module C(X) of X is defined as follows

 $C(X) := \inf\{C(U) : U \text{ balanced, bounded, neighbourhood of o in } X\}.$

Theorem 3.2.16. Let (X, τ) be a TVS. Then (X, τ) is lb if and only if τ is induced by some α -norm for some $0 < \alpha \leq 1$.

Proof. The sufficiency is given by the previous corollary. As for the necessity, it is possible to show that if (X, τ) is lb then there exists a bounded α -convex neighbourhood of the origin for all $0 < \alpha < \alpha_0$, where $\alpha_0 := \frac{\log 2}{\log C(X)}$ (see Sheet 5). Hence, the conclusion follows by Proposition 3.2.13.

In the context of lb algebras, it might happen that the α -norm defining the topology is actually submultiplicative. This is actually the case if the considered algebra is complete.

Definition 3.2.17. An α -normed algebra is a \mathbb{K} -algebra endowed with the topology induced by a submultiplicative α -norm.

Theorem 3.2.18. Any lb Hausdorff complete algebra can be made into an α -normed algebra for some $0 < \alpha \leq 1$.

Proof. Sketch

Let (X, τ) be a Hausdorff complete lb algebra. For convenience let us assume that X is unital but the proof can be adapted also to the non-unital case.

As (X, τ) is lb, Theorem 3.2.16 ensures that the exists $0 < \alpha \leq 1$ such that τ is induced by an α -norm q. Consider the space L(X) of all linear continuous operators on X equipped with pointwise addition and scaler multiplication and with the composition as multiplication. Then the operator norm on L(X) defined by $\|\ell\| := \sup_{x \in X \setminus \{o\}} \frac{q(\ell(x))}{q(x)}$ for all $\ell \in L(X)$ is a submultiplicative α -norm. Since (X, q) is complete, it is possible to show that it is topologically isomorphic to $(L(X), \|\cdot\|)$. If we denote by φ such an isomorphism, we then get that (X, p) with $p(x) := \|\varphi(x)\|$ for all $x \in X$ is an α -normed algebra. \Box

Proposition 3.2.19. Let (X, τ) be an lb Hausdorff TA. Show that if (X, τ) has jointly continuous multiplication, then (X, τ) is α -normable.

Proof. (see Sheet 6)

3.3 Projective limit algebras

The class of topological algebras which we are going to introduce in this section consisits of algebras obtained as a projective limit of a family of TAs and then endowed with the so-called projective topology associated to the natural system of maps given by the projective limit construction. Therefore, we are first going to introduce in general the notion of projective topology w.r.t. a family of maps, then we will focus on the projective limit construction from both an algebraic and topological point of view.

3.3.1 Projective topology

Let $\{(E_{\alpha}, \tau_{\alpha}) : \alpha \in I\}$ be a family of TVSs over \mathbb{K} (*I* is an arbitrary index set). Let *E* be a vector space over the same field \mathbb{K} and, for each $\alpha \in I$, let $f_{\alpha} : E \to E_{\alpha}$ be a linear mapping. The *projective topology* τ_{proj} on *E* w.r.t. the family $\{(E_{\alpha}, \tau_{\alpha}), f_{\alpha}, I\}$ is the coarsest topology on E for which all the mappings f_{α} ($\alpha \in I$) are continuous.

It is easy to check that (E, τ_{proj}) is a TVS and that a basis of neighbourhoods of the origin is given by:

$$\mathcal{B}_{proj} := \left\{ \bigcap_{\alpha \in F} f_{\alpha}^{-1}(U_{\alpha}) : F \subseteq I \text{ finite, } U_{\alpha} \in \mathcal{B}_{\alpha}, \forall \alpha \in F \right\}, \qquad (3.7)$$

where \mathcal{B}_{α} is a basis of neighbourhoods of the origin in $(E_{\alpha}, \tau_{\alpha})$.

Remark 3.3.1. Note that the projective topology τ_{proj} coincides with the initial topology given by the map

$$\begin{array}{rccc} \varphi : & E & \to & \left(\prod_{\alpha \in I} E_{\alpha}\right) \\ & x & \mapsto & \left(f_{\alpha}(x)\right)_{\alpha \in I}. \end{array}$$

Recall that the initial topology is defined as the coarsest topology on E such that φ is continuous or equivalently as the topology on E generated by the collection of all $\varphi^{-1}(U)$ when U is a neighbourhood of the origin in $(\prod_{\alpha \in I} E_{\alpha}, \tau_{prod})$.

Let us first introduce some properties of the projective topology in the TVS setting.

Lemma 3.3.2. Let E be a vector space over \mathbb{K} endowed with the projective topology τ_{proj} w.r.t. the family $\{(E_{\alpha}, \tau_{\alpha}), f_{\alpha}, I\}$, where each $(E_{\alpha}, \tau_{\alpha})$ is a TVS over \mathbb{K} and each f_{α} a linear mapping from E to E_{α} . Let (F, τ) be an arbitrary TVS and g a linear mapping from F into E. The mapping $g : F \to E$ is continuous if and only if, for each $\alpha \in I$, $f_{\alpha} \circ g : F \to E_{\alpha}$ is continuous.

Proof.

Suppose that $g: F \to E$ is continuous. Since by definition of τ_{proj} all f_{α} 's are continuous, we have that for each $\alpha \in I$, $f_{\alpha} \circ g: F \to E_{\alpha}$ is continuous.

Conversely, suppose that for each $\alpha \in I$ the map $f_{\alpha} \circ g : F \to E_{\alpha}$ is continuous and let U be a neighbourhood of the origin in (E, τ_{proj}) . Then there exists a finite subset F of I and for each $\alpha \in F$ there exists $U_{\alpha} \in \mathcal{B}_{\alpha}$ such that $\bigcap_{\alpha \in F} f_{\alpha}^{-1}(U_{\alpha}) \subseteq U$. Therefore, we obtain

$$g^{-1}(U) \supseteq g^{-1}\left(\bigcap_{\alpha \in F} f_{\alpha}^{-1}(U_{\alpha})\right) = \bigcap_{\alpha \in F} g^{-1}\left(f_{\alpha}^{-1}(U_{\alpha})\right) = \bigcap_{\alpha \in F} (f_{\alpha} \circ g)^{-1}(U_{\alpha}),$$

which yields that $g^{-1}(U)$ is a neighbourhood of the origin in (F, τ) since the continuity of all $f_{\alpha} \circ g$'s ensures that $(f_{\alpha} \circ g)^{-1}(U_{\alpha})$ is a neighbourhood of the origin in (F, τ) .

Proposition 3.3.3. Let *E* be a vector space over \mathbb{K} endowed with the projective topology τ_{proj} w.r.t. the family $\{(E_{\alpha}, \tau_{\alpha}), f_{\alpha}, I\}$, where each $(E_{\alpha}, \tau_{\alpha})$ is a TVS over \mathbb{K} and each f_{α} a linear mapping from *E* to E_{α} . Then τ_{proj} is Hausdorff if and only if for each $0 \neq x \in E$, there exists an $\alpha \in I$ and a neighbourhood U_{α} of the origin in $(E_{\alpha}, \tau_{\alpha})$ such that $f_{\alpha}(x) \notin U_{\alpha}$.

Proof.

Suppose that (E, τ_{proj}) is Hausdorff and let $0 \neq x \in E$. By Proposition 1.3.2, there exists a neighbourhood U of the origin in E not containing x. Then, by (3.7), there exists a finite subset $F \subseteq I$ and, for any $\alpha \in F$, there exists U_{α} neighbourhood of the origin in $(E_{\alpha}, \tau_{\alpha})$ s.t. $\bigcap_{\alpha \in F} f_{\alpha}^{-1}(U_{\alpha}) \subseteq U$. Hence, as $x \notin U$, there exists $\alpha \in F$ s.t. $x \notin f_{\alpha}^{-1}(U_{\alpha})$, i.e. $f_{\alpha}(x) \notin U_{\alpha}$. Conversely, suppose that there exists $\alpha \in I$ and a neighbourhood V_{α} of the origin in $(E_{\alpha}, \tau_{\alpha})$ such that $f_{\alpha}(x) \notin V_{\alpha}$. Let \mathcal{B}_{α} be a basis of neighbourhoods of the origin in $(E_{\alpha}, \tau_{\alpha})$. Then there exists $U_{\alpha} \in \mathcal{B}_{\alpha}$ such that $U_{\alpha} \subseteq V_{\alpha}$. Hence, $x \notin f_{\alpha}^{-1}(U_{\alpha})$ and $f_{\alpha}^{-1}(U_{\alpha}) \in \mathcal{B}_{proj}$ (see (3.7)), that is, we have found a neighbourhood of the origin in (E, τ_{proj}) not containing x. This implies, by Proposition 1.3.2, that τ_{proj} is a Hausdorff topology.

Coming back to the context of TAs, we have the following result.

Theorem 3.3.4. Let E be a \mathbb{K} -algebra endowed with the projective topology τ_{proj} w.r.t. the family $\{(E_{\alpha}, \tau_{\alpha}), f_{\alpha}, I\}$, where each $(E_{\alpha}, \tau_{\alpha})$ is a TA over \mathbb{K} (resp. a TA with continuous multiplication) and each f_{α} a homomorphism from E to E_{α} . Then (E, τ_{proj}) is a TA (resp. a TA with continuous multiplication).

Proof.

As each $(E_{\alpha}, \tau_{\alpha})$ is a TVS, it is easy to verify that (E, τ_{proj}) is a TVS. Therefore, it remains to show that left and right multiplication are both continuous. For any $x \in E$, consider the left multiplication $\ell_x : E \to E$. For each $\alpha \in I$ we get that:

$$\forall y \in E, \ (f_{\alpha} \circ \ell_x)(y) = f_{\alpha}(xy) = f_{\alpha}(x)f_{\alpha}(y) = \ell_{f_{\alpha}(x)}(f_{\alpha}(y)) = (\ell_{f_{\alpha}(x)} \circ f_{\alpha})(y).$$
(3.8)

Since $f_{\alpha}(x) \in E_{\alpha}$ and $(E_{\alpha}, \tau_{\alpha})$ is a TA, we have that $\ell_{f_{\alpha}(x)} : E_{\alpha} \to E_{\alpha}$ is continuous and so $\ell_{f_{\alpha}(x)} \circ f_{\alpha}$ is continuous. Hence, by (3.8), we have that $f_{\alpha} \circ \ell_x$ is continuous for all $\alpha \in I$ and so by the previous lemma we have that ℓ_x is continuous. Similarly, we get the continuity of the right multiplication in E. Hence, (E, τ_{proj}) is a TA.

If each $(E_{\alpha}, \tau_{\alpha})$ is a TA with continuous multiplication, then by combining Remark 3.3.1 and Proposition 1.4.1 we can conclude that (E, τ_{proj}) is a TA.

Proposition 3.3.5. Let E be a \mathbb{K} -algebra endowed with the projective topology τ_{proj} w.r.t. the family $\{(E_{\alpha}, \tau_{\alpha}), f_{\alpha}, I\}$, where each $(E_{\alpha}, \tau_{\alpha})$ is an lc (resp. lmc) algebra over \mathbb{K} and each f_{α} a homomorphism from E to E_{α} . Then (E, τ_{proj}) is an lc (resp. lmc) algebra.

Proof.

By assumption, we know that each $(E_{\alpha}, \tau_{\alpha})$ is a TA and so Theorem 3.3.4 ensures that (E, τ_{proj}) is a TA, too. Moreover, as each $(E_{\alpha}, \tau_{\alpha})$ is an lc (resp. lmc) algebra, there exists a basis \mathcal{B}_{α} of convex (resp. m-convex) neighbourhoods of the origin in $(E_{\alpha}, \tau_{\alpha})$. Then the corresponding \mathcal{B}_{proj} (see (3.7)) also consists of convex (resp. m-convex) neighbourhoods of the origin in (E, τ_{proj}) . In fact, any $B \in \mathcal{B}_{proj}$ is of the form $B = \bigcap_{\alpha \in F} f_{\alpha}^{-1}(U_{\alpha})$ for some $F \subseteq I$ finite and $U_{\alpha} \in \mathcal{B}_{\alpha}, \forall \alpha \in F$. Since all the U_{α} 's are convex (resp. m-convex) and the preimage of a convex (resp. m-convex) set under a homomorphism is convex (resp. m-convex by Proposition 2.1.3-d)), we get that B is a finite intersection of convex (resp. m-convex) sets and so it is convex (resp. m-convex).

Corollary 3.3.6. Let (A, τ) be an lc (resp. lmc) algebra and M a subalgebra of A. If we endow M with the relative topology τ_M induced by A, then (M, τ_M) is an lc (resp. lmc) algebra.

Proof.

Recalling that τ_M coincides with the projective topology on M induced by id : $M \to A$ (see Corollary 1.4.2), the conclusion easily follows from the previous proposition (applied for $I = \{1\}, E_1 = A$ and $\tau_1 = \tau, E = M$ and $f_1 = id$).

Corollary 3.3.7. Any subalgebra of a Hausdorff TA is a Hausdorff TA.

Proof. This is a direct application of Proposition 3.3.3 and Corollary 1.4.2.

Example 3.3.8. Let $\{(E_{\alpha}, \tau_{\alpha}) : \alpha \in I\}$ be a family of TAs over \mathbb{K} . Then the Cartesian product $F = \prod_{\alpha \in I} E_{\alpha}$ equipped with coordinatewise operation is a \mathbb{K} -algebra. Consider the canonical projections $\pi_{\alpha} : F \to E_{\alpha}$ defined by $\pi_{\alpha}(x) := x_{\alpha}$ for any $x = (x_{\beta})_{\beta \in I} \in F$, which are all homomorphisms. Then the product topology τ_{prod} on F is the coarsest topology for which all the canonical projections are continuous and so coincides with the projective topology on F w.r.t. $\{(E_{\alpha}, \tau_{\alpha}), \pi_{\alpha} : \alpha \in I\}^{2}$. Hence, by Theorem 3.3.4 we have that (F, τ_{prod}) is a TA.

Recalling that a cartesian product of complete Hausdorff TAs endowed with the product topology is a complete Hausdorff TA and applying Proposition 3.3.5, Corollary 3.3.6 and Proposition 3.3.3 to the previous example, we can easily prove the following properties

- any Cartesian product of lc (resp. lmc) algebras endowed with the product topology is an lc (resp. lmc) algebra
- any subalgebra of a Cartesian product of lc (resp. lmc) endowed with the relative topology is a TA of the same type
- a cartesian product of Hausdorff TAs endowed with the product topology is a Hausdorff TA.

3.3.2 Projective systems of TAs and their projective limit

In this section we are going to discuss the concept of projective system (resp. projective limit) first for just K-algebras and then for TAs.

Definition 3.3.9. Let (I, <) be a directed partially ordered set (i.e. for all $\alpha, \beta \in I$ there exists $\gamma \in I$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$). A projective system of algebras $\{E_{\alpha}, f_{\alpha\beta}, I\}$ is a family of \mathbb{K} -algebras $\{E_{\alpha}, \alpha \in I\}$ together with a family of homomorphisms $f_{\alpha\beta} : E_{\beta} \to E_{\alpha}$ defined for all $\alpha \leq \beta$ in I such that $f_{\alpha\alpha}$ is the identity on E_{α} and $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$ for all $\alpha \leq \beta \leq \gamma$, i.e. the diagram



commutes.

 $^2{\rm We}$ could have also directly showed that the equivalence of the two topologies using their basis of neighbourhoods of the origin. Indeed

$$\mathcal{B}_{proj} \stackrel{(\mathbf{3.7})}{=} \left\{ \bigcap_{\alpha \in F} \pi_{\alpha}^{-1}(U_{\alpha}) : F \subseteq I \text{ finite, } U_{\alpha} \in \mathcal{B}_{\alpha}, \forall \alpha \in F \right\}$$
$$= \left\{ \prod_{\alpha \in F} U_{\alpha} \times \prod_{\alpha \in I \setminus F} E_{\alpha} : F \subseteq I \text{ finite, } U_{\alpha} \in \mathcal{B}_{\alpha}, \forall \alpha \in F \right\} = \mathcal{B}_{prod}.$$

Definition 3.3.10. Given a projective system of algebras $S := \{E_{\alpha}, f_{\alpha\beta}, I\}$, we define the projective limit of S (or the projective limit algebra associated with S) to be the triple $\{E_S, f_{\alpha}, I\}$, where

$$E_{\mathcal{S}} := \left\{ x := (x_{\alpha})_{\alpha \in I} \in \prod_{\alpha \in I} E_{\alpha} : x_{\alpha} = f_{\alpha\beta}(x_{\beta}), \ \forall \alpha \le \beta \ in \ I \right\}$$

and, for any $\alpha \in I$, $f_{\alpha} : E_{\mathcal{S}} \to E_{\alpha}$ is defined by $f_{\alpha} := \pi_{\alpha} \upharpoonright_{E_{\mathcal{S}}}$ (where $\pi_{\alpha} : \prod_{\beta \in I} E_{\beta} \to E_{\alpha}$ is the canonical projection, see Example 3.3.8).

It is easy to see from the previous definitions that $E_{\mathcal{S}}$ is a subalgebra of $\prod_{\alpha \in I} E_{\alpha}$. Indeed, for any $x, y \in E_{\mathcal{S}}$ and for any $\lambda \in \mathbb{K}$ we have that for all $\alpha \leq \beta$ in I the following hold

$$\lambda x_{\alpha} + y_{\alpha} = \lambda f_{\alpha\beta}(x_{\beta}) + f_{\alpha\beta}(y_{\beta}) = f_{\alpha\beta}(\lambda x_{\beta} + y_{\beta})$$

and

$$x_{\alpha}y_{\alpha} = f_{\alpha\beta}(x_{\beta})f_{\alpha\beta}(y_{\beta}) = f_{\alpha\beta}(x_{\beta}y_{\beta})$$

i.e. $\lambda x + y, xy \in E_{\mathcal{S}}$. Note that the f_{α} 's are not necessarily surjective and also that

$$f_{\alpha} = f_{\alpha\beta} \circ f_{\beta}, \, \forall \alpha \leq \beta \text{ in } I,$$

since for all $x := (x_{\alpha})_{\alpha \in I} \in E_{\mathcal{S}}$ we have $f_{\alpha}(x) = x_{\alpha} = f_{\alpha\beta}(x_{\beta}) = f_{\alpha\beta}(f_{\beta}(x))$.

Also, we can show that $\{E_{\mathcal{S}}, f_{\alpha}, I\}$ satisfies the following universal property: given a \mathbb{K} -algebra A and a family of homomorphism $\{g_{\alpha} : A \to E_{\alpha}, \alpha \in I\}$ such that $g_{\alpha} = f_{\alpha\beta} \circ g_{\beta}$ for all $\alpha \leq \beta$ in I, there exists a unique homomorphism $\varphi : A \to E_{\mathcal{S}}$ such that $g_{\alpha} = f_{\alpha} \circ \varphi$ for all $\alpha \in I$, i.e. the diagram



commutes. In fact, the map $\varphi : A \to E_{\mathcal{S}}$ defined by $\varphi(a) := (g_{\alpha}(a))_{\alpha \in I}$ for all $a \in A$ is a homomorphism such that $(f_{\alpha} \circ \varphi)(a) = (\varphi(a))_{\alpha} = g_{\alpha}(a)$, for all $a \in A$. Moreover, if there exists $\varphi' : A \to E_{\mathcal{S}}$ such that $g_{\alpha} = f_{\alpha} \circ \varphi'$ for all $\alpha \in I$, then for all $a \in A$ we get

$$\varphi(a) = (g_{\alpha}(a))_{\alpha \in I} = \left((f_{\alpha} \circ \varphi')(a) \right)_{\alpha \in I} = \left((\varphi'(a))_{\alpha} \right)_{\alpha \in I} = \varphi'(a),$$
i.e. $\varphi' \equiv \varphi$ on A.

These considerations allows to easily see that one can give the following more general definition of projective limit algebra.

Definition 3.3.11. Given a projective system of algebras $S := \{E_{\alpha}, f_{\alpha\beta}, I\}$, a projective limit of S (or a projective limit algebra associated with S) is a triple $\{E, h_{\alpha}, I\}$, where E is a \mathbb{K} -algebra; for any $\alpha \in I$, $h_{\alpha} : E \to E_{\alpha}$ is a homomorphisms such that $h_{\alpha} = f_{\alpha\beta} \circ h_{\beta}$, $\forall \alpha \leq \beta$ in I; and the following universal property holds: for any \mathbb{K} -algebra A and any family of homomorphism $\{g_{\alpha} : A \to E_{\alpha}, \alpha \in I\}$ such that $g_{\alpha} = f_{\alpha\beta} \circ g_{\beta}$ for all $\alpha \leq \beta$ in I, there exists a unique homomorphism $\varphi : A \to E$ such that $g_{\alpha} = h_{\alpha} \circ \varphi$ for all $\alpha \in I$.

It is easy to show that $\{E, h_{\alpha}, I\}$ is unique up to (algebraic) isomorphisms, i.e. if $\{\tilde{E}, \tilde{h}_{\alpha}, I\}$ fulfills Definition 3.3.11 then there exists a unique isomorphism between E and \tilde{E} . This justifies why in Definition 3.3.10 we have called $\{E_{\mathcal{S}}, f_{\alpha}, I\}$ the projective limit of \mathcal{S} . (Indeed, we have already showed that $\{E_{\mathcal{S}}, f_{\alpha}, I\}$ fulfills Definition 3.3.11.)

The definitions introduced above for algebras can be easily adapted to the category of TAs.

Definition 3.3.12. Let (I, <) be a directed partially ordered set. A projective system of TAs $\{(E_{\alpha}, \tau_{\alpha}), f_{\alpha\beta}, I\}$ is a family of K-algebras $\{(E_{\alpha}, \tau_{\alpha}) : \alpha \in I\}$ together with a family of continuous homomorphisms $f_{\alpha\beta} : E_{\beta} \to E_{\alpha}$ defined for all $\alpha \leq \beta$ in I such that $f_{\alpha\alpha}$ is the identity on E_{α} and $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$ for all $\alpha \leq \beta \leq \gamma$, i.e. the diagram



commutes. Equivalently, a projective system of TAs is a projective system of algebras $\{E_{\alpha}, f_{\alpha\beta}, I\}$ in which each E_{α} is endowed with a topology τ_{α} making $(E_{\alpha}, \tau_{\alpha})$ into a TA and all the homomorphisms $f_{\alpha\beta}$ continuous.

Definition 3.3.13. Given a projective system $S := \{(E_{\alpha}, \tau_{\alpha}), f_{\alpha\beta}, I\}$ of TAs, we define the projective limit of S (or the projective limit TA associated with S) to be the triple $\{(E_S, \tau_{proj}), f_{\alpha}, I\}$ where $\{E_S, f_{\alpha}, I\}$ is the projective limit algebra associated with $\{E_{\alpha}, f_{\alpha\beta}, I\}$ and τ_{proj} is the projective topology on E_S w.r.t. the family $\{(E_{\alpha}, \tau_{\alpha}), f_{\alpha} : \alpha \in I\}$.

Similarly, to the algebraic case, one could give the following more general definition of projective limit TA.

Definition 3.3.14. Given a projective system of TAs $S := \{(E_{\alpha}, \tau_{\alpha}), f_{\alpha\beta}, I\}$, a projective limit of S (or a projective limit TA associated with S) is a triple $\{(E, \tau), h_{\alpha}, I\}$ where (E, τ) is a TA; for any $\alpha \in I$, $h_{\alpha} : E \to E_{\alpha}$ is a continuous homomorphism such that $h_{\alpha} = f_{\alpha\beta} \circ h_{\beta}$, for all $\alpha \leq \beta$ in I; and the following universal property holds: for any TA (A, ω) and any family of continuous homomorphism $\{g_{\alpha} : A \to E_{\alpha}, \alpha \in I\}$ such that $g_{\alpha} = f_{\alpha\beta} \circ g_{\beta}$ for all $\alpha \leq \beta$ in I, there exists a unique continuous homomorphism $\varphi : A \to E$ such that $g_{\alpha} = h_{\alpha} \circ \varphi$ for all $\alpha \in I$.

It is easy to show that $\{(E, \tau), h_{\alpha}, I\}$ is unique up to topological isomorphisms. We have already showed that $E_{\mathcal{S}}$ is an algebra such that the family of all $f_{\alpha} := \pi_{\alpha} \upharpoonright_{E_{\mathcal{S}}} (\alpha \in I)$ fulfills $f_{\alpha} = f_{\alpha\beta} \circ f_{\beta}, \forall \alpha \leq \beta$ in I. Endowing $E_{\mathcal{S}}$ with the projective topology τ_{proj} w.r.t. $\{(E_{\alpha}, \tau_{\alpha}), f_{\alpha}, I\}$, we get by Theorem 3.3.4 that $(E_{\mathcal{S}}, \tau_{proj})$ is a TA and that all f_{α} 's are continuous. Also, for any TA (A, ω) and any family of continuous homomorphism $\{g_{\alpha} : A \to E_{\alpha}, \alpha \in I\}$ such that $g_{\alpha} = f_{\alpha\beta} \circ g_{\beta}$ for all $\alpha \leq \beta$ in I, we have already showed that $\varphi : A \to E_{\mathcal{S}}$ defined by $\varphi(a) := (g_{\alpha}(a))_{\alpha \in I}$ for all $a \in A$ is the unique homomorphism such that $g_{\alpha} = f_{\alpha} \circ \varphi$ for all $\alpha \in I$. But φ is also continuous because for any $U \in \mathcal{B}_{proj}$ we have $U = \bigcap_{\alpha \in F} f_{\alpha}^{-1}(U_{\alpha})$ for some $F \subset I$ finite and some $U_{\alpha} \in \mathcal{B}_{\alpha}$ for all $\alpha \in F$ and so $\varphi^{-1}(U) = \bigcap_{\alpha \in F} \varphi^{-1}(f_{\alpha}^{-1}(U_{\alpha})) = \bigcap_{\alpha \in F} (f_{\alpha} \circ \varphi)^{-1}(U_{\alpha}) = \bigcap_{\alpha \in F} g_{\alpha}^{-1}(U_{\alpha}) \in \mathcal{B}_{\omega}$. Hence, $\{(E_{\mathcal{S}}, \tau_{proj}), f_{\alpha}, I\}$ satisfies Definiton 3.3.14.

Remark 3.3.15.

From the previous definitions one can easily deduce the following:

- a) the projective limit of a projective system of Hausdorff TAs is a Hausdorff TA (easily follows by Proposition 3.3.3).
- b) the projective limit of a projective system of Hausdorff TAs $\{(E_{\alpha}, \tau_{\alpha}), f_{\alpha,\beta}, I\}$ is a closed subalgebra of $(\prod_{\alpha \in I} E_{\alpha}, \tau_{\text{prod}})$ (see Sheet 6).
- c) the projective limit of a projective system of complete Hausdorff TAs is a complete Hausdorff TA (see Sheet 6).

Corollary 3.3.16. A projective limit of lmc algebras is an lmc algebra.

Proof.

Let $\{(E_{\alpha}, \tau_{\alpha}), f_{\alpha,\beta}, I\}$ be a projective system of lmc algebras. Then its projective limit $\{(E_{\mathcal{S}}, \tau_{proj}), f_{\alpha}, I\}$ is an lmc algebra by Proposition 3.3.5.

This easy corollary brings us to a very natural but fundamental question: can any lmc algebra be written as a projective limit of a projective system of lmc algebras or at least as a subalgebra of such a projective limit? The whole next section will be devoted to show a positive answer to this question.

3.3.3 Arens-Michael decomposition

This section will be devoted to the Arens-Michael decomposition theorem, which was independently conceived by Arens and Michael in the early days of the theory of TAs (1952). This result is so important because it provides a device to reduce basic questions about lmc algebra to analogous ones for the corresponding factor Banach algebras. Since the theory of Banach algebras has been heavily studied, being able to reduce to Banach algebras is very advantageous and so much desirable.

Before stating the Arens-Michael decomposition theorem, let us recall the completion theorem for TVS and two useful lemmas about projective limit algebras.

Theorem 3.3.17.

Let X be a Haudorff TVS. Then there exists a complete Hausdorff TVS \hat{X} and a mapping $i: X \to \hat{X}$ with the following properties:

- a) The mapping i is a topological monomorphism.
- b) The image of X under i is dense in X.
- c) For every complete Hausdorff TVS Y and for every continuous linear map $f: X \to Y$, there is a continuous linear map $\hat{f}: \hat{X} \to Y$ such that the diagram



is commutative. Furthermore:

I) Any other pair (\hat{X}_1, i_1) , consisting of a complete Hausdorff TVS \hat{X}_1 and of a mapping $i_1 : X \to \hat{X}_1$ such that properties (a) and (b) hold substituting \hat{X} with \hat{X}_1 and i with i_1 , is topologically isomorphic to (\hat{X}, i) . This means that there is a topological isomorphism j of \hat{X} onto \hat{X}_1 such that the diagram

$$\begin{array}{c} X \xrightarrow{i_1} \hat{X_1} \\ \downarrow^i \swarrow^j \\ \hat{X} \end{array}$$

 $is \ commutative.$

II) Given Y and f as in property (c), the continuous linear map \hat{f} is unique.

Lemma 3.3.18. Let $\{(A_{\mathcal{S}}, \tau_{proj}), g_{\alpha}, J\}$ be the projective limit of the projective system $\mathcal{S} := \{(A_{\alpha}, \tau_{\alpha}), g_{\alpha\beta}, J\}$ of TAs. Then a basis of neighbourhoods of the

origin in $(A_{\mathcal{S}}, \tau_{proj})$ is given by

$$\widetilde{\mathcal{B}}_{proj} := \{ g_{\alpha}^{-1}(V_{\alpha}) : V_{\alpha} \in \mathcal{B}_{\alpha}, \alpha \in J \},\$$

where each \mathcal{B}_{α} is a basis of neighbourhoods of the origin in $(A_{\alpha}, \tau_{\alpha})$.

Proof.

By the continuity of the g_{α} 's, we know that $\widetilde{\mathcal{B}}_{proj}$ is a collection of neighbourhoods of the origin in $(A_{\mathcal{S}}, \tau_{proj})$. We want to show that it is a basis.

By (3.7), we have that a basis of neighbourhoods of the origin in (A_S, τ_{proj}) is given by

$$\mathcal{B}_{proj} := \left\{ \bigcap_{\beta \in F} g_{\beta}^{-1}(V_{\beta}) : F \subseteq I \text{ finite, } V_{\beta} \in \mathcal{B}_{\beta}, \forall \beta \in F \right\}.$$

As J is directed, for any finite subset F of I there exists $\alpha \in J$ such that $\beta \leq \alpha$ for all $\beta \in F$. Then we have that $g_{\beta} = g_{\beta\alpha} \circ g_{\alpha}$ for all $\beta \in F$ and so

$$\bigcap_{\beta \in F} g_{\beta}^{-1}(V_{\beta}) = \bigcap_{\beta \in F} (g_{\beta\alpha} \circ g_{\alpha})^{-1}(V_{\beta}) = \bigcap_{\beta \in F} g_{\alpha}^{-1}(g_{\beta\alpha}^{-1}(V_{\beta})) = g_{\alpha}^{-1} \left(\bigcap_{\beta \in F} g_{\beta\alpha}^{-1}(V_{\beta}) \right).$$

Set $W_{\alpha} := \bigcap_{\beta \in F} g_{\beta\alpha}^{-1}(V_{\beta})$. Since for all $\beta \in F$ the map $g_{\beta\alpha} : A_{\alpha} \to A_{\beta}$ is continuous, we get that for all $\beta \in F$ the set $g_{\beta\alpha}^{-1}(V_{\beta})$ is a neighbourhood of the origin in $(A_{\alpha}, \tau_{\alpha})$ and so is W_{α} . Then there exists $V_{\alpha} \in \mathcal{B}_{\alpha}$ such that $V_{\alpha} \subseteq W_{\alpha}$. Hence, we obtain

$$\bigcap_{\beta \in F} g_{\beta}^{-1}(V_{\beta}) = g_{\alpha}^{-1}(W_{\alpha}) \supseteq g_{\alpha}^{-1}(V_{\alpha})$$

and so we have showed that for any $M \in \mathcal{B}_{proj}$ there exists $\widetilde{M} \in \widetilde{\mathcal{B}}_{proj}$ such that $\widetilde{M} \subseteq M$, i.e. $\widetilde{\mathcal{B}}_{proj}$ is a basis of neighbourhoods of the origin in $(A_{\mathcal{S}}, \tau_{proj})$. \Box

Lemma 3.3.19. Let $\{(A_{\mathcal{S}}, \tau_{proj}), g_{\alpha}, J\}$ be the projective limit of the projective system $\mathcal{S} := \{(A_{\alpha}, \tau_{\alpha}), g_{\alpha\beta}, J\}$ of TAs and W a linear subspace of $A_{\mathcal{S}}$. Then

$$\overline{W}^{\tau_{proj}} = \bigcap_{\alpha \in J} g_{\alpha}^{-1} \left(\overline{g_{\alpha}(W)}^{\tau_{\alpha}} \right) = \operatorname{projlim}(\mathcal{S}_1),$$

where S_1 denotes the projective system $\left\{\overline{g_{\alpha}(W)}^{\tau_{\alpha}}, g_{\alpha\beta} \upharpoonright_{\overline{g_{\beta}(W)}}, J\right\}$ of TAs (here $\overline{g_{\alpha}(W)}^{\tau_{\alpha}}$ is intended as endowed with the relative topology induced by τ_{α}).

In particular, if W is closed in $(A_{\mathcal{S}}, \tau_{proj})$ then

 $W = \operatorname{projlim}(\mathcal{S}_2) = \operatorname{projlim}(\mathcal{S}_1),$

where S_2 denotes the projective system $\{g_{\alpha}(W), g_{\alpha\beta} \upharpoonright_{g_{\beta}(W)}, J\}$ of TAs (here $g_{\alpha}(W)$ is intended as endowed with the relative topology induced by τ_{α}).

Proof.

Since S is a projective system of TAs, by Definition 3.3.12, we have that for any $\alpha \leq \beta$ the map $g_{\alpha\beta} : A_{\beta} \to A_{\alpha}$ is a continuous homomorphism fulfilling

$$g_{\alpha\alpha} = \mathrm{id}, \ \forall \alpha \in J \tag{3.9}$$

and

$$g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma}, \ \forall \alpha \le \beta \le \gamma \text{ in } J.$$
 (3.10)

Also, by Definition 3.3.14 we have that for any $\alpha \in J$ the map $g_{\alpha} : A_{\mathcal{S}} \to A_{\alpha}$ is a continuous homomorphism such that

$$g_{\alpha} = g_{\alpha\beta} \circ g_{\beta}, \,\forall \alpha \le \beta \text{ in } J.$$

$$(3.11)$$

For any $\alpha \in J$, we have that $g_{\alpha}(W) \subseteq A_{\alpha}$ and so (3.9) provides that $g_{\alpha\alpha} \upharpoonright_{g_{\alpha}(W)} = \text{id} \upharpoonright_{g_{\alpha}(W)}$. Moreover, for any $\alpha \leq \beta \leq \gamma$ in J, the relation (3.11) implies that $g_{\beta\gamma}(g_{\gamma}(W)) \subseteq g_{\beta}(W)$, which in turn gives that for any $x \in g_{\gamma}(W)$:

$$g_{\alpha\beta}\restriction_{g_{\beta}(W)} \left(g_{\beta\gamma}\restriction_{g_{\gamma}(W)}(x)\right) = g_{\alpha\beta}(g_{\beta\gamma}(x)) \stackrel{(3.10)}{=} g_{\alpha\gamma}(x) = g_{\alpha\gamma}\restriction_{g_{\gamma}(W)}(x).$$

Endowing each $g_{\beta}(W)$ with the subspace topology induced by τ_{β} , we have that $g_{\alpha\beta} \upharpoonright_{g_{\beta}(W)}$ is continuous for any $\alpha \leq \beta$ in J. Hence, we have showed that S_2 is a projective system of TAs.

By the continuity of the $g_{\alpha\beta}$'s for all $\alpha \leq \beta$ in J, we also get that

$$g_{\beta\gamma}(\overline{g_{\gamma}(W)}) \subseteq \overline{g_{\beta\gamma}(g_{\gamma}(W))} \stackrel{(3.11)}{=} \overline{g_{\beta}(W)}, \forall \beta \le \gamma \text{ in } J$$
(3.12)

Therefore, for any $\alpha \leq \beta \leq \gamma$ in J and for any $x \in \overline{g_{\gamma}(W)}$ we obtain that

$$g_{\alpha\beta}\restriction_{\overline{g_{\beta}(W)}}\left(g_{\beta\gamma}\restriction_{\overline{g_{\gamma}(W)}}(x)\right) = g_{\alpha\beta}(g_{\beta\gamma}(x)) \stackrel{(3.10)}{=} g_{\alpha\gamma}(x) = g_{\alpha\gamma}\restriction_{\overline{g_{\gamma}(W)}}(x)$$

Hence, we have showed that S_1 is a projective system of TAs, too.

Then

$$\operatorname{projlim}(\mathcal{S}_{1}) = \begin{cases} x := (x_{\alpha})_{\alpha \in J} : x_{\alpha} \in \overline{g_{\alpha}(W)}, \ \forall \alpha \in J \text{ and} \\ x_{\alpha} = g_{\alpha\beta}(x_{\beta}), \ \forall \alpha \leq \beta \text{ in } J \end{cases}$$

$$\stackrel{(3.12)}{=} \begin{cases} x := (x_{\alpha})_{\alpha \in J} : x_{\alpha} \in \overline{g_{\alpha}(W)}, \alpha \in J \end{cases}$$

$$= \{x \in A_{\mathcal{S}} : g_{\alpha}(x) \in \overline{g_{\alpha}(W)}, \forall \alpha \in J \} = \bigcap_{\alpha \in J} g_{\alpha}^{-1}(\overline{g_{\alpha}(W)})$$

and similarly $\operatorname{projlim}(\mathcal{S}_2) = \bigcap_{\alpha \in J} g_{\alpha}^{-1}(g_{\alpha}(W)).$

For any $\alpha \in J$, the continuity of g_{α} provides that $g_{\alpha}(\overline{W}) \subseteq \overline{g_{\alpha}(W)}$ and so $\overline{W} \subseteq g_{\alpha}^{-1}(\overline{g_{\alpha}(W)})$. Hence,

$$\overline{W} \subseteq \bigcap_{\alpha \in J} g_{\alpha}^{-1}(\overline{g_{\alpha}(W)})) = \operatorname{projlim}(\mathcal{S}_1)$$

Conversely, suppose that $x \in \operatorname{projlim}(S_1)$. Then $x \in g_{\alpha}^{-1}(\overline{g_{\alpha}(W)}))$ for all $\alpha \in J$, that means $g_{\alpha}(x) \in \overline{g_{\alpha}(W)}$ for all $\alpha \in J$. Hence, for each $\alpha \in J$, we have that for any neighbourhood V_{α} of the origin in $(A_{\alpha}, \tau_{\alpha})$, the following holds $(g_{\alpha}(x) + V_{\alpha}) \cap g_{\alpha}(W) \neq \emptyset$ and so $(x + g_{\alpha}^{-1}(V_{\alpha})) \cap W \neq \emptyset$. This gives by Lemma 3.3.18 that for any U neighbourhood of the origin in $(A_{\mathcal{S}}, \tau_{proj})$ the sets x + U and W have non-empty intersection, i.e. $x \in \overline{W}$. We have therefore showed that $\overline{W} = \operatorname{projlim}(S_1)$.

If W is closed, then $W = \overline{W}$. However, we have

$$W \subseteq \bigcap_{\alpha \in J} g_{\alpha}^{-1}(g_{\alpha}(W)) \subseteq \bigcap_{\alpha \in J} g_{\alpha}^{-1}(\overline{g_{\alpha}(W)}) = \operatorname{projlim}(\mathcal{S}_{1}) = \overline{W} = W$$

i.e. $W = \operatorname{projlim}(\mathcal{S}_2) = \operatorname{projlim}(\mathcal{S}_1)$.

Suppose now that (E, τ) is a Hausdorff lmc algebra. Then, by Theorem 2.1.11 there exists a basis $\mathcal{M} := \{U_{\alpha}\}_{\alpha \in I}$ of neighbourhoods of the origin in (E, τ) consisting of m-barrels. For each $\alpha \in I$, let p_{α} be the Minkowski functional of U_{α} . Then we have showed in Section 2.2 that $\{p_{\alpha}\}_{\alpha \in I}$ is a family of submultiplicative seminorms on E generating τ . For each $\alpha \in I$, we define $N_{\alpha} := \ker(p_{\alpha})$ which is a closed ideal in (E, τ) . Then we can take the quotient $E_{\alpha} := E/N_{\alpha}$ and endow it with the quotient norm $q_{\alpha}(\rho_{\alpha}(x)) := \inf_{y \in N_{\alpha}} p_{\alpha}(x - y)$ where $\rho_{\alpha} : E \to E_{\alpha}$ denotes the corresponding quotient map. With a similar proof to the one of Proposition 1.4.9 we can prove that is (E_{α}, q_{α}) is a normed algebra. Taking the completion $(\hat{E}_{\alpha}, \hat{q}_{\alpha})$ of each (E_{α}, q_{α}) , we get a family of Banach algebras. If we denote by $i_{\alpha}: E_{\alpha} \to \hat{E}_{\alpha}$ the canonical injection (which is an injective continuous and open homomorphism), then $\overline{\rho_{\alpha}}:=i_{\alpha}\circ\rho_{\alpha}$ is a continuous open homomorphism. For convenience, from now on we will just denote (E_{α}, q_{α}) by E_{α} and $(\hat{E}_{\alpha}, \hat{q}_{\alpha})$ by \hat{E}_{α} .

We define a partial order on I by setting:

$$\alpha \leq \beta \Leftrightarrow U_{\beta} \subseteq U_{\alpha} \Leftrightarrow p_{\alpha}(x) \leq p_{\beta}(x), \forall x \in X.$$

Then (I, \leq) is directed because \mathcal{M} is a basis and so for any $\alpha, \beta \in I$ we have $U_{\alpha} \cap U_{\beta} \in \mathcal{M}$, i.e. there exists $\gamma \in I$ such that $U_{\gamma} \subseteq U_{\alpha} \cap U_{\beta}$ and so $U_{\gamma} \subseteq U_{\alpha}$ and $U_{\gamma} \subseteq U_{\beta}$, i.e. $\alpha \leq \gamma$ and $\beta \leq \gamma$. Also, for any $\alpha \leq \beta$ in I we have $N_{\beta} \subseteq N_{\alpha}$ and hence

$$\begin{array}{rcccc} f_{\alpha\beta}: & E_{\beta} & \to & E_{\alpha} \\ & & x+N_{\beta} & \mapsto & x+N_{\alpha} \end{array}$$

is a well-defined surjective homomorphism and the following holds

$$\rho_{\alpha} = f_{\alpha\beta} \circ \rho_{\beta}, \ \forall \alpha \le \beta \text{ in } I.$$
(3.13)

Then all $f_{\alpha\beta}$'s are continuous homomorphisms and for any $\alpha \leq \beta \leq \gamma$ in Iand any $x \in E_{\beta}$, we have

$$f_{\alpha\beta}(f_{\beta\gamma}(x+N_{\gamma})) = f_{\alpha\beta}(x+N_{\beta}) = x + N_{\alpha} = f_{\alpha\gamma}(x+N_{\gamma})$$

i.e. $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$.

Hence, $\{(E_{\alpha}, q_{\alpha}), f_{\alpha\beta}, I\}$ is a projective system of normed algebras. Moreover, for any $\alpha \leq \beta$ in I there exists $\overline{f_{\alpha\beta}} : \hat{E}_{\beta} \to \hat{E}_{\alpha}$ continuous and linear such that $\overline{f_{\alpha\beta}} \circ i_{\beta} = i_{\alpha} \circ f_{\alpha\beta}$ where i_{α} (resp. i_{β}) denotes the embedding of E_{α} (resp. E_{β}) in \hat{E}_{α} (resp. \hat{E}_{β}). Then it is easy to check that $\{(\hat{E}_{\alpha}, \hat{q}_{\alpha}), \overline{f_{\alpha\beta}}, I\}$ is a projective system of Banach algebras.

We are ready now for the Arens-Michael decomposition theorem.

Theorem 3.3.20. Let (E, τ) be a Hausdorff lmc algebra and $\mathcal{M} := \{U_{\alpha}\}_{\alpha \in I}$ a basis of neighbourhoods of the origin in (E, τ) consisting of m-barrels. Consider the projective system $\{E_{\alpha}, f_{\alpha\beta}, I\}$ of normed algebras and the projective system $\{\hat{E}_{\alpha}, \overline{f_{\alpha\beta}}, I\}$ of Banach algebras introduced above. Then there exist the following topological monomorphisms

$$E \hookrightarrow \operatorname{projlim}\{(E_{\alpha}, q_{\alpha}), f_{\alpha\beta}, I\} \hookrightarrow \operatorname{projlim}\{(\tilde{E}_{\alpha}, q_{\alpha}), \overline{f_{\alpha\beta}}, I\} \cong \tilde{E}.$$
(3.14)

If in addition (E, τ) is complete, then the maps in (3.14) are all topological isomorphisms. In this case, the expression $E = \text{projlim}\{(\hat{E}_{\alpha}, q_{\alpha}), \overline{f_{\alpha\beta}}, I\}$ is called the Arens-Michael decomposition of E w.r.t. \mathcal{M} .

Proof.

For convenience, let us denote by \mathcal{P} and $\hat{\mathcal{P}}$ the projective systems $\{E_{\alpha}, f_{\alpha\beta}, I\}$ and $\{\hat{E}_{\alpha}, \overline{f_{\alpha\beta}}, I\}$, respectively.

For any $x \in E$, let us define $\phi(x) := (\rho_{\alpha}(x))_{\alpha \in I}$. Then $\phi(E) \subseteq \operatorname{projlim}(\mathcal{P})$. Indeed, for any $x \in E$ and $\alpha \leq \beta$ in I we have

$$\left(\phi(x)\right)_{\alpha} = \rho_{\alpha}(x) \stackrel{(3.13)}{=} f_{\alpha\beta}(\rho_{\beta}(x)) = f_{\alpha\beta}\left(\left(\phi(x)\right)_{\beta}\right).$$

Then the following hold:

• ϕ is a homomorphism, as each ρ_{α} is a homomorphism and $\operatorname{projlim}(\mathcal{P})$ is equipped with coordinatewise operations. Let us just show that ϕ is multiplicative: for all $x, y \in E$,

$$\phi(xy) = (\rho_{\alpha}(xy))_{\alpha \in I} = (\rho_{\alpha}(x)\rho_{\alpha}(y))_{\alpha \in I} = (\rho_{\alpha}(x))_{\alpha \in I} (\rho_{\alpha}(y))_{\alpha \in I} = \phi(x)\phi(y).$$

• ϕ is injective, because

$$\phi(x) = 0 \Rightarrow \rho_{\alpha}(x) = 0, \ \forall \alpha \in I \Rightarrow x \in N_{\alpha}, \ \forall \alpha \in I \Rightarrow p_{\alpha}(x) = 0, \ \forall \alpha \in I \Rightarrow x = 0,$$

where in the last implication we used that E is Hausdorff and so $\{p_{\alpha}\}_{\alpha \in I}$ is a separating family of seminorms.

• ϕ is continuous, because Lemma 3.3.18 guarantees that for any neighbourhood U of the origin in projlim(\mathcal{P}), there exist $\alpha \in I$ and a neighbourhood V_{α} of the origin in E_{α} such that $f_{\alpha}^{-1}(V_{\alpha}) \subseteq U$. Then $\rho_{\alpha}^{1}(V_{\alpha}) = \phi^{-1}(f_{\alpha}^{-1}(V_{\alpha})) \subseteq$ $\phi^{-1}(U)$ and so, by the continuity of ρ_{α} we have that $\phi^{-1}(U)$ is a neighborhood of the origin in E.

• ϕ is an open map. Indeed, recalling that $\mathcal{M} := \{U_{\alpha}\}_{\alpha \in I}$ is a basis of neighbourhoods of the origin in (E, τ) , we can show that for any $\alpha \in I$ the set $V := \left(\rho_{\alpha}\left(\frac{1}{2}U_{\alpha}\right) \times \prod_{\beta \in I \setminus \{\alpha\}} E_{\beta}\right) \cap \phi(E)$ is a neighbourhood of the origin in projlim(\mathcal{P}) such that $V \subseteq \phi(U_{\alpha})$. Fix $\alpha \in I$. Then the openness of ρ_{α} implies that $\rho_{\alpha}(\frac{1}{2}U_{\alpha})$ is a neighbourhood of the origin in E_{α} and so $\rho_{\alpha}\left(\frac{1}{2}U_{\alpha}\right) \times \prod_{\beta \in I \setminus \{\alpha\}} E_{\beta}$ is a neighbourhood of the origin in $\prod_{\gamma \in I} E_{\gamma}$ endowed with the product topology. Hence, V is a neighbourhood of the origin in projlim(\mathcal{P}).

Moreover, for any $x := (x_{\gamma})_{\gamma \in I} \in V$ we have that: a) $x \in \phi(E)$, i.e. there exists $y \in E$ such that $\phi(y) = x$ b) $x_{\alpha} \in \rho_{\alpha} \left(\frac{1}{2}U_{\alpha}\right)$ c) $x_{\beta} \in E_{\beta}$ for all $\beta \neq \alpha$ in I. Then () $x_{\beta} \in L_{\beta} = 0$ for all $\beta \neq \alpha$ in I.

$$\rho_{\alpha}(y) = f_{\alpha}(\phi(y)) \stackrel{(a)}{=} f_{\alpha}(x) = x_{\alpha} \stackrel{(b)}{\in} \rho_{\alpha}\left(\frac{1}{2}U_{\alpha}\right),$$

which implies that there exists $z \in \frac{1}{2}U_{\alpha}$ such that $\rho_{\alpha}(y) = \rho_{\alpha}(z)$. Therefore, y = z + w for some $w \in N_{\alpha}$, which gives in turn

$$|p_{\alpha}(y) - p_{\alpha}(z)| \le p_{\alpha}(y - z) \le p_{\alpha}(w) = 0,$$

and so $p_{\alpha}(y) = p_{\alpha}(z) \leq \frac{1}{2} < 1$, i.e. $y \in U_{\alpha}$. Hence, $x \stackrel{(a)}{=} \phi(y) \in \phi(U_{\alpha})$, that gives $V \subseteq \phi(U_{\alpha})$.

We have then just showed that $\phi : E \hookrightarrow \operatorname{projlim}(\mathcal{P})$ is a topological monomorphism.

Now, by using Theorem 3.3.17, we get that for any $\alpha \leq \beta$ in I the diagram

$$\begin{array}{ccc} E_{\beta} & \stackrel{i_{\beta}}{\longrightarrow} & \hat{E}_{\beta} \\ f_{\alpha\beta} & & & & & \\ f_{\alpha\beta} & & & & & \\ E_{\alpha} & \stackrel{i_{\alpha}}{\longrightarrow} & \hat{E}_{\alpha} \end{array}$$

commutes, where i_{α} and i_{β} are topological monomorphisms such that $\overline{i_{\alpha}(E_{\alpha})} = \hat{E}_{\alpha}$ and $\overline{i_{\beta}(E_{\beta})} = \hat{E}_{\beta}$. Then [4, E.III.53, Corollary 1] ensures that there exists a unique topological monomorphism j: projlim(\mathcal{P}) \hookrightarrow projlim($\hat{\mathcal{P}}$) such that the following diagram commutes

$$\begin{array}{ccc} \operatorname{projlim}(\mathcal{P}) & \stackrel{\mathcal{I}}{\longrightarrow} & \operatorname{projlim}(\hat{\mathcal{P}}) \\ & & & & \downarrow_{\overline{f_{\alpha}}} \\ & & & \downarrow_{\overline{f_{\alpha}}} \\ & & & E_{\alpha} & \stackrel{i_{\alpha}}{\longrightarrow} & \hat{E}_{\alpha} \end{array}$$
 (3.15)

Setting $\psi = j \circ \phi$ we get a topological monomorphism from E to $\operatorname{projlim}(\hat{\mathcal{P}})$ and so $\psi(E)$ is a linear subspace of $\operatorname{projlim}(\hat{\mathcal{P}})$. Therefore, Lemma 3.3.19 provides that $\overline{\psi(E)} = \operatorname{projlim}(\mathcal{Q})$, where $\mathcal{Q} := \left\{\overline{\overline{f_{\alpha}}(\psi(E))}, \overline{f_{\alpha\beta}} \upharpoonright_{\overline{f_{\beta}}(\psi(E))}, I\right\}$. By the commutativity of the diagram (3.15), we know that

$$\overline{f}_{\alpha}(\psi(E)) = \overline{f}_{\alpha}(j(\phi(E))) = i_{\alpha}(f_{\alpha}(\phi(E))) = i_{\alpha}(\rho_{\alpha}(E)) = i_{\alpha}(E_{\alpha}).$$

Hence, $\overline{\overline{f_{\alpha}}(\psi(E))} = \overline{i_{\alpha}(E_{\alpha})} = \hat{E}_{\alpha}$ and so

$$\overline{\psi(E)} = \operatorname{projlim}(\mathcal{Q}) = \operatorname{projlim}(\hat{\mathcal{P}}).$$

75

This together with the fact that $\operatorname{projlim}(\hat{\mathcal{P}})$ is complete (see Remark 3.3.15-c)) implies that \hat{E} is topologically isomorphic to $\operatorname{projlim}(\hat{\mathcal{P}})$ by Theorem 3.3.17-I). Therefore, we have proved that

$$E \stackrel{\phi}{\hookrightarrow} \operatorname{projlim}(\mathcal{P}) \stackrel{j}{\hookrightarrow} \operatorname{projlim}(\hat{\mathcal{P}}) \cong \hat{E}.$$

If in addition E is complete, then $E = \hat{E}$ and so ϕ and j must be also isomorphisms.

Using Remark 3.3.15, we can easily derive from Theorem 3.3.20 the following

Corollary 3.3.21.

- a) Every Hausdorff lmc algebra can be topologically embedded in a cartesian product of Banach algebras.
- b) Every Fréchet lmc algebra is topologically isomorphic to the projective limit of a sequence of Banach algebras.

Theorem 3.3.22. Let (E, τ) be a Hausdorff complete lmc algebra and $\mathcal{M} := \{U_{\alpha}\}_{\alpha \in I}$ a basis of neighbourhoods of the origin in (E, τ) consisting of mbarrels. Then:

- a) E is unital if and only if each component of its Arens-Michael decomposition w.r.t. M is a unital Banach algebra.
- b) $x \in E$ is invertible if and only if its image into each component of the its Arens-Michael decomposition of E w.r.t. \mathcal{M} is invertible.

Proof.

Let $E = \text{projlim}\{\hat{E}_{\alpha}, \overline{f_{\alpha\beta}}, I\}$ be the Arens-Michael decomposition of E w.r.t. \mathcal{M} (see Theorem 3.3.20).

a) Suppose that there exists $u \in E$ s.t. for all $y \in E$ we have $u \cdot y = y = y \cdot u$. For any $\alpha \in I$, set $u_{\alpha} := \overline{\rho}_{\alpha}(u) \in \hat{E}_{\alpha}$. By the surjectivity of $\overline{\rho}_{\alpha}$, we know that for any $x_{\alpha} \in \hat{E}_{\alpha}$ there exists $x \in E$ such that $\overline{\rho}_{\alpha}(x) = x_{\alpha}$ and so we get that:

$$x_{\alpha} \cdot u_{\alpha} = \overline{\rho}_{\alpha}(x)\overline{\rho}_{\alpha}(u) = \overline{\rho}_{\alpha}(x \cdot u) = \overline{\rho}_{\alpha}(x) = x_{\alpha}$$

and similarly we obtain $u_{\alpha}x_{\alpha} = x_{\alpha}$, i.e. each \hat{E}_{α} is unital.

Conversely, suppose that for any $\alpha \in I$ there exists $u_{\alpha} \in \hat{E}_{\alpha}$ s.t. $y \cdot u_{\alpha} = y = u_{\alpha} \cdot y$ for all $y \in \hat{E}_{\alpha}$. Then $u := (u_{\alpha})_{\alpha \in I}$ belongs to projlim $\{\hat{E}_{\alpha}, \overline{f_{\alpha\beta}}, I\}$ since for all $\alpha \leq \beta$ in I and for all $x_{\alpha} \in \hat{E}_{\alpha}$ we get:

$$\begin{aligned} x_{\alpha} \cdot \overline{f_{\alpha\beta}}(u_{\beta}) &= \overline{\rho}_{\alpha}(x) \cdot \overline{f_{\alpha\beta}}(u_{\beta}) = \overline{f_{\alpha\beta}}(\overline{\rho}_{\beta}(x)) \cdot \overline{f_{\alpha\beta}}(u_{\beta}) \\ &= \overline{f_{\alpha\beta}}(\overline{\rho}_{\beta}(x) \cdot u_{\beta}) = \overline{f_{\alpha\beta}}(\overline{\rho}_{\beta}(x)) = \overline{\rho}_{\alpha}(x) = x_{\alpha} \end{aligned}$$

i.e. $\overline{f_{\alpha\beta}}(u_{\beta}) = u_{\alpha}$. As the multiplication in projlim $\{\hat{E}_{\alpha}, \overline{f_{\alpha\beta}}, I\}$ is defined coordinatewise, it is then clear that $u := (u_{\alpha})_{\alpha \in I}$ is the identity element of the multiplication in projlim $\{\hat{E}_{\alpha}, \overline{f_{\alpha\beta}}, I\}$, which is therefore a unital algebra.

b) Suppose that u is the identity element of the multiplication in E and that $x \in E$ is invertible, i.e. there exists $y \in E$ s.t. $x \cdot y = u = y \cdot x$. For each $\alpha \in I$, we have already showed that $u_{\alpha} := \overline{\rho}_{\alpha}(u)$ is the identity element of the multiplication in \hat{E}_{α} . Hence, we have

$$\overline{\rho}_{\alpha}(x) \cdot \overline{\rho}_{\alpha}(y) = \overline{\rho}_{\alpha}(x \cdot y) = \overline{\rho}_{\alpha}(u) = u_{\alpha},$$

i.e. $\overline{\rho}_{\alpha}(x)$ is invertible in \hat{E}_{α} .

Conversely, suppose that $x \in \operatorname{projlim}\{\hat{E}_{\alpha}, \overline{f_{\alpha\beta}}, I\}$ is s.t. for each $\alpha \in I$ the element $\overline{\rho}_{\alpha}(x)$ is invertible. Then for each $\alpha \in I$ there exists $y_{\alpha} \in \hat{E}_{\alpha}$ s.t. $\overline{\rho}_{\alpha}(x) \cdot y_{\alpha} = u_{\alpha} = y_{\alpha} \cdot \overline{\rho}_{\alpha}(x)$, where u_{α} is the identity element of the multiplication in \hat{E}_{α} . Now as we have already showed that $u := (u_{\alpha})_{\alpha \in I}$ is the identity element of the (coordinatewise) multiplication in $\operatorname{projlim}\{\hat{E}_{\alpha}, \overline{f_{\alpha\beta}}, I\}$, it is enough to prove that $(y_{\alpha})_{\alpha \in I} \in \operatorname{projlim}\{\hat{E}_{\alpha}, \overline{f_{\alpha\beta}}, I\}$. This is indeed true since for all $\alpha \leq \beta$ in I the following holds

$$\overline{\rho}_{\alpha}(x) \cdot \overline{f_{\alpha\beta}}(y_{\beta}) = \overline{f_{\alpha\beta}}(\overline{\rho}_{\beta}(x)) \cdot \overline{f_{\alpha\beta}}(y_{\beta}) = \overline{f_{\alpha\beta}}(\overline{\rho}_{\beta}(x) \cdot y_{\beta}) = \overline{f_{\alpha\beta}}(u_{\beta}) = y_{\alpha},$$

and, hence, $\overline{f_{\alpha\beta}}(y_{\beta}) = y_{\alpha}.$

Chapter 4

Symmetric tensor algebras

As usual, we consider only vector spaces over the field \mathbb{K} of real numbers or of complex numbers. The aim of this section is to present a way to explicitly construct an lmc algebra starting from the symmetric tensor algebra of a lc TVS. For this purpose, we will preliminarily introduce the concept of tensor product of vector spaces and then endow it with one of the many topologies which can be defined when the starting space carries an lc structure.

4.1 Tensor product of vector spaces

Let us start with a notion which is central in the definition of tensor product.

Definition 4.1.1.

Let E, F, M be three vector spaces over \mathbb{K} and $\phi : E \times F \to M$ be a bilinear map. E and F are said to be ϕ -linearly disjoint if:

(LD) For any $r, s \in \mathbb{N}$, x_1, \ldots, x_r linearly independent in E and y_1, \ldots, y_s linearly independent in F, the set $\{\phi(x_i, y_j) : i = 1, \ldots, r, j = 1, \ldots, s\}$ consists of linearly independent vectors in M.

or equivalently if:

- **(LD)'** For any $r \in \mathbb{N}$, any $\{x_1, \ldots, x_r\}$ finite subset of E and any $\{y_1, \ldots, y_r\}$ finite subset of F s.t. $\sum_{i=1}^r \phi(x_i, y_j) = 0$, we have that both the following conditions hold:
 - if x_1, \ldots, x_r are linearly independent in E, then $y_1 = \cdots = y_r = 0$
 - if y_1, \ldots, y_r are linearly independent in F, then $x_1 = \cdots = x_r = 0$.

Definition 4.1.2. A tensor product of two vector spaces E and F over \mathbb{K} is a pair (M, ϕ) consisting of a vector space M over \mathbb{K} and of a bilinear map $\phi: E \times F \to M$ (canonical map) s.t. the following conditions are satisfied:

- **(TP1)** The image of $E \times F$ spans the whole space M.
- **(TP2)** E and F are ϕ -linearly disjoint.

The following theorem guarantees that the tensor product of any two vector spaces always exists, satisfies the "universal property" and it is unique up to isomorphisms. For this reason, the tensor product of E and F is usually denoted by $E \otimes F$ and the canonical map by $(x, y) \mapsto x \otimes y$.

Theorem 4.1.3. Let E, F be two vector spaces over \mathbb{K} .

- (a) There exists a tensor product of E and F.
- (b) Let (M, ϕ) be a tensor product of E and F. Let G be any vector space over \mathbb{K} , and b any bilinear mapping of $E \times F$ into G. There exists a unique linear map $\tilde{b}: M \to G$ such that the diagram



is commutative.

(c) If (M_1, ϕ_1) and (M_2, ϕ_2) are two tensor products of E and F, then there is a bijective linear map u such that the diagram



is commutative.

Proof. (see [16, Theorem 4.1.4])

Examples 4.1.4.

1. Let $n, m \in \mathbb{N}$, $E = \mathbb{K}^n$ and $F = \mathbb{K}^m$. Then $E \otimes F = \mathbb{K}^{nm}$ is a tensor product of E and F whose canonical bilinear map ϕ is given by:

$$\phi: \quad E \times F \qquad \to \quad \mathbb{K}^{nm} \\ \left((x_i)_{i=1}^n, (y_j)_{j=1}^m \right) \quad \mapsto \quad (x_i y_j)_{1 \le i \le n, 1 \le j \le m}$$

2. Let X and Y be two sets. For any functions $f: X \to \mathbb{K}$ and $g: Y \to \mathbb{K}$, we define:

$$\begin{array}{rcccc} f \otimes g : & X \times Y & \to & \mathbb{K} \\ & & (x,y) & \mapsto & f(x)g(y). \end{array}$$

Let E (resp. F) be the linear space of all functions from X (resp. Y) to \mathbb{K} endowed with the pointwise addition and multiplication by scalars. We

denote by M the linear subspace of the space of all functions from $X \times Y$ to \mathbb{K} spanned by the elements of the form $f \otimes g$ for all $f \in E$ and $g \in F$. Then M is actually a tensor product of E and F, i.e. $M = E \otimes F$.

Similarly to how we defined the tensor product of two vector spaces we can define the tensor product of an arbitrary number of vector spaces.

Definition 4.1.5. Let $n \in \mathbb{N}$ with $n \geq 2$ and E_1, \ldots, E_n vector spaces over \mathbb{K} . A tensor product of E_1, \ldots, E_n is a pair (M, ϕ) consisting of a vector space M over \mathbb{K} and of a multilinear map $\phi : E_1 \times \cdots \times E_n \to M$ (canonical map) s.t. the following conditions are satisfied:

(TP1) The image of $E_1 \times \cdots \times E_n$ spans the whole space M.

(TP2) E_1, \ldots, E_n are ϕ -linearly disjoint, i.e. for any $r_1, \ldots, r_n \in \mathbb{N}$ and for any $x_1^{(i)}, \ldots, x_{r_i}^{(i)}$ linearly independent in E_i $(i = 1, \ldots, n)$, the set

$$\left\{\phi\left(x_{j_1}^{(1)},\ldots,x_{j_n}^{(n)}\right): j_1=1,\ldots,r_1,\ldots,j_n=1,\ldots,r_n\right\}$$

consists of linearly independent vectors in M.

Recall that a map is multilinear if it is linear in each of its variables. As for the case n = 2 it is possible to show that:

- (a) There always exists a tensor product of E_1, \ldots, E_n .
- (b) The universal property holds for $E_1 \otimes \cdots \otimes E_n$.
- (c) $E_1 \otimes \cdots \otimes E_n$ is unique up to isomorphisms.

4.2 The π -topology on the tensor product of lc TVS

Given two locally convex TVS E and F, there are various ways to construct a topology on the tensor product $E \otimes F$ which makes the vector space $E \otimes F$ in a TVS. Indeed, starting from the topologies on E and F, one can define a topology on $E \otimes F$ either relying directly on the seminorms on E and F, or using an embedding of $E \otimes F$ in some space related to E and F over which a natural topology already exists. The first method leads to the so-called π -topology. The second method may lead instead to a variety of topologies, which we are not going to investigate in this course.

Definition 4.2.1 (π -topology).

Given two locally convex TVS E and F, we define the π -topology (or projective topology) on $E \otimes F$ to be the finest locally convex topology on this vector space for which the canonical mapping $E \times F \to E \otimes F$ is continuous. The space $E \otimes F$ equipped with the π -topology will be denoted by $E \otimes_{\pi} F$. A basis of neighbourhoods of the origin in $E \otimes_{\pi} F$ is given by the family:

$$\mathcal{B} := \{ \operatorname{conv}_b(U_\alpha \otimes V_\beta) : U_\alpha \in \mathcal{B}_E, V_\beta \in \mathcal{B}_F \},\$$

where \mathcal{B}_E (resp. \mathcal{B}_F) is a basis of neighbourhoods of the origin in E (resp. in F), $U_{\alpha} \otimes V_{\beta} := \{x \otimes y \in E \otimes F : x \in U_{\alpha}, y \in V_{\beta}\}$. In fact, on the one hand, the π -topology is by definition locally convex and so it has a basis \mathcal{B} of convex balanced neighbourhoods of the origin in $E \otimes F$. Then, as the canonical mapping ϕ is continuous w.r.t. the π -topology, we have that for any $C \in \mathcal{B}$ there exist $U_{\alpha} \in \mathcal{B}_E$ and $V_{\beta} \in \mathcal{B}_F$ s.t. $U_{\alpha} \times V_{\beta} \subseteq \phi^{-1}(C)$. Hence, $U_{\alpha} \otimes V_{\beta} = \phi(U_{\alpha} \times V_{\beta}) \subseteq C$ and so $\operatorname{conv}_b(U_{\alpha} \otimes V_{\beta}) \subseteq \operatorname{conv}_b(C) = C$ which yields that the topology generated by \mathcal{B}_{π} is finer than the π -topology. On the other hand, the canonical map ϕ is continuous w.r.t. the topology generated by \mathcal{B}_{π} , because for any $U_{\alpha} \in \mathcal{B}_E$ and $V_{\beta} \in \mathcal{B}_F$ we have that $\phi^{-1}(\operatorname{conv}_b(U_{\alpha} \otimes V_{\beta})) \supseteq \phi^{-1}(U_{\alpha} \otimes V_{\beta}) = U_{\alpha} \times V_{\beta}$ which is a neighbourhood of the origin in $E \times F$. Hence, the topology generated by \mathcal{B}_{π} is coarser than the π -topology.

The π -topology on $E \otimes F$ can be described by means of the seminorms defining the locally convex topologies on E and F.

Theorem 4.2.2. Let E and F be two locally convex TVS and let \mathcal{P} (resp. \mathcal{Q}) be a family of seminorms generating the topology on E (resp. on F). The π -topology on $E \otimes F$ is generated by the family of seminorms

$$\{p \otimes q : p \in \mathcal{P}, q \in \mathcal{Q}\},\$$

where for any $p \in \mathcal{P}, q \in \mathcal{Q}, \theta \in E \otimes F$ we define:

$$(p \otimes q)(\theta) := \inf \left\{ \sum_{k=1}^r p(x_k)q(y_k) : \theta = \sum_{k=1}^r x_k \otimes y_k, \, , x_k \in E, y_k \in F, r \in \mathbb{N} \right\}.$$

Proof. (see [16, Proposition 4.3.10 and Theorem 4.3.11])

The seminorm $p \otimes q$ on $E \otimes F$ defined in the previous proposition is called tensor product of the seminorms p and q (or projective cross seminorm)

Proposition 4.2.3. Let E and F be two locally convex TVS. $E \otimes_{\pi} F$ is Hausdorff if and only if E and F are both Hausdorff.

In analogy with the algebraic case (see Theorem 4.1.3-b), we also have a universal property for the space $E \otimes_{\pi} F$.

Proposition 4.2.4.

Let E, F be locally convex spaces. The π -topology on $E \otimes_{\pi} F$ is the unique locally convex topology on $E \otimes F$ such that the following property holds:

(UP) For every locally convex space G, the algebraic isomorphism between the space of bilinear mappings from $E \times F$ into G and the space of all linear mappings from $E \otimes F$ into G (given by Theorem 4.1.3-b) induces an algebraic isomorphism between B(E, F; G) and $L(E \otimes F; G)$, where B(E, F; G) denotes the space of all continuous bilinear mappings from $E \times F$ into G and $L(E \otimes F; G)$ the space of all continuous linear mappings from $E \otimes F$ into G.

Proof. Let τ be a locally convex topology on $E \otimes F$ such that the property (UP) holds. Then (UP) holds in particular for $G = (E \otimes F, \tau)$. Therefore, by Theorem 4.1.3-b) the identity id : $E \otimes F \to E \otimes F$ is the unique linear map such that the diagram

commutes. Hence, we get that $\phi: E \times F \to E \otimes_{\tau} F$ has to be continuous.

This implies that $\tau \subseteq \pi$ by definition of π -topology. On the other hand, (UP) also holds for $G = (E \otimes F, \pi)$.

$$E \times F \xrightarrow{\phi} E \otimes_{\pi} F$$

$$\downarrow^{\phi} \xrightarrow{\text{id}}$$

$$E \otimes_{\pi} F$$

Hence, since by definition of π -topology $\phi : E \times F \to E \otimes_{\pi} F$ is continuous, the $id : E \otimes_{\tau} F \to E \otimes_{\pi} F$ has to be also continuous. This means that $\pi \subseteq \tau$, which completes the proof.

4.3 Tensor algebra and symmetric tensor algebra of a vs

Let V be a vector space over K. For any $k \in \mathbb{N}$, we define the k-th tensor power of V as

$$V^{\otimes k} := \underbrace{V \otimes \cdots \otimes V}_{\text{k-times}}$$

and we take by convention $V^{\otimes 0} := \mathbb{K}$. Then it is possible to show that there exists the following algebraic isomorphism:

$$\forall n, m \in \mathbb{N}, \quad V^{\otimes n} \otimes V^{\otimes m} \cong V^{\otimes (n+m)} \tag{4.1}$$

We can pack together all tensor powers of V in a unique vector space:

$$T(V) := \bigoplus_{k=0}^{\infty} V^{\otimes k}$$

We define a multiplication over T(V) which makes it into a unital \mathbb{K} -algebra.

First of all, let us observe that for any $k \in \mathbb{N}_0$ there is a natural embedding $i_k : V^{\otimes k} \to T(V)$. For sake of notational convenience, in the following we will identify each $g \in V^{\otimes k}$ with $i_k(g)$. Then every element $f \in T(V)$ can be expressed as $f = \sum_{k=0}^{N} f_k$ for some $N \in \mathbb{N}_0$ and $f_k \in V^{\otimes k}$ for $k = 0, \ldots, N$. Using the isomorphism given by (4.1), for any $j, k \in \mathbb{N}$ we can define the following bilinear operation:

$$: V^{\otimes k} \times V^{\otimes j} \to V^{\otimes (k+j)}$$

$$((v_1 \otimes \cdots \otimes v_k), (w_1 \otimes \cdots \otimes w_j)) \mapsto v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_j.$$

$$(4.2)$$

Hence, we get a multiplication $\cdot: T(V) \times T(V) \to T(V)$ just by defining for all $f, g \in T(V)$, say $f = \sum_{k=0}^{N} f_k$ and $g = \sum_{j=0}^{M} g_j$ for some $N, M \in \mathbb{N}_0$, $f_k \in V^{\otimes k}, g_j \in V^{\otimes j}$,

$$f \cdot g := \sum_{k=0}^{N} \sum_{j=0}^{M} f_k \cdot g_j$$

where $f_k \cdot g_j$ is the one defined in (4.2). Then we easily see that:

- a) \cdot is bilinear on $T(V) \times T(V)$ as it is bilinear on each $V^{\otimes k} \times V^{\otimes j}$ for all $j, k \in \mathbb{N}_0$.
- b) \cdot is associative, i.e. $\forall f, g, h \in T(V), (f \cdot g) \cdot h = f \cdot (g \cdot h)$. Indeed, if $f = \sum_{k=0}^{N} f_k, g = \sum_{j=0}^{M} g_j, h = \sum_{l=0}^{S} h_l$ with $N, M, S \in \mathbb{N}_0, f_k \in V^{\otimes k}, g_j \in V^{\otimes j}, h_l \in V^{\otimes l}$, then

$$(f \cdot g) \cdot h = \sum_{k=0}^{N} \sum_{j=0}^{M} \sum_{l=0}^{S} (f_k \cdot g_j) \cdot h_l = \sum_{k=0}^{N} \sum_{j=0}^{M} \sum_{l=0}^{S} f_k \cdot (g_j \cdot h_l) = f \cdot (g \cdot h),$$

where we have just used that $V^{\otimes (k+j)} \otimes V^{\otimes l} \cong V^{\otimes (k+j+l)} \cong V^{\otimes k} \otimes V^{\otimes (j+l)}$ by (4.1).

c) $1 \in \mathbb{K}$ is the identity for the multiplication \cdot , since $\mathbb{K} = V^{\otimes 0}$ and for all $f = \sum_{k=0}^{N} f_k \in T(V)$ we have $1 \cdot f = \sum_{k=0}^{n} (1 \cdot f_k) = \sum_{k=0}^{N} f_k = f$.

Hence, $(T(V), \cdot)$ is a unital K-algebra, which is usually called the *tensor* algebra of V.

Remark 4.3.1. If $\{x_i\}_{i\in\Omega}$ is a basis of the vector space V, then each element of $V^{\otimes k}$ can be identified with a polynomial of degree k in the non-commuting variables $\{x_i\}_{i\in\Omega}$ and with coefficients in \mathbb{K} . Hence, T(V) is identified with the non-commutative polynomial ring $\mathbb{K}\langle x_i, i \in \Omega \rangle$.

Proposition 4.3.2. Let V be a vector space over \mathbb{K} . For any unital \mathbb{K} -algebra (A, *) and any linear map $f: V \to A$, there exists a unique \mathbb{K} -algebra homomorphism $\overline{f}: T(V) \to A$ such that the following diagram commutes



where i_1 is the natural embedding of $V = V^{\otimes 1}$ into T(V).

Proof. For any $k \in \mathbb{N}$, we define

$$f_k: \underbrace{V \times \cdots \times V}_{\substack{k \text{ times} \\ (v_1, \dots, v_k)}} \to A$$

which is multilinear by the linearity of f. For k = 0 we define

By the universal property of $V^{\otimes k}$, we have that there exists a unique linear map $\sigma_k : V^{\otimes k} \to A$ s.t. $\sigma_k(v_1 \otimes \cdots \otimes v_k) = f_k(v_1, \ldots, v_k) = f(v_1) * \cdots * f(v_k)$ and for k = 0 we have $\sigma_0(r) = f_0(r) = r \mathbf{1}_A$, $\forall r \in \mathbb{K}$. Then, by the universal property of the direct sum, we get that there exists a unique linear map $\overline{f}: T(V) \to A$ such that $\overline{f}(i_k(v_1 \otimes \cdots \otimes v_k)) = \sigma_k(v_1 \otimes \cdots \otimes v_k)$



85

In particular, for k = 1 we obtain that $\overline{f}(i_1(v)) = \sigma_1(v) = f(v)$.

It remains to show that \overline{f} is a \mathbb{K} -algebra homomorphism from T(V) to A. By construction of \overline{f} , we clearly have that \overline{f} is linear and

$$\bar{f}(1_{T(V)}) = \bar{f}(i_0(1)) = \sigma_0(1) = f_0(1) = 1_A.$$

Let us prove now that for any $x, y \in T(V)$ we get $\overline{f}(x \cdot y) = \overline{f}(x) * \overline{f}(y)$. As \overline{f} is linear, it is enough to show that for any $n, m \in \mathbb{N}$, any $x_1, \ldots, x_n \in V$ and any $y_1, \ldots, y_m \in V$, we get

$$\bar{f}((x_1 \otimes \cdots \otimes x_n) \cdot (y_1 \otimes \cdots \otimes y_m)) = \bar{f}(x_1 \otimes \cdots \otimes x_n) * \bar{f}(y_1 \otimes \cdots \otimes y_m).$$
(4.3)

Indeed, by just applying the properties of \bar{f} , we obtain that:

$$\overline{f}(x_1 \otimes \cdots \otimes x_n) = \sigma_n(x_1 \otimes \cdots \otimes x_n) = f(x_1) * \cdots * f(x_n)$$

and

$$\overline{f}(y_1\otimes\cdots\otimes y_m)=\sigma_m(y_1\otimes\cdots\otimes y_m)=f(y_1)*\cdots*f(y_m).$$

These together with the definition of multiplication in T(V) give that:

$$\bar{f}(x_1 \otimes \cdots \otimes x_n) * \bar{f}(y_1 \otimes \cdots \otimes y_m) = f(x_1) * \cdots * f(x_n) * f(y_1) * \cdots * f(y_m)$$
$$= \bar{f}((x_1 \otimes \cdots \otimes x_n) \cdot (y_1 \otimes \cdots \otimes y_m)).$$

Consider now the ideal I in $(T(V), \cdot)$ generated by the elements $v \otimes w - w \otimes v$, for all $v, w \in V$. The tensor algebra T(V) factored by this ideal I is denoted by S(V) and called the *symmetric (tensor) algebra* of V. If we denote by π the quotient map from T(V) to S(V), then for any $k \in \mathbb{N}_0$ and any element $f = \sum_{i=1}^n f_{i1} \otimes \cdots \otimes f_{ik} \in V^{\otimes k}$ (here $n \in \mathbb{N}$, $f_{ij} \in V$ for $i = 1, \ldots, n, j = 1, \ldots, k$ and $n \geq 1$) we have that

$$\pi\left(\sum_{i=1}^n f_{i1}\otimes\cdots\otimes f_{ik}\right)=\sum_{i=1}^n f_{i1}\cdots f_{ik}.$$

We define the k-th homogeneous component of S(V) to be the image of $V^{\otimes k}$ under π and we denoted it by $S(V)_k$. Note that $S(V)_0 = \mathbb{K}$ and $S(V)_1 = V$. Hence, we have

$$S(V) = \bigoplus_{k=0}^{\infty} S(V)_k$$

and so every element $f \in S(V)$ can expressed as $f = \sum_{k=0}^{N} f_k$ for some $N \in \mathbb{N}$, $f_k \in S(V)_k$ for $k = 0, \ldots, N$.

Remark 4.3.3. If $\{x_i\}_{i\in\Omega}$ is a basis of the vector space V, then each element of $S(V)_k$ can be identified with a polynomial of degree k in the commuting variables $\{x_i\}_{i\in\Omega}$ and with coefficients in K. Hence, S(V) is identified with the commutative polynomial ring $\mathbb{K}[x_i: i \in \Omega]$.

The universal property of S(V) easily follows from the universal property of T(V).

Proposition 4.3.4. Let V be a vector space over \mathbb{K} . For any unital commutative \mathbb{K} -algebra (A, *) and any linear map $\psi : V \to A$, there exists a unique \mathbb{K} -algebra homomorphism $\overline{\psi} : S(V) \to A$ such that the following diagram commutes



i.e. $\bar{\psi} \upharpoonright_V = \psi$.

Corollary 4.3.5. Let V be a vector space over \mathbb{K} . The algebraic dual V^* of V is algebraically isomorphic to $\operatorname{Hom}(S(V), \mathbb{K})$.

Proof. For any $\alpha \in \text{Hom}(S(V), \mathbb{K})$ we clearly have $\alpha \upharpoonright_V \in V^*$. On the other hand, by Proposition 4.3.4, for any $\ell \in V^*$ there exists a unique $\bar{\ell} \in \text{Hom}(S(V), \mathbb{K})$ such that $\bar{\ell} \upharpoonright_V = \ell$.

4.4 An Imc topology on the symmetric algebra of a lc TVS

Let V be a vector space over \mathbb{K} . In this section we are going to explain how a locally convex topology τ on V can be naturally extended to a locally convex topology $\overline{\tau}$ on the symmetric algebra S(V) (see [14]). Let us start by considering the simplest possible case, i.e. when τ is generated by a single seminorm.

Suppose now that ρ is a seminorm on V. Starting from the seminorm ρ on V, we are going to construct a seminorm $\bar{\rho}$ on S(V) in three steps:

1. For $k \in \mathbb{N}$, let us consider the projective tensor seminorm on $V^{\otimes k}$ see Theorem 4.2.2, i.e.

$$\rho^{\otimes k}(g) := (\underbrace{\rho \otimes \cdots \otimes \rho}_{k \text{ times}})(g)$$
$$= \inf \left\{ \sum_{i=1}^{N} \rho(g_{i1}) \cdots \rho_k(g_{ik}) : g = \sum_{i=1}^{N} g_{i1} \otimes \cdots \otimes g_{ik}, \ g_{ij} \in V, \ N \in \mathbb{N} \right\}.$$

2. Denote by $\pi_k : V^{\otimes k} \to S(V)_k$ the quotient map π restricted to $V^{\otimes k}$ and define $\overline{\rho}_k$ to be the quotient seminorm on $S(V)_k$ induced by $\rho^{\otimes k}$, i.e.

$$\overline{\rho}_k(f) := \inf \{ \rho^{\otimes k}(g) : g \in V^{\otimes k}, \ \pi_k(g) = f \}$$
$$= \inf \left\{ \sum_{i=1}^N \rho(f_{i1}) \cdots \rho(f_{ik}) : f = \sum_{i=1}^N f_{i1} \cdots f_{ik}, f_{ij} \in V, N \in \mathbb{N} \right\}.$$

Define $\overline{\rho}_0$ to be the usual absolute value on \mathbb{K} .

3. For any $h \in S(V)$, say $h = h_0 + \cdots + h_\ell$, $f_k \in S(V)_k$, $k = 0, \ldots, \ell$, define

$$\overline{\rho}(f) := \sum_{k=0}^{\ell} \overline{\rho}_k(f_k).$$

We refer to $\overline{\rho}$ as the projective extension of ρ to S(V).

Proposition 4.4.1. $\overline{\rho}$ is a seminorm on S(V) extending the seminorm ρ on V and $\overline{\rho}$ is also submultiplicative i.e. $\overline{\rho}(f \cdot g) \leq \overline{\rho}(f)\overline{\rho}(g), \ \forall f, g \in S(V)$

To prove this result we need an essential lemma:

Lemma 4.4.2. Let $i, j \in \mathbb{N}$, $f \in S(V)_i$ and $g \in S(V)_j$. If k = i + j then $\overline{\rho}_k(fg) \leq \overline{\rho}_i(f)\overline{\rho}_j(g)$.

Proof.

Let us consider a generic representation of $f \in S(V)_i$ and $g \in S(V)_j$, i.e. $f = \sum_p f_{p1} \cdots f_{pi}$ with $f_{pk} \in V$ for $k = 1, \ldots, i$ and $g = \sum_q g_{q1} \cdots g_{qj}$ with $g_{ql} \in V$ for $l = 1, \ldots, j$. Then $f \cdot g = \sum_{p,q} f_{p1} \cdots f_{pi} g_{q1} \cdots g_{qj}$, and so

$$\overline{\rho}_k(f \cdot g) \leq \sum_{p,q} \rho(f_{p1}) \cdots \rho(f_{pi}) \rho(g_{q1}) \cdots \rho(g_{qj})$$
$$= \left(\sum_p \rho(f_{p1}) \cdots \rho(f_{pi})\right) \left(\sum_q \rho(g_{q1}) \cdots \rho(g_{qj})\right)$$

Since this holds for any representation of f and g, we get $\overline{\rho}_k(fg) \leq \overline{\rho}_i(f)\overline{\rho}_j(g)$.

Proof. (of Proposition 4.4.1).

It is quite straightforward to show that $\overline{\rho}$ is a seminorm on S(V). Indeed

• Let $k \in \mathbb{K}$ and $f \in S(V)$. Consider any representation of f, say we take $f = \sum_{j=0}^{n} f_j$ with $n \in \mathbb{N}$ and $f_j \in S(V)_j$ for $j = 0, \ldots, n$. Then using the definition of $\overline{\rho}$ and the fact that $\overline{\rho}_k$ is a seminorm on $S(V)_k$ we get:

$$\overline{\rho}(kf) = \overline{\rho}\left(\sum_{j=0}^{n} kf_{j}\right) = \sum_{j=0}^{n} \overline{\rho}_{j}(kf_{j}) = |k| \sum_{j=0}^{n} \overline{\rho}_{j}(f_{j}) = |k|\overline{\rho}(f).$$

• Let $f, g \in S(V)$. Consider any representation of f and g, say we take $f = \sum_{j=0}^{n} f_j, g = \sum_{i=0}^{m} g_i$ with $n, m \in \mathbb{N}, f_j \in S(V)_j$ for $j = 0, \ldots, n$ and $g_i \in S(V)_i$ for $i = 0, \ldots, m$. Take $N := \max\{n, m\}$. Then we can rewrite $f = \sum_{j=0}^{N} f_j$ and $g = \sum_{i=0}^{N} g_i$, where $f_j = 0$ for $j = n+1, \ldots, N$ and $g_i = 0$ for $i = m+1, \ldots, N$. Therefore, using the definition of $\overline{\rho}$ and the fact that $\overline{\rho}_k$ is a seminorm on $S(V)_k$, we have

$$\overline{\rho}(f+g) = \overline{\rho}\left(\sum_{j=0}^{N} (f_j + g_j)\right) \le \sum_{j=0}^{N} \overline{\rho}_j(f_j) + \sum_{j=0}^{N} \overline{\rho}_j(g_j) = \overline{\rho}(f) + \overline{\rho}(g).$$

Also, $\overline{\rho}_1 = \rho$, so $\overline{\rho}$ restricted to V coincides with ρ . Let us finally show that $\overline{\rho}$ is submultiplicative. Let $f = \sum_{i=0}^{m} f_i$, $g = \sum_{j=0}^{n} g_j$, $f_i \in S(V)_i$, $g_j \in S(V)_j$ and set $T := \{0, \ldots, m\} \times \{0, \ldots, n\}$. Then by using the definition of $\overline{\rho}$, the fact that $\overline{\rho}_k$ is a seminorm on $S(V)_k$ and Lemma 4.4.2 we obtain

$$\overline{\rho}(f \cdot g) = \overline{\rho}\left(\sum_{i=0}^{m} \sum_{j=0}^{n} f_{i}g_{j}\right) = \overline{\rho}\left(\sum_{k=0}^{m+n} \sum_{\substack{(i,j) \in T \\ i+j=k}} f_{i}g_{j}\right) = \sum_{k=0}^{m+n} \overline{\rho}_{k}\left(\sum_{\substack{(i,j) \in T \\ i+j=k}} f_{i}g_{j}\right)$$

$$\leq \sum_{k=0}^{m+n} \sum_{\substack{(i,j) \in T \\ i+j=k}} \overline{\rho}_{k}(f_{i}g_{j}) \leq \sum_{k=0}^{m+n} \sum_{\substack{(i,j) \in T \\ i+j=k}} \overline{\rho}_{i}(f_{i})\overline{\rho}_{j}(g_{j}) = \sum_{i=0}^{m} \sum_{j=0}^{n} \overline{\rho}_{i}(f_{i})\overline{\rho}_{j}(g_{j})$$

$$= \left(\sum_{i=0}^{m} \overline{\rho}_{i}(f_{i})\right) \left(\sum_{j=0}^{n} \overline{\rho}_{j}(g_{j})\right) = \overline{\rho}(f)\overline{\rho}(g).$$

Let us now consider $(S(V), \overline{\rho})$ and any other submultiplicative seminormed unital commutative \mathbb{K} -algebra (A, σ) . If $\alpha : (S(V), \overline{\rho}) \to (A, \sigma)$ is linear and continuous, then clearly $\alpha \upharpoonright_{V} : (V, \rho) \to (A, \sigma)$ is also continuous. However, if $\psi : (V, \rho) \to (A, \sigma)$ is linear and continuous, then the unique extension $\overline{\psi}$ given by Proposition 4.3.4 need not be continuous. All one can say in general is the following lemma. **Lemma 4.4.3.** If $\psi : (V, \rho) \to (A, \sigma)$ is linear and continuous, namely $\exists C > 0$ such that $\sigma(\psi(v)) \leq C\rho(v) \ \forall v \in V$, then for any $k \in \mathbb{N}$ we have $\sigma(\overline{\psi}(g)) \leq C^k \overline{\rho}_k(g) \ \forall g \in S(V)_k$.

Proof.

Let $k \in \mathbb{N}$ and $g \in S(V)_k$. Suppose $g = \sum_{i=1}^N g_{i1} \cdots g_{ik}$ with $g_{ij} \in V$ for $j = 1, \dots, N$. Then $\overline{\psi}(g) = \sum_{i=1}^N \psi(g_{i1}) \cdots \psi(g_{ik})$, and so

$$\sigma(\overline{\psi}(g)) \le \sigma\left(\sum_{i=1}^{N} \psi(g_{i1}) \cdots \psi(g_{ik})\right) \le \sum_{i=1}^{N} \sigma(\psi(f_{i1})) \cdots \sigma(\psi(g_{ik}))$$
$$\le \sum_{i=1}^{N} C\rho(g_{i1}) \cdots C\rho(g_{ik}) = C^k \sum_{i=1}^{N} \rho(g_{i1}) \cdots \rho(g_{ik}).$$

As this holds for any representation of g, we get $\sigma(\overline{\psi}(g)) \leq C^k \overline{\rho}_k(g)$. \Box

Proposition 4.4.4. If $\psi : (V, \rho) \to (A, \sigma)$ has operator norm ≤ 1 , then the induced algebra homomorphism $\overline{\psi} : (S(V), \overline{\rho}) \to (A, \sigma)$ has operator norm $\leq \sigma(1)$.

Recall that given a linear operator L between two seminormed spaces (W_1, q_1) and (W_2, q_2) we define the operator norm of L as follows:

$$||L|| := \sup_{\substack{w \in W_1 \\ q_1(w) \le 1}} q_2(L(w)).$$

Proof.

Suppose $\sigma \neq 0$ on A (if this is the case then there is nothing to prove). Then there exists $a \in A$ such that $\sigma(a) > 0$. This together with the fact that σ is a submultiplicative seminorm gives that

$$\sigma(1) \ge 1. \tag{4.4}$$

Since $\|\psi\| \leq 1$, we have that $\sigma(\psi(v)) \leq \rho(v), \forall v \in V$. Then we can apply Lemma 4.4.3 and get that

$$\forall k \in \mathbb{N}, \ g \in S(V)_k, \ \sigma(\overline{\psi}(g)) \le \overline{\rho}_k(g) \tag{4.5}$$

Now let $f \in S(V)$, i.e. $f = \sum_{k=0}^{m} f_k$ with $f_k \in S(V)_k$ for $k = 0, \dots, m$. Then

$$\sigma(\overline{\psi}(f)) = \sigma\left(\sum_{k=0}^{m} \overline{\psi}(f_k)\right) \leq \sum_{k=0}^{m} \sigma(\overline{\psi}(f_k)) \leq \sigma(\overline{\psi}(f_0)) + \sum_{k=1}^{m} \overline{\rho}_k(f_k)$$
$$= \sigma(f_0) + \sum_{k=1}^{m} \overline{\rho}_k(f_k) \leq \sigma(1)\overline{\rho}(f_0) + \sum_{k=1}^{m} \overline{\rho}_k(f_k)$$
$$\stackrel{(4.4)}{\leq} \sigma(1)\overline{\rho}(f_0) + \sum_{k=1}^{m} \sigma(1)\overline{\rho}_k(f_k) = \sigma(1)\sum_{k=0}^{m} \overline{\rho}_k(f_k) = \sigma(1)\overline{\rho}(f).$$

Hence, $\|\overline{\psi}\| \leq \sigma(1)$.

Using the properties we have showed for the projective extension $\bar{\rho}$ of ρ to S(V), we can easily pass to the case when V is endowed with a locally convex topology τ (generated by more than one seminorm) and to study how to extend this topology to S(V) in a such a way that the latter becomes an lmc TA.

Let τ be any locally convex topology on a vector space V over \mathbb{K} and let \mathcal{P} be a directed family of seminorms generating τ . Denote by $\overline{\tau}$ the topology on S(V) determined by the family of seminorms $\mathcal{Q} := \{\overline{n\rho} : \rho \in \mathcal{P}, n \in \mathbb{N}\}.$

Proposition 4.4.5. $\overline{\tau}$ is an lmc topology on S(V) extending τ and is the finest lmc topology on S(V) having this property.

Proof. By definition of $\overline{\tau}$ and by Proposition 4.4.1, it is clear that Q is a directed family of submultiplicative seminorms and so that $\overline{\tau}$ is an lmc topology on S(V) extending τ . It remains to show that τ is the finest lmc topology with extending τ to S(V). Let μ an lmc topology on S(V) s.t. $\mu \upharpoonright_{V} = \tau$, i.e. μ extends τ to S(V). Suppose that μ is finer than $\overline{\tau}$. Let S be a directed family of submultiplicative seminorms generating μ and consider the identity map id : $(V, \tau) \to (V, \mu \upharpoonright_{V})$. As by assumption $\mu \upharpoonright_{V} = \tau$, we have that id is continuous and so by Theorem 4.6.3-TVS-I (applied for directed families of seminorms) we get that:

$$\forall s \in \mathcal{S}, \exists n \in \mathbb{N}, \exists \rho \in \mathcal{P} : s(v) = s((id(v)) \le n\rho(v), \forall v \in V.$$

Consider the embedding $i : (V, n\rho) \to (S(V), q)$. Then $||i|| \leq 1$ and so, by Proposition 4.4.4, the unique extension $\overline{i} : (S(V), \overline{n\rho}) \to (S(V), s)$ of i is continuous with $||\overline{i}|| \leq q(1)$. This gives that

$$s(f) \le s(1)\overline{n\rho}(f), \,\forall f \in S(V).$$

Hence, all $s \in \mathcal{F}$ are continuous w.r.t. $\overline{\tau}$ and so μ must be coarser than $\overline{\tau}$. \Box

Chapter 5

Short overview on the moment problem

In this chapter we are going to consider always *Radon measures* on Hausdorff topological spaces, i.e. non-negative Borel measures which are locally finite and inner regular.

5.1 The classical finite-dimensional moment problem

Let μ be a Radon measure on \mathbb{R} . We define the *n*-th moment of μ as

$$m_n^{\mu} := \int_{\mathbb{R}} x^n \mu(dx)$$

If all moments of μ exist and are finite, then we can associate to μ the sequence of real numbers $(m_n^{\mu})_{n \in \mathbb{N}_0}$, which is said to be the *moment sequence* of μ . The moment problem exactly addresses the inverse question:

Problem 5.1.1 (The one-dimensional K-Moment Problem (KMP)). Given a closed subset K of \mathbb{R} and a sequence $m = (m_n)_{n \in \mathbb{N}_0}$ of real numbers, does there exist a Radon measure μ on \mathbb{R} s.t. for any $n \in \mathbb{N}_0$ we have $m_n = m_n^{\mu}$ and μ is supported on K, i.e.

$$m_n = \underbrace{\int_{\mathbb{R}} x^n \mu(dx)}_{n-th \ moment \ of \ \mu}, \ \forall \ n \in \mathbb{N}_0 \quad and \quad supp(\mu) \subseteq K?$$

If such a measure μ does exist we say that μ is a *K*-representing measure for *m* or that *m* is represented by μ on *K*.

Note that there is a bijective correspondence between the set $\mathbb{R}^{\mathbb{N}_0}$ of all sequences of real numbers and the set $(\mathbb{R}[x])^*$ of all linear functional from $\mathbb{R}[x]$ to \mathbb{R} .

In virtue of this correspondence, we can always reformulate the KMP in terms of linear functionals

Problem 5.1.2 (The one-dimensional K-Moment Problem (KMP)). Given a closed subset K of \mathbb{R} and a linear functional $L : \mathbb{R}[x] \to \mathbb{R}$, does there exists a Radon measure μ on \mathbb{R} s.t.

$$L(p) = \int_{\mathbb{R}} p(x)\mu(dx), \, \forall p \in \mathbb{R}[x] \quad and \quad supp(\mu) \subseteq K?$$

As before, if such a measure exists we say that μ is a *K*-representing measure for *L* and that it is a solution to the *K*-moment problem for *L*.

Clearly one can generalize the one-dimensional KMP to higher dimension by considering $\mathbb{R}[\mathbf{x}] := \mathbb{R}[x_1, \ldots, x_d]$ for some $d \in \mathbb{N}$ (see [15, Section 5.2.2]).

Problem 5.1.3 (The *d*-dimensional K-Moment Problem (KMP)). Given a closed subset K of \mathbb{R}^d and a linear functional $L : \mathbb{R}[\mathbf{x}] \to \mathbb{R}$, does there exists a Radon measure μ on \mathbb{R}^d s.t.

$$L(p) = \int_{\mathbb{R}^d} p(\mathbf{x}) \mu(d\mathbf{x}), \, \forall p \in \mathbb{R}[\mathbf{x}] \quad and \quad supp(\mu) \subseteq K?$$

It is then very natural to ask the following:

Questions

- What if we have infinitely many variables, i.e. we consider $\mathbb{R}[x_i : i \in \Omega]$ where Ω is an infinite index set?
- What if instead of real variables we consider variables in a generic \mathbb{R} -vector space V (even infinite dimensional)?
- What if instead of the polynomial ring ℝ[**x**] we take any unital commutative ℝ-algebra A?

All these possible generalization of the moment problem usually go under the name of *infinite dimensional moment problem*.

5.2 Moment problem for commutative \mathbb{R} -algebras

In this section we are going to give a formulation of the moment problem general enough to encompass all the possible generalizations addressed in the previous section. Let us start by introducing some notation and terminology.

Given a unital commutative \mathbb{R} -algebra A, we denote by $\mathcal{X}(A)$ the character space of A (see Definition 2.4.6). For any $a \in A$, we define the Gelfand transform $\hat{a} : \mathcal{X}(A) \to \mathbb{R}$ as $\hat{a}(\alpha) := \alpha(a), \forall \alpha \in \mathcal{X}(A)$. We endow the character space $\mathcal{X}(A)$ with the weakest topology $\tau_{\mathcal{X}(A)}$ s.t. all Gelfand transforms are continuous, i.e. \hat{a} is continuous for all $a \in A$.

Remark 5.2.1. $\mathcal{X}(A)$ can be seen as a subset of \mathbb{R}^A via the embedding:

$$\begin{array}{rccc} \pi: & \mathcal{X}(A) & \to & \mathbb{R}^A \\ & \alpha & \mapsto & \pi(\alpha) := (\alpha(a))_{a \in A} = (\hat{a}(\alpha))_{a \in A} \,. \end{array}$$

If we equip \mathbb{R}^A with the product topology τ_{prod} , then it can be showed (see [19, Section 5.7]) that $\tau_{\mathcal{X}(A)}$ coincides with the topology induced by π on $\mathcal{X}(A)$ from $(\mathbb{R}^A, \tau_{prod})$, i.e.

$$\tau_{\mathcal{X}(A)} \equiv \left\{ \pi^{-1}(O) : O \in \tau_{prod} \right\}.$$

The space $(\mathcal{X}(A), \tau_{\mathcal{X}(A)})$ is therefore Hausdorff.

Problem 5.2.2 (The *KMP* for unital commutative \mathbb{R} -algebras). Given a closed subset $K \subseteq \mathcal{X}(A)$ and a linear functional $L : A \to \mathbb{R}$, does there exist a Radon measure μ on $\mathcal{X}(A)$ s.t. we have

$$L(a) = \int_{\mathcal{X}(A)} \hat{a}(\alpha)\mu(d\alpha), \forall a \in A \quad and \quad supp(\mu) \subseteq K?$$

lc

$$\alpha(p) = \alpha \left(\sum_{\beta \in \mathbb{N}_0^d} p_\beta \mathbf{x}^\beta \right) = \sum_{\beta \in \mathbb{N}_0^d} \alpha(p_\beta) \alpha(x_1)^{\beta_1} \cdots \alpha(x_d)^{\beta_d}$$
$$= \sum_{\beta \in \mathbb{N}_0^d} p_\beta \alpha(x_1)^{\beta_1} \cdots \alpha(x_d)^{\beta_d} = p\left(\alpha(x_1), \dots, \alpha(x_d)\right).$$

Conversely, for any $\mathbf{y} \in \mathbb{R}^d$ we can define the functional $\alpha_{\mathbf{y}} : \mathbb{R}[\mathbf{x}] \to \mathbb{R}$ by $\alpha_{\mathbf{y}}(p) := p(\mathbf{y})$ for any $p \in \mathbb{R}[\mathbf{x}]$, which is clearly a \mathbb{R} -algebra homomorphism. Hence, we have showed that $X(\mathbb{R}[\mathbf{x}]) \cong \mathbb{R}^d$ and via this isomorphism we have that, for any $p \in \mathbb{R}[\mathbf{x}]$, the Gelfand transform \hat{p} is identified with the polynomial p itself. Using these identifications, we get that Problem 5.2.2 for $A = \mathbb{R}[\mathbf{x}]$ is nothing but Problem 5.1.3.

Let us come back to the general KMP 5.2.2. Fixed a subset K of $\mathcal{X}(A)$, we denote by

$$\operatorname{Pos}(K) := \{ a \in A : \hat{a} \ge 0 \text{ on } K \}.$$

A necessary condition for the existence of a solution to the KMP 5.2.2 is clearly that L is nonnegative on Pos(K). In fact, if there exists a K-representing measure μ for L then for all $a \in Pos(K)$ we have

$$L(a) = \int_{\mathcal{X}(A)} \hat{a}(\alpha) \mu(d\alpha) \ge 0$$

since μ is nonnegative and supported on K and \hat{a} is nonnegative on K.

It is then natural to ask if the nonnegativity of L on Pos(K) is also sufficient. For $A = \mathbb{R}[\mathbf{x}]$ a positive answer is provided by the so-called Riesz-Haviland theorem (see [15, Theorem 5.2.5]). An analogous result also holds in this general setting:

Theorem 5.2.3 (Generalized Riesz-Haviland Theorem). Let $K \subseteq \mathcal{X}(A)$ closed and $L : A \to \mathbb{R}$ linear. Suppose there exists $p \in A$ such that $\hat{p} \ge 0$ on Kand for all $n \in \mathbb{N}$ the set $\{\alpha \in K : \hat{p}(\alpha) \le n\}$ is compact. Then L has a K-representing measure if and only if $L(\operatorname{Pos}(K)) \subseteq [0, +\infty)$.

This theorem provides a complete solution for the K- moment problem 5.2.2 but it is somehow unpractical! In fact, it reduces the solvability of the K-moment problem to the problem of characterizing Pos(K). To approach to this problem we will try to approximate elements in Pos(K) with elements of A whose Gelfand transform is "more evidently" non-negative, e.g. sum of even powers of elements of A. In this spirit we consider 2d-power modules of the algebra A for $d \in \mathbb{N}$.

Definition 5.2.4 (2*d*-power module). Let $d \in \mathbb{N}$. A 2*d*-power module of A is a subset M of A satisfying $1 \in M$, $M + M \subseteq M$ and $a^{2d}M \subseteq M$ for each $a \in A$.

In the case d = 1, 2d-power modules are referred to as *quadratic modules*. We denote by $\sum A^{2d}$ the set of all finite sums $\sum a_i^{2d}$, $a_i \in A$. $\sum A^{2d}$ is the smallest 2d-power module of A.

Definition 5.2.5 (Generated 2*d*-power module).

Let $\{p_j\}_{j\in J}$ be an arbitrary subset of elements in A (J can have also infinite cardinality). The 2d-power module of A generated by $\{p_j\}_{j\in J}$ is defined as

$$M := \{\sigma_0 + \sigma_1 p_{j_1} + \ldots + \sigma_s p_{j_s} : s \in \mathbb{N}, j_1, \ldots, j_s \in J, \sigma_0, \ldots, \sigma_s \in \sum A^{2d}\}.$$

For any subset M of A, we set

$$X_M := \{ \alpha \in \mathcal{X}(A) : \hat{a}(\alpha) \ge 0, \ \forall a \in M \},\$$

which is a closed subset of $(\mathcal{X}(A), \tau_{\mathcal{X}(A)})$. If $M = \sum A^{2d}$ then $X_M = \mathcal{X}(A)$. If M is the 2*d*-power module of A generated by $\{p_j\}_{j\in J}$ then $X_M := \{\alpha \in \mathcal{X}(A) : \hat{p}_j(\alpha) \ge 0, \forall j \in J\}$.

Given a 2d-power module M, let us consider the X_M -moment problem for a linear functional $L : A \to \mathbb{R}$. If there exists a X_M -representing measure μ for L, then it is clear that $L(M) \subseteq [0, +\infty)$ since $M \subseteq \text{Pos}(X_M)$. Under which assumptions does the converse hold?

The answer is positive when the module M is Archimedean. The main ingredient of the proof of this result is the the so-called *Jacobi Positivstellensatz*, which holds for Archimedean power modules and provides that $Pos(X_M) \subseteq \overline{M}^{\varphi}$, where φ is the finest locally convex topology on A. This inclusion together with Proposition 2.4.8 allows to get the desired conclusion by applying of Hahn-Banach and Riesz-Haviland theorems.

Theorem 5.2.6. Let M be an archimedean 2d-power module of A and L: $A \to \mathbb{R}$ a linear functional. L has a X_M -representing measure if and only if $L(M) \subseteq [0, +\infty)$.

Proof. See [13, Corollary 2.6]. The conclusion can be also obtained as a consequence of [11, Theorem 5.5]. \Box

A 2d-power module M in A is said to be *archimedean* if for each $a \in A$ there exists an integer N such that $N \pm a \in M$. If M is a 2d-power module of A which is archimedean then X_M is compact. The converse is false in general (see [19, Section 7.3]).

Does Theorem 5.2.6 still hold when M is not Archimedean? Can we find other topologies τ rather than the finest lc topology φ on A such that $\operatorname{Pos}(X_M) \subseteq \overline{M}^{\tau}$ so that we can get a similar result for τ -continuous linear functionals on A? In order to attack those questions we are going to investigate the KMP for linear functionals on some special kind of topological \mathbb{R} -algebras.

5.3 Moment problem for submultiplicative seminormed \mathbb{R} -algebras

In this section we are going to present some results about Problem 5.2.2 when A a submultiplicative seminormed \mathbb{R} -algebra (for more details see [12]).

Let A be a unital commutative \mathbb{R} -algebra and σ be a submultiplicative seminorm on an \mathbb{R} -algebra A, i.e. $\sigma(a \cdot b) \leq \sigma(a)\sigma(b)$ for all $a, b \in A$ (\cdot denotes the multiplication in A). The algebra A together with such a σ is called a submultiplicative seminormed \mathbb{R} -algebra and is denoted by (A, σ) .

We denote the set of all σ -continuous \mathbb{R} -algebra homomorphisms from A to \mathbb{R} by $\mathfrak{sp}(\sigma)$, which we refer to as the *Gelfand spectrum* of (A, σ) , i.e.

$$\mathfrak{sp}(\sigma) := \{ \alpha \in \mathcal{X}(A) : \alpha \text{ is } \sigma - \text{continuous} \}$$

We endow $\mathfrak{sp}(\sigma)$ with the subspace topology induced by $(\mathcal{X}(A), \tau_{\mathcal{X}(A)})$. Then one can show the following two results (see [12] for a proof).

Lemma 5.3.1.

For any submultiplicative seminormed \mathbb{R} -algebra (A, σ) we have:

$$\mathfrak{sp}(\sigma) = \{ \alpha \in \mathcal{X}(A) : |\alpha(a)| \le \sigma(a) \text{ for all } a \in A \}.$$

Corollary 5.3.2. The Gelfand spectrum of any submultiplicative seminormed \mathbb{R} -algebra (A, σ) is compact.

An important closure result useful for the Problem 5.2.2 when A a submultiplicative seminormed \mathbb{R} -algebra was proved by M. Ghasemi, S. Kuhlmann and M. Marshall in [12, Theorem 3.7]. We just state it here but we show in details how this result helps to get better conditions than the ones provided by the Generalized Riesz-Haviland theorem.

Theorem 5.3.3. Let (A, σ) be a submultiplicative seminormed \mathbb{R} -algebra and M is a 2d-power module of A (not necessarily Archimedean). Then

$$\overline{M}^{\rho} = \operatorname{Pos}(X_M \cap \mathfrak{sp}(\sigma)).$$

Corollary 5.3.4. Let (A, σ) be a submultiplicative seminormed \mathbb{R} -algebra, M is a 2d-power module of A and $L : A \to \mathbb{R}$ a linear functional. L has a representing measure supported on $X_M \cap \mathfrak{sp}(\sigma)$ if and only if L is σ -continuous and $L(M) \subseteq [0, +\infty)$.

Proof.

(\Leftarrow) By our hypothesis and Theorem 5.3.3, L is nonnegative on $\text{Pos}(X_M \cap \mathfrak{sp}(\sigma))$. Hence, by applying Theorem 5.2.3, L has a $(X_M \cap \mathfrak{sp}(\sigma))$ -representing measure.¹

¹ Note that we can apply the Generalized Riesz-Haviland Theorem since $X_M \cap \mathfrak{sp}(\sigma)$ is compact in $(\mathcal{X}(A), \tau_{\mathcal{X}(A)})$. (This is a direct consequence of Corollary 5.3.2 and of the fact that X_M is a closed subset of $(\mathcal{X}(A), \tau_{\mathcal{X}(A)})$).

 (\Rightarrow) Suppose that L has a representing measure μ supported on $X_M \cap \mathfrak{sp}(\sigma)$. Then for all $b \in M$ we have

$$L(b) = \int_{X_M \cap \mathfrak{sp}(\sigma)} \hat{b}(\alpha) \mu(d\alpha) \ge 0$$

since μ is nonnegative and supported on a subset of X_M . Therefore, we have got $L(M) \subseteq [0, +\infty)$. Also, we have that for all $a \in A$:

$$\begin{aligned} |L(a)| &\leq \int_{X_M \cap \mathfrak{sp}(\sigma)} |\hat{a}(\alpha)| \mu(d\alpha) \\ &= \int_{X_M \cap \mathfrak{sp}(\sigma)} |\alpha(a)| \mu(d\alpha) \\ &\stackrel{\text{Lemma 5.3.1}}{\leq} \int_{X_M \cap \mathfrak{sp}(\sigma)} \sigma(a) \mu(d\alpha) = \sigma(a) \mu\left(X_M \cap \mathfrak{sp}(\sigma)\right) \end{aligned}$$

Note that $\mu(X_M \cap \mathfrak{sp}(\sigma))$ is finite since μ is Radon and $X_M \cap \mathfrak{sp}(\sigma)$ compact. Hence, L is σ -continuous.

5.4 Moment problem for symmetric algebras of lc spaces

In this section we are going to present some results about Problem 5.2.2 when A is the symmetric algebra S(V) of a locally convex space V over \mathbb{R} (for more details see [14]).

Let us start with the simplest case, i.e. when V is a \mathbb{R} -vector space endowed with a seminorm ρ . In Section 5.2, we have showed how to extend the seminorm ρ to a seminorm $\overline{\rho}$ on S(V), which we proved to be submultiplicative by Proposition 4.4.1. Therefore, $(S(V), \overline{\rho})$ is a submultiplicative seminormed \mathbb{R} -algebra and so we can apply Corollary 5.3.4, obtaining the following result.

Proposition 5.4.1. Let (V, ρ) be a seminormed \mathbb{R} -vector space, M a 2d-power module of S(V) and $L : S(V) \to \mathbb{R}$ a linear functional. L is $\overline{\rho}$ -continuous and $L(M) \subseteq [0, +\infty)$ if and only if $\exists ! \mu$ on V^* : $L(f) = \int_{V^*} \hat{f}(\alpha)\mu(d\alpha)$ and

 $\sup \mu \subseteq X_M \cap \overline{B}_1^{\|\cdot\|_{\rho}}, \text{ where } \|\cdot\|_{\rho} \text{ denotes the operator norm on } V^*, \text{ i.e.} \\ \|\beta\|_{\rho} := \sup_{\substack{v \in V \\ \rho(v) \leq 1}} |\beta(v)| \text{ and } \overline{B}_1^{\|\cdot\|_{\rho}} := \{\beta \in V^* : \|\beta\|_{\rho} \leq 1\}.$

Proof.

By Proposition 4.4.1, we can apply Corollary 5.3.4 to $(S(V), \overline{\rho})$ and obtain that: L is $\overline{\rho}$ -continuous and $L(M) \subseteq [0, +\infty)$ if and only if $\exists ! \mu$ on X(S(V)): $L(f) = \int_{X(S(V))} \hat{f}(\alpha)\mu(d\alpha) \text{ and } \operatorname{supp} \mu \subseteq X_M \cap \mathfrak{sp}(\overline{\rho}). \text{ Now by Corollary 4.3.5 we}$ know that $\operatorname{Hom}(S(V), \mathbb{R}) \cong V^*$, i.e. $X(S(V)) \cong V^*$. Using this isomorphism we can get that $\mathfrak{sp}(\overline{\rho}) \cong \overline{B}_1^{\|\cdot\|_{\rho}}$ and so the desired conclusion.

Let us prove that $\mathfrak{sp}(\overline{\rho}) \cong \overline{B}_1^{\|\cdot\|_{\rho}}$. Suppose that $\alpha \in \mathfrak{sp}(\overline{\rho})$. Then by Lemma 5.3.1 we have that $|\alpha(f)| \leq \overline{\rho}(f) \forall f \in S(V)$. Clearly this implies that $|\alpha(v)| \leq \rho(v) \forall v \in V$, so $\|\alpha \upharpoonright_V \|_{\rho} \leq 1$, i.e. $\alpha \in \overline{B}_1^{\|\cdot\|_{\rho}}$. Conversely, suppose that $\beta \in V^*$ s.t. $\|\beta\|_{\rho} \leq 1$. Denote by $\overline{\beta}$ the unique extension of β to an \mathbb{R} -algebra homomorphism $\overline{\beta} : S(V) \to \mathbb{R}$. Then, by Proposition 4.4.4, we get that $\|\overline{\beta}\|_{\overline{\rho}} \leq 1$ and so that $|\overline{\beta}(f)| \leq \overline{\rho}(f) \forall f \in S(V)$. Thus $\overline{\beta} \in \mathfrak{sp}(\overline{\rho})$. \Box

We can generalize this result to (V, τ) locally convex TVS over \mathbb{R} by using Proposition 4.4.5, which provides an extension of τ to an lmc topology $\overline{\tau}$ to S(V).

Proposition 5.4.2. Let (V, τ) be a lmc TVS over \mathbb{R} whose topology is generated by a directed family of seminorms \mathcal{P} . Let M be a 2d-power module of S(V) and $L : S(V) \to \mathbb{R}$ a linear functional. L is $\overline{\tau}$ -continuous and $L(M) \subseteq [0, +\infty)$ if and only if $\exists ! \mu$ on V^* : $L(f) = \int_{V^*} \hat{f}(\alpha)\mu(d\alpha)$ and $\sup p \mu \subseteq X_M \cap \overline{B}_n^{\|\cdot\|_{\rho}}$, for some $n \in \mathbb{N}$ and $\rho \in \mathcal{P}$.

Proof.

By Proposition 4.4.5, we know that $\overline{\tau}$ is a lmc topology on S(V) generated by the family $\mathcal{Q} := \{\overline{n\rho} : \rho \in \mathcal{P}, n \in \mathbb{N}\}$. Then Proposition 4.6.1 in TVS-I guarantees that L is $\overline{\tau}$ -continuous if and only if there exists $q \in \mathcal{Q}$ s.t. L is q-continuous, i.e. there exists $n \in \mathbb{N}$ and $\rho \in \mathcal{P}$ s.t. L is $\overline{n\rho}$ -continuous. Thus we reduced to the case of one single seminorm and so we can apply Proposition 5.4.1 and get that: L is $\overline{\tau}$ -continuous and $L(M) \subseteq [0, +\infty)$ if and only if $\exists ! \mu$ on V^* : $L(f) = \int_{V^*} \hat{f}(\alpha)\mu(d\alpha)$ and $\operatorname{supp} \mu \subseteq X_M \cap \overline{B}_1^{\|\cdot\|_{n\rho}}$. This yields the conclusion as $\overline{B}_1^{\|\cdot\|_{n\rho}} = \overline{B}_n^{\|\cdot\|_{\rho}}$.

What happens when the assumption of continuity of L is weakened? Can we get results for this moment problem for measures which are not compactly supported? Some results in this direction have been obtained in [17] for the case when $V = C_c^{\infty}(\mathbb{R}^d)$ endowed with the projective topology introduced in Section 1.4. However, there are still many open questions concerning the moment problem in this general framework and we are still far from a complete understanding of the infinite dimensional moment problem.

Bibliography

- [1] R. Arens, The space L^{ω} and convex topological rings, Bull. Amer. Math. Soc. 52, (1946), 931–935.
- V. K. Balachandran, Topological algebras. Reprint of the 1999 original. North-Holland Mathematics Studies, 185. North-Holland Publishing Co., Amsterdam, 2000.
- [3] E. Beckenstein, L. Narici, C. Suffel, Topological algebras. North-Holland Mathematics Studies, Vol. 24. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.
- [4] N. Bourbaki, lments de mathmatique, Thorie des ensembles, Nouvelle dition (rimpression de l'dition Hermann), Masson diffusion C.C.L.S., 1970.
- [5] A. C. Cochran, R. Keown, C. R. Williams, On a class of topological algebras. Pacific J. Math. 34 (1970),17–25.
- [6] I. Gelfand, On normed rings, C. R. (Doklady) Acad. Sci. URSS (N.S.) 25, (1939), 430–432.
- [7] I. Gelfand, To the theory of normed rings. II: On absolutely convergent trigonometrical series and integrals, C. R. (Doklady) Acad. Sci. URSS (N.S.) 25, (1939), 570–572.
- [8] I. Gelfand, To the theory of normed rings. III. On the ring of almost periodic functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) 25, (1939). 573–574.
- [9] I. Gelfand, Normierte Ringe, (German) Rec. Math. [Mat. Sbornik] N. S. 9 (51), (1941), 3–24.
- [10] I. Gelfand, D. Raikov and G. Silov, Commutative normed rings (Chelsea 1964), Russian edition 1960.

- [11] M. Ghasemi, S. Kuhlmann, Closure of the cone of sums of 2d-powers in real topological algebras, J. Funct. Analysis, 264 (2013), 413–427.
- [12] M. Ghasemi, S. Kuhlmann, M. Marshall, Application of Jacobi's representation theorem to locally multiplicatively convex topological ℝalgebras, J. Funct. Analysis, 266, no. 2, (2014), 1041–1049.
- [13] M. Ghasemi, M. Marshall, Sven Wagner, Closure of the cone of sums of 2*d*-powers in certain weighted ℓ_1 -seminorm topologies, *Canad. Math. Bull.* 57 no. 2 (2014), 289–302.
- [14] M. Ghasemi, M. Infusino, S. Kuhlmann, M. Marshall, Moment problem for symmetric algebras of locally convex spaces, Integr. Equ. Oper. Theory (2018) 90:29.
- [15] M. Infusino, Lecture notes on topological vector spaces I, Universität Konstanz, Summer Semester 2017, http://www.math.unikonstanz.de/ infusino/TVS-SS17/Note2017(July29).pdf.
- [16] M. Infusino, Lecture notes on topological vector spaces II, Universität Konstanz, Winter Semester 2017-2017, http://www.math.unikonstanz.de/ infusino/TVS-WS17-18/Note2017-TVS-II.pdf.
- [17] M. Infusino, T. Kuna, A. Rota, The full infinite dimensional moment problem on semi-algebraic sets of generalized functions, J. Funct. Analysis 267 no. 5 (2014),1382–1418.
- [18] A. Mallios, Topological algebras. Selected topics. 109. North-Holland Publishing Co.1986.
- [19] M. Marshall, Positive Polynomials and Sums of Squares, 146, Math. Surveys & Monographs, AMS, 2008.
- [20] M. J. Meyer, Some algebras without submultiplicative norms or positive functionals. Studia Math. 116 (1995), no. 3, 299–302.
- [21] E. Michael, Locally multiplicatively-convex topological algebras, Mem. Amer. Math. Soc., No. 11 (1952), 79 pp.
- [22] M. Reed, B, Simon, Methods of modern mathematical physics. I. Functional analysis. Second edition. Academic Press, Inc., New York, 1980.
- [23] H.H. Schaefer, M. P. Wolff, *Topological vector spaces*, second edition, Graduate Texts in Mathematics, 3.
- [24] F. Tréves, Topological vector spaces, distributions, and kernels, Academic Press, 1967.
- [25] W. Zelazko, Selected topics in topological algebras. Lecture Notes Series, No. 31. Matematisk Institut, Aarhus Universitet, Aarhus, 1971.