The core variety of a multisequence in the truncated moment problem

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The Classical Truncated K-Moment Problem (TKMP)

Given an n-dimensional multisequence of degree m,

$$\beta \equiv \beta^{(m)} = \{\beta_i : i \in \mathbb{Z}_+^n,\ |i| \leq m\},$$

and a closed set $K \subseteq \mathbb{R}^n$, TKMP seeks conditions on β such that there exists a positive Borel measure μ on $\mathbb{R}^n$, with *supp* $\mu \subseteq K$, satisfying

$$\beta_i = \int_{\mathbb{R}^n} x^i d\mu(x)\ (|i| \leq m)$$

$(x \equiv (x_1, \ldots, x_n),\ i \equiv (i_1, \ldots, i_n) \in \mathbb{Z}_+^n,\ x^i := x_1^{i_1} \cdots x_n^{i_n}).$

μ is a *K-representing measure*. For $K = \mathbb{R}^n$, we have the *Truncated Moment Problem (TMP)* and μ is a *representing measure*.

Positivity of the Riesz functional

Let $\mathcal{P} := \mathbb{R}[x_1, \ldots, x_n]$ and let $\mathcal{P}_k := \{p \in \mathcal{P} : deg\ \ p \leq k\}$. Given $\beta \equiv \beta^{(m)}$, define the *Riesz functional* $L_\beta : \mathcal{P}_m \longrightarrow \mathbb{R}$ by

$$\sum a_i x^i \longmapsto L_\beta(\sum a_i x^i) := \sum a_i \beta_i$$

If β has a K-representing measure μ, then L_β is *K-positive*, i.e.,

$$p \in \mathcal{P}_m,\ \ p|K \geq 0 \Longrightarrow L_\beta(p) \geq 0\ .$$

(Proof: $L_\beta(p) = \int_K p d\mu \geq 0$.)

Solution to Classical Full K-Moment Problem

We first briefly review the classical Full Multivariable K-Moment Problem for $\beta \equiv \beta^{(\infty)}$.

Theorem [M. Riesz ($n = 1$) [1923], Haviland ($n > 1$) [1935]]

$\beta \equiv \beta^{(\infty)}$ has a K-representing measure if and only if the corresponding functional L_β is K-positive.

Concrete conditions for K-positivity are known in some special cases, e.g., in the solutions of Stieltjes [1896] for $K = [0, +\infty)$, Hamburger [1921] for $K = \mathbb{R}$, Hausdorff [1923] for $K = [0, 1]$, and K. Schmüdgen for K a compact basic semialgebriac set $\mathcal{S}$ [1991]. In these cases, there is a concrete description of the polynomials that are positive on K. M. Dritshel talk: Fejer-Riesz describes positive trigonometric polynomials

A connection between the Full and Truncated Moment Problems:

Theorem [Stochel, 2001]

$\beta^{(\infty)}$ has a K-representing measure if and only if $\beta^{(m)}$ has a K-representing measure for each $m \geq 1$.

Moment matrices

For a multisequence $\beta \equiv \beta^{(2d)}$ with Riesz functional L_β, the *moment matrix* M_d has rows and columns X^i indexed by the monomials in $\mathcal{P}_d$, $x^i \equiv x_1^{i_1} \cdots x_n^{i_n}$ ($|i| \equiv i_1 + \cdots i_n \leq d$) in degree-lexicographic order. For $p, q \in \mathcal{P}_d$, with coefficient vectors $\widehat{p}, \widehat{q}$ relative to the basis of monomials, we have

$$\langle M_d \widehat{p}, \widehat{q} \rangle := L_\beta(pq).$$

In the "concrete" cases of Riesz-Haviland for the full K-moment problem, weighted sums-of-squares decompositions for positive polynomials permit the equivalence of K-positivity for L_β with positive semidefiniteness conditions for a finite family of infinite moment matrices and associated "localizing matrices". In the Schmüdgen decomposition, there are no degree bounds ([Scheiderer]), so the decomposition cannot directly be used in TKMP. Moment matrix extension methods (flat extensions) apply in principle, but may be difficult to apply.(The quartic theorem in Aljaz Zalar's talk does not extend to degree 6.)

More on the role of K-positivity in TKMP

Tchakaloff's Theