A combinatorial expression for the moment sequence in $\mathbb{R}^2$ via Fibonacci sequence

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AIM

An alternate approach to the truncated moment problems on the real 2D plane by bound to the Fibonacci sequence

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- In the selfsame spirit that it was established by Ben Taher-Rachidi et al in "Bull . London Math. Soc. 33 (2001) 425-432", the connection between the 1 dimensional truncated moment problem and Fibonacci sequence, we provide a closed link between the real 2 dimensional truncated moment problem and the bi-indexed Fibonacci sequence.

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- Using The combinatorial expression of generalized Fibonacci sequences as tool to establish a combinatorial expression both for each term of the associated moment matrix. And so yields the terms of the extension of the truncated moment problem in $\mathbb{R}^2$ to the full moment problem.

- Introduce the notion of Fibonacci sequences on the measures, that leads to arise a characterisation of full momemt problem in $\mathbb{R}^2$ admitting a finitely atomic measure.

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  Let μ be a measure on $\mathbb{R}^d$ having convergent moments up to at least degree n. Then there exists a quadrature rule for μ of degree $n-1$ with size $\leq 1 + N_{n-1,d;\mu}$, $(N_{n,d;\mu} := dim\{P|supp\mu : p \in \mathbb{R}_{n,d}[t]\})$.

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- By Bayer-Teichmann ,
  Whether a positive measure μ solution of the truncated multivariable moment problem is found, then μ is a finitely-atomic representing measure. And such measure may be presented as the sum $\mu = \sum_{k=1}^{d} \rho_k \delta_{x_k}$, where $1 \leq d < +\infty$, $\rho_k > 0$ for $k = 1, \ldots, d$, and δ_{x_k} is the point mass at $x_k \in \mathbb{R}^N$.

plan

In what follows in this talk , we apply the general setting provided by Bayer-Teichmann in the case of $N = 2$

Let $\beta \equiv \beta^{(2d)} \equiv \{\beta_{ij}\}_{\{(i,j)\in\mathbb{Z}_+^2,\ \ i+j\leq 2d\}}$, be a 2-dimensional real multisequence. Let $K \subset \mathbb{R}^2$ be a closed subset, the K-moment problem (KMP for short) for the sequence β consists of finding a positive Borel measure μ on $\mathbb{R}^2$ such that, In what follows in this talk , we apply the general setting provided by Bayer-Teichmann in the case of $N = 2$

Let $\beta \equiv \beta^{(2d)} \equiv \{\beta_{ij}\}_{\{(i,j)\in\mathbb{Z}_+^2,\ \ i+j\leq 2d\}}$, be a 2-dimensional real multisequence. Let $K \subset \mathbb{R}^2$ be a closed subset, the K-moment problem (KMP for short) for the sequence β consists of finding a positive Borel measure μ on $\mathbb{R}^2$ such that,

$$\beta_{ij} = \int_{\mathbb{R}^2} x^i y^j d\mu(x,y), \ \ (0 \leq i+j \leq 2d) \text{ with } supp(\mu) \subset K. \tag{1.1}$$

A measure satisfying (1.1) is said a representing (or *K*-representing) measure for the sequence $\beta \equiv \beta^{(2d)}$.

- if $d = +\infty$, The *K*- moment problem (1.1) is called *full momemt*.

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- if $d < +\infty$, The K- moment problem (1.1) is called *truncated moment*

Associated with β is a moment matrix $\mathcal{M}_d \equiv \mathcal{M}_d(\beta)$, defined by $M_d = (B[i,j])_{0 \leq i,j \leq d}$, where

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$$B[i,j] = \begin{pmatrix} \beta_{i+j,0} & \cdots & \beta_{i,j} \\ \vdots & \vdots & \vdots \\ \beta_{j,i} & \cdots & \beta_{0,i+j} \end{pmatrix}.$$

It follows from Bayer-Teichmann-2006 that β admits a finitely-atomic representing measure μ, which therefore has finite moments of all orders.
As a consequence, $\mathcal{M}_d$ admits a positive recursively generated moment matrix extensions of all orders, namely $\mathcal{M}_{d+1}[\mu], \cdots$, $\mathcal{M}_{d+k}[\mu], \cdots$.

The full moment problem on $\mathbb{R}^2$ is more exploited in the literature, our main idea here is to give an approach by formulating the crucial bridge between the truncated moment problem in $\mathbb{R}^2$ and the linear generalized Fibonacci sequence.

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Way to get around the study to the full moment problem

To define the notion of the *bi-indexed Fibonacci sequence*. For reason of clarity, let first recall some basic notions of the 1 dimensional case , that means the generalized Fibonacci sequences. I provide some needed properties about these sequences, notably The Analytic formula and the Combinatorial expression. These two formulas are of great importance for giving off the new combinatorial expression of the entries of the associated moment matrix.

plan

1. Introduction
   - The K- moment problem
   - analytic formula and Combinatorial expression of generalized Fibonacci sequences
   - The bi-indexed Fibonacci sequence
2. A combinatorial expression for the variable in $\mathbb{R}^2$ moment sequence via Fibonacci sequence
3. Application to the quartic moment problem

Let consider the sequence $\{V_n\}_{n\geq 0}$ defined by $V_n = \alpha_n$ for $0 \leq n \leq r-1$ and the linear recurrence relation of order r, Let consider the sequence $\{V_n\}_{n\geq 0}$ defined by $V_n = \alpha_n$ for $0 \leq n \leq r-1$ and the linear recurrence relation of order r,

$$V_{n+1} = a_0 V_n + \cdots + a_{r-1} V_{n-r+1}, \quad n \geq r, \qquad (1.2)$$

is called a *r-generalized Fibonacci sequence*.
$a_0, a_1, \cdots, a_{r-1}$ and $\alpha_0, \alpha_1, \cdots, \alpha_{r-1}$: the *coefficients* and the *initial conditions*

the polynomial $P(X) = X^r - a_0 X^{r-1} - \cdots - a_{r-1}$ is called the characterisic polynomial $\{V_n\}_{n\geq 0}$, and its roots are called the *characteristic roots* of the sequence

The Analytic *formula* of $\{V_n\}_{n\geq 0}$

Set $\{\lambda_i\}_{1\leq i\leq s}$ the characteristic roots of multiplicity m_i respectively.

$$V_n = \sum_{i=1}^{s}\left(\sum_{j=0}^{m_i-1}\gamma_{i,j}n^j\right)\lambda_i^n, \quad (n \geq 0) \tag{1.3}$$

We can determine $\gamma_{i,j}$ by solving the genaralized Vandemond system of r linear equations

$$\sum_{i=1}^{s}\left(\sum_{j=0}^{m_i-1}\beta_{i,j}n^j\right)\lambda_i^n = \alpha_n, \quad n = 0, 1, \cdots, r-1.$$

The Combinatorial expression of $\{V_n\}_{n\geq 0}$

For $n \geq r$,

$$V_n = \rho(n,r)w_0 + \rho(n-1,r)w_1 + \cdots + \rho(n-r+1,r)w_{r-1}, \quad (1.4)$$

where $w_s = a_{r-1}v_s + \cdots + a_s v_{r-1}$ for $s = 0, 1, \cdots, r-1$ and
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$$\rho(n,r) = \sum_{k_0+2k_1+\ldots+rk_{r-1}=n-r} \frac{(k_0+\ldots+k_{r-1})!}{k_0!\ldots k_{r-1}!} a_0^{k_0}\ldots a_{r-1}^{k_{r-1}}. \quad (1.5)$$

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$$\begin{cases} \beta_{(k_1+1,k_2)} = \sum_{i=0}^{r-1} a_i\beta_{(k_1-i,k_2)} \\ \beta_{(k_1,k_2+1)} = \sum_{j=0}^{s-1} b_j\beta_{(k_1,k_2-j)}, \end{cases} \quad (1.6)$$

where $k_1 \geq r-1$ $(r \geq 2)$ and $k_2 \geq s-1$ $(s \geq 2)$.

- For a fixed $j \in \mathbb{Z}_+$, the sequences $\{\beta_{i,j}\}_{i\in\mathbb{Z}_+}$ are the generalized Fibonacci sequences of order r, of characteristic polynomial $P(x) = x^r - a_0x^{r-1} - \cdots - a_{r-1}$.

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- for a fixed $i \in \mathbb{Z}_+$, sequences $\{\beta_{i,j}\}_{j \in \mathbb{Z}_+}$ are also generalized Fibonacci sequences of order s of characteristic polynomial $Q(y) = y^s - b_0 y^{s-1} - \cdots - b_{s-1}$.

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- for a fixed $i \in \mathbb{Z}_+$, sequences $\{\beta_{i,j}\}_{j\in\mathbb{Z}_+}$ are also generalized Fibonacci sequences of order s of characteristic polynomial $Q(y) = y^s - b_0y^{s-1} - \cdots - b_{s-1}$.
- P and Q will be called the (characteristic) polynomials associated to the *bi-indexed Fibonacci sequence* $\{\beta_{i,j}\}_{(i,j)\in\mathbb{Z}_+^2}$.

Now, given $\{\beta_{i,j}\}_{(i,j)\in\mathbb{Z}_+^2}$ a bi-indexed (r, s) generalized Fibonacci sequence , such that its associated characteristic polynomials P and Q admit distinct roots $x_1, \cdots, x_r$ and $y_1, \cdots, y_s$ (respectively).

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$$\mu = \sum_{1\leq i\leq r,\ 1\leq j\leq s} \rho_{i,j}\delta_{(x_i,y_j)},$$

where the coefficients $\rho_{i,j}$ are solution of the following system of $r\times s$ equations, Now, given $\{\beta_{i,j}\}_{(i,j)\in\mathbb{Z}_+^2}$ a bi-indexed (r, s) generalized Fibonacci sequence , such that its associated characteristic polynomials P and Q admit distinct roots $x_1, \cdots, x_r$ and $y_1, \cdots, y_s$ (respectively).

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where the coefficients $\rho_{i,j}$ are solution of the following system of $r \times s$ equations,

$$\sum_{1\leq i\leq r, 1\leq j\leq s} \rho_{i,j}x_i^n y_j^m = \beta_{n,m}, \quad 0 \leq n \leq r-1, \quad 0 \leq m \leq s-1. \tag{2.1}$$

The determinant of this system (of Vandermonde type) is nonzero ($\prod_{1\leq i<j\leq r}(x_i - x_j) \neq 0, \prod_{1\leq i<j\leq s}(y_i - y_j) \neq 0$).

We observe that the The powers X^n and Y^m, columns of moment matrix $M_d(\beta)$ satisfy the r-th and s-th linear relations respectively :

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We observe that the The powers X^n and Y^m, columns of moment matrix $M_d(\beta)$ satisfy the r-th and s-th linear relations respectively :

$$\begin{cases} X^{n+1} = a_0 X^n + \cdots + a_{r-1} X^{n-r+1}, \quad for \quad n \geq r-1 \\ and \\ Y^{m+1} = b_0 Y^m + \cdots + b_{s-1} Y^{m-s+1}, \quad for \quad m \geq s-1. \end{cases}$$

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$$\begin{cases} X^n = \sum_{k=0}^{r-1}\left(\sum_{i=0}^{k} a_{r-k+i-1}\rho(n-i,r)\right)X^k, \\ Y^m = \sum_{\ell=0}^{s-1}\left(\sum_{j=0}^{\ell} b_{s-\ell+j-1}\phi(m-j,s)\right)Y^\ell, \end{cases} \quad (2.2)$$

for any $n \geq r$ and $m \geq s$, where

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for any $n \geq r$ and $m \geq s$, where

$$\begin{cases} \rho(n,r) = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n-r} \frac{(k_0+\cdots+k_{r-1})!}{k_0!k_1!\ldots k_{r-1}!} a_0^{k_0} a_1^{k_1} \ldots a_{r-1}^{k_r-1}, \\ and \\ \phi(m,s) = \sum_{\ell_0+2\ell_1+\cdots+s\ell_{s-1}=m-s} \frac{(\ell_0+\cdots+\ell_{s-1})!}{\ell_0!\ell_1!\ldots\ell_{s-1}!} b_0^{\ell_0} b_1^{\ell_1} \ldots b_{s-1}^{\ell_s-1}. \end{cases}$$

Since $\{\rho(n,r)\}_{n\geq 0}$ and $\{\phi(m,s)\}_{n\geq 0}$ are Fibonacci sequences of characteristic polynomials P and Q respectively, Ben Taher-Rachidi (2002,LAA), using the Analytic formula, showed that for every $n \geq r+1$ and $m \geq s+1$ we have, Since $\{\rho(n,r)\}_{n\geq 0}$ and $\{\phi(m,s)\}_{n\geq 0}$ are Fibonacci sequences of characteristic polynomials P and Q respectively, Ben Taher-Rachidi (2002,LAA), using the Analytic formula, showed that for every $n \geq r+1$ and $m \geq s+1$ we have,

$$\rho(n,r) = \sum_{i=1}^{r} \frac{x_i^{n-1}}{P'(x_i)}, \quad \phi(m,s) = \sum_{j=1}^{s} \frac{y_j^{m-1}}{Q'(y_j)}$$

.

Therefore, we show that every column of the form X^nY^m with $n \geq r$ or/and $m \geq s$), may be expressed, in terms of X^kY^ℓ with $k \leq r-1$ and $\ell \leq s-1$. We have the following three situations,

- If $n \leq r-1$ and $m \geq s$,

$$X^nY^m = \sum_{\ell=0}^{s-1}\left(\sum_{j=0}^{\ell} b_{s-\ell+j-1}\phi(m-j,s)\right)X^nY^\ell,$$

- If $n \geq r$ and $m \leq s-1$,

$$X^nY^m = \sum_{k=0}^{r-1}\left(\sum_{i=0}^{k} a_{r-k+i-1}\rho(n-i,r)\right)X^kY^m,$$

- If $n \geq r$ and $m \geq s$,

$$X^nY^m = \sum_{k=0}^{r-1}\sum_{\ell=0}^{s-1} \Delta_{k,\ell} X^k Y^\ell,$$

where

$$\Delta_{k,\ell} = \sum_{i=0}^{k}\sum_{j=0}^{\ell} a_{r-k+i-1} b_{s-\ell+j-1} \rho(n-i, r)\phi(m-j, s)$$

.

Since $\beta_{n,m}$ $(n, m \geq 0)$ is the entry in the row X^n and the column Y^m, based on the three previous cases, we give the explicit expression of the $\beta_{n,m}$, for every n and m, in terms of the $\{\beta_{k,l}\}_{0\leq k\leq r-1, 0\leq l\leq s-1}$.

Theorem

Let $\{\beta_{i,j}\}_{(i,j)\in\mathbb{Z}_+^2}$ be a bi-indexed generalized Fibonacci sequence of order (r,s), such that its associated characteristic polynomials P and Q admit distinct roots. Then, we have

$$\begin{cases} 1)\beta_{nm} = \langle X^n, Y^m\rangle = \sum_{l=0}^{s-1}\left(\sum_{j=0}^{l} b_{s-l+j-1}\phi(m-j,s)\right)\beta_{nl}, \\ \text{for} \;\; n \leq r-1 \;\; \text{and} \;\; m \geq s \\ 2)\beta_{nm} = \langle X^n, Y^m\rangle = \sum_{k=0}^{r-1}\left(\sum_{i=0}^{k} a_{r-k+i-1}\rho(n-i,r)\right)\beta_{km}, \\ \text{for} \;\; n \geq r \;\; \text{and} \;\; m \leq s-1 \\ 3)\beta_{nm} = \langle X^n, Y^m\rangle = \sum_{k=0}^{r-1}\sum_{l=0}^{s-1}\Delta_{k,l}\beta_{kl}, \\ \text{for} \;\; n \geq r \;\; \text{and} \;\; m \geq s, \end{cases} \tag{2.3}$$

where $\Delta_{k,l} = \sum_{i=0}^{k}\sum_{j=0}^{l} a_{r-k+i-1}b_{s-l+j-1}\rho(n-i,r)\phi(m-j,s)$, $\rho(n,r)$ and $\phi(m,s)$ are given as above.

Return to the interpolating measure μ for the sequence β,
$\mu = \sum_{1\leq i\leq r,\ 1\leq j\leq s} \rho_{i,j}\delta_{(x_i,y_j)}$.

Find the positivity of $\rho_{i,j}$.

Consider the moment matrix $\mathcal{M}_{r+s-2}(\beta)$ associated to the sequence β. $\mathcal{M}_{r+s-1}(\beta)$ is a matrix extension of $\mathcal{M}_{r+s-2}(\beta)$ with the same *rank*. Indeed,

$$\mathcal{M}_{r+s-1}(\beta) = \left[\begin{array}{c|c} \mathcal{M}_{r+s-2} & B(r+s-1) \\ \hline B(r+s-1)^T & C(r+s-1) \end{array}\right],$$

where $B(r+s-1) = (B[i, r+s-1])_{0\leq i\leq r+s-2}$ and $C = B[r+s-1, r+s-1]$. Clearly, the columns of $B(r+s-1)$ will be denoted by $X^{r+s-1}, \cdots, Y^{r+s-1}$.

We remark that,

- The columns appearing in $B(r+s-1)$, may be expressed in terms of X^aY^b, with $a, b \in \mathbb{N}$ and $a+b \leq r+s-2$ (see the three cases above)
- The entries of $\mathcal{M}_{r+s-1}(\beta)$ are expressed in terms of those of $M_{r+s-2}(\beta)$.

For every $\mathcal{M}_{r+s-2+k}(\beta)$ (with $k \in \mathbb{N}^*$) extension of $\mathcal{M}_{r+s-2}(\beta)$, the columns $X^a Y^b$ (with $a, b \in \mathbb{N}$ and $a + b \geq r + s - 1$) are completely expressed in terms of the columns $1, X, Y, \cdots, X^{r+s-2}, \cdots, Y^{r+s-2}$.
To sum up for all $k \geq 1$, we have

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To sum up for all $k \geq 1$, we have

$$rank[\mathcal{M}_{r+s-2}(\beta)] = rank[\mathcal{M}_{r+s-2+k}(\beta)]$$

.

Using

Theorem

(Smulj'an, 1959)

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \Leftrightarrow \begin{cases} A \geq 0 \\ AW = B \\ C \geq W^* AW \end{cases}$$

Moreover, $rank \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = rank(A) \Leftrightarrow C = W^* AW$

Furthermore, if $rank \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = rank(A)$. Then

$$A \geq 0 \Leftrightarrow \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0$$

Also, we have recourse to

Lemma

(Curto-Fialkow-1996)

Let $\gamma \equiv \gamma^{(2d)}$ be a complex sequence, admitting an r-atomic interpolating measure ν, with $r \leq k+1$. If $M(k)(\gamma) \geq 0$, then $\nu \geq 0$.

We manage to have,

Theorem

Let $\{\beta_{i,j}\}_{(i,j)\in\mathbb{Z}_+^2}$ be a bi-indexed (r,s) generalized Fibonacci sequence , such that its associated characteristic polynomials P and Q admit distinct roots. If $\mathcal{M}_{r+s-2}(\beta) \geq 0$, then the full moment sequence $\beta \equiv \{\beta_{i,j}\}_{(i,j)\in\mathbb{Z}_+^2}$ has a representing measure and rank$[\mathcal{M}_{r+s-2}(\beta)] \leq \frac{(r+s-1)(r+s)}{2} - (r+s-2)$.

Remark

- Curto-Fialkow (2013) showed that the matricial positivity condition is not sufficient. Indeed, by modifying an example of K.Schmüdgen, they built a family $\beta_{00}, \beta_{01}, \beta_{10}, \cdots, \beta_{06}, \cdots, \beta_{60}$ with positive invertible moment matrix $M(3)$ but no representation measure.

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- The data of bi-indexed (r, s) generalized Fibonacci sequence makes the positivity condition sufficient.

Example

Let $\beta = \{\beta_{ij}\}_{i,j\geq 0}$ *be a bi-indexed generalized Fibonacci sequence, whose associated characteristic polynomials are* $Q(y) = y^2 - 4y + 3$ *and* $P(x) = x^3 - 2x^2 - x + 2$*. The initial data* $\{\beta_{00}, \beta_{01}, \beta_{10}, \beta_{11}, \beta_{20}, \beta_{21}\}$ *are sufficient for the determination of the explicit expression of the terms* β_{ij}*. A direct calculation leads to have* $\rho(n,3) = \frac{-1}{2} + \frac{(-1)^{n-1}}{6} + 2^{n-2}$ *for* $n \geq 3$ *and* $\phi(m,2) = \frac{-1}{2} + \frac{3^{m-1}}{2}$*, for* $m \geq 2$*.*

Assume that
$\beta_{00} = 1, \beta_{01} = 0, \beta_{10} = 0, \beta_{11} = 1, \beta_{20} = 1, \beta_{21} = 0$. Thence, we obtain,

1) For $m \geq 2$,

$$\beta_{0,m} = \frac{3 - 3^m}{2}\ , \beta_{1,m} = \frac{-1 + 3^m}{2} \text{ and } \beta_{2,m} = \frac{3 - 3^m}{2}$$

2) For $n \geq 3$,

$$\beta_{n,0} = \frac{1}{2} + \frac{(-1)^{n-2}}{2} \text{ and } \beta_{n,1} = \frac{1}{2} + \frac{(-1)^{n-1}}{2}$$

3) For $n \geq 3$ and $m \geq 2$,

$$\beta_{n,m} = \left(\frac{-3}{2} + 3\frac{(-1)^{n-1}}{2}\right)\left(\frac{-1}{2} + \frac{3^{m-1}}{2}\right) + \left(\frac{1}{2} + \frac{(-1)^{n-1}}{2}\right)\left(\frac{-1}{2} + \frac{3^{m}}{2}\right).$$

$$\mathcal{M}_3(\beta) =$$

$$\begin{pmatrix} \mathbf{1} & X & Y & X^2 & XY & Y^2 & X^3 & X^2Y & XY^2 & Y^3 \\ 1 & 0 & 0 & 1 & 1 & -3 & 0 & 0 & 4 & -12 \\ 0 & 1 & 1 & 0 & 0 & 4 & 1 & 1 & -3 & 13 \\ 0 & 1 & -3 & 0 & 4 & -12 & 1 & -3 & 13 & -39 \\ 1 & 0 & 0 & 1 & 1 & -3 & 0 & 0 & 4 & -12 \\ 1 & 0 & 4 & 1 & -3 & 13 & 0 & 4 & -12 & 40 \\ -3 & 4 & -12 & -3 & 13 & -39 & 4 & -12 & 40 & -120 \\ 0 & 1 & 1 & 0 & 0 & 4 & 1 & 1 & -3 & 13 \\ 0 & 1 & -3 & 0 & 4 & -12 & 1 & -3 & 13 & -39 \\ 4 & -3 & 13 & 4 & -12 & 40 & -3 & 13 & -39 & 121 \\ -12 & 13 & -39 & -12 & 40 & -120 & 13 & -39 & 121 & -363 \end{pmatrix}.$$

A calculation via FreeMat permits to differ that $rank(\mathcal{M}_3(\beta)) = 3$ and its eigenvalues are -456.9186, -0.0000, -0.0000, -0.0000, -0.0000, 0.0000, 0.0000, 0.0000, 4.5121, 6.4065. Thereby, the sequence $\beta = \{\beta_{ij}\}_{i,j\geq 0}$ does not admit a representing measure.

A characterisation of full moment problem admitting a finitely measure

Now , we state a result formulating a a necessary and sufficient condition satisfying by the two sequences of measures $\{\mu_n\}_{n\geq 0}$ and $\{\nu_m\}_{m\geq 0}$, where μ_n is the representing measure of the subsequence $\{\beta_{n,j}\}_{j\geq 0}$ and ν_m is the representing measure of the subsequence $\{\beta_{i,m}\}_{i\geq 0}$, for getting $\beta \equiv \{\beta_{i,j}\}_{i,j\geq 0}$. Thus, we have the following property.

Proposition

Let $\{\beta_{i,j}\}_{i,\, j\geq 0}$ *be a 2-dimensional real multisequence. Then, the two following affirmations are equivalent,*

1. $\{\beta_{i,j}\}_{i,j\geq 0}$ *has a finitely atomic representing measure* μ *.*
2. *Each of the two sequences of measures* $\{\mu_n\}_{n\geq 0}$ *and* $\{\nu_m\}_{m\geq 0}$*, representing measures of the two sequences* $\{\beta_{n,j}\}_{j\geq 0}$ *and* $\{\beta_{i,m}\}_{i\geq 0}$*, for fixed n, m (respectively), is a Fibonacci sequence of order r , s (respectively) such that the roots of their characteristic polynomial are distinct . In addition, we have* $\mathcal{M}_{r+s-2}(\beta)\geq 0$*.*

We observe that for i fixed, we get $\beta_{i,m} = \int x^i d\nu_m$, and for j fixed, $\beta_{n,j} = \int y^j d\mu_n$. we obtain, For each fixed n,m (respectively) ; the sequences $\{\beta_{n,j}\}_{j\geq 0}$ and $\{\beta_{i,m}\}_{i\geq 0}$(respectively) are generalized Fibonacci sequences admitting the same characteristic polynomial of order r, s (respectively)

Let $\beta = \{\beta_{i,j}\}_{i,j\geq 0}$ be a bi-indexed generalized Fibonacci sequence, such that the characteristic polynomial $Q(y)$ associated to the family of sequences $\{\beta_{i,j}\}_{j\geq 0}$ ($i \geq 0$ fixed) owns two distinct roots y_1, y_2 in $\mathbb{R}$ i.e $Q(y) = (y - y_1)(y - y_2) = y^2 - b_0 y - b_1$. And assume that $P(x)$, the characteristic polynomial associated to the family of sequences $\{\beta_{i,j}\}_{i\geq 0}$ ($j \geq 0$ fixed), owns also two distinct roots x_1, x_2 in $\mathbb{R}$ i.e $P(x) = (x - x_1)(x - x_2) = x^2 - a_0 x - a_1$.

We consider the moment matrix $\mathcal{M}_2(\beta)$

$$\mathcal{M}_2(\beta) = \begin{bmatrix} \beta_{00} & \beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} \\ \beta_{10} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} \\ \beta_{01} & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} \\ \beta_{20} & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} \\ \beta_{11} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} \\ \beta_{02} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} \end{bmatrix}.$$

We may rewrite $\mathcal{M}_2(\beta)$ in terms of $\mathcal{M}_1(\beta)$ and the blocs $B(2)$ and $C(2)$, we have

$$\mathcal{M}_2 = \left[\begin{array}{c|c} \mathcal{M}_1(\beta) & B(2) \\ \hline B(2)^T & C(2) \end{array}\right],$$

where the matrices $\mathcal{M}_1(\beta)$, $B(2)$ and $C(2)$ are given by,

$$\mathcal{M}_1(\beta) = \begin{bmatrix} \beta_{00} & \beta_{10} & \beta_{01} \\ \beta_{10} & \beta_{20} & \beta_{11} \\ \beta_{01} & \beta_{11} & \beta_{02} \end{bmatrix}, B(2) = \begin{bmatrix} \beta_{20} & \beta_{11} & \beta_{02} \\ \beta_{30} & \beta_{21} & \beta_{12} \\ \beta_{21} & \beta_{12} & \beta_{03} \end{bmatrix},$$

$$C(2) = \begin{bmatrix} \beta_{40} & \beta_{30} & \beta_{21} \\ \beta_{31} & \beta_{22} & \beta_{13} \\ \beta_{22} & \beta_{13} & \beta_{04} \end{bmatrix}.$$

Proposition

Assume that the family of columns $\{\mathbf{1}, X, Y, XY\}$ *is linearly independent. Then, we have rank*$[\mathcal{M}_2(\beta)] = 4$, *and* $\mathcal{M}_2(\beta) \geq 0$ *if and only if* $\beta = \{\beta_{i,j}\}_{i,j\geq 0}$ *admits a representing measure owning exactly* 4 *atoms, with*

$$supp(\mu) = \{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)\},$$

where x_1, x_2 *and* y_1, y_2 *are (respectively) the roots of the characteristic polynomials* P, Q *of the bi-indexed generalized Fibonacci sequence* $\beta = \{\beta_{i,j}\}_{i,j\geq 0}$.

Proposition

Assume that the family of columns $\{\mathbf{1}, X, Y, XY\}$ *is linearly dependent. Then, we have* $\mathcal{M}_1(\beta) > 0 \iff \beta = \{\beta_{i,j}\}_{i,j\geq 0}$ *admits a finitely atomic representing measure.*

Proposition

Assume that the family of columns $\{1, X, Y, XY\}$ *is linearly dependent and* $\mathcal{M}_1(\beta) \geq 0$*. Then, we have*

1. $rank[\mathcal{M}_1(\beta)] = 3 \iff \beta = \{\beta_{i,j}\}_{i,j\geq 0}$ *is a full moment sequence satisfying the compatibility condition*
$$\beta_{22} = \begin{pmatrix} \beta_{11} \\ \beta_{21} \\ \beta_{12} \end{pmatrix}^T \mathcal{M}_1(\beta)^{-1} \begin{pmatrix} \beta_{11} \\ \beta_{21} \\ \beta_{12} \end{pmatrix}.$$
2. $rank[\mathcal{M}_1(\beta)] = 2$*, put* $Y = \alpha 1 + \lambda X$*. Then, the sequence* $\beta = \{\beta_{i,j}\}_{i,j\geq 0}$ *is a full moment sequence satisfying* $rank[\mathcal{M}_1(\beta)] = rank[\mathcal{M}_2(\beta)]$ *if and only if* $\beta_{12} = (\alpha + \lambda a_0)\beta_{11} + \lambda a_1 \beta_{01}$.
3. $rank[\mathcal{M}_1(\beta)] = 1 \iff \beta = \{\beta_{i,j}\}_{i,j\geq 0}$ *is a full moment sequence satisfying the following rank condition* $rank[\mathcal{M}_2(\beta)] = 1$

Open question

- When we take the general setting provided by Bayer-Teichmann in the case $N > 2$, in the selfsame spirit, which linear recurrence relations of type Fibonacci can be considered ?.
- Why of Fibonacci type ?
- The analytic Formula and the combinatorial expression of the terms of GFS can make several problems easier to handle.

References I

R. Ben Taher and M. Rachidi, *Truncated moment problems in $\mathbb{R}^2$ and recursiveness*, Operators and Matrices, Volume 11, Number 4 (2017), pp. 953-968.

R. Ben Taher, M. Rachidi and E.H. Zerouali, *Recursive subnormal completion and the truncated moment problem* , Bull. London Math. Soc. 33 (2001) 425-432.

R. Ben Taher and M. Rachidi, *Some explicit formulas for the polynomial decomposition of the matrix exponential and applications,Linear Algebra and Its Applications.* 350(1-3) (2002), p. 171-184.

References II

- R. E. Curto and L. A. Fialkow, *Recursiveness, positivity and truncated moment problem*, *Houston J. Math.* 17 no 4 (1991), pp. 603-635.
- R. Curto and L. Fialkow, *Recursively generated weighted shifts and the subnormal completion problem*, Integral Equations and Operator Theory 17 (1993), pp. 202-246.
- R. Curto and L. Fialkow, *Solution of the truncated complex moment problem for flat data*, Mem. Amer. Math. Soc. 119 (1996), number 568.
- R. Curto and L. Fialkow, *Recursively determined representing measures for bivariate truncated moment sequences*, J. Operator Th. 70(2013), 401-436.

References III

J. A. Shohat and J. D. Tomakin, *The moment problems*, Amer. Math. Soc. Surveys, **2**, Amercican Mathematical Society.

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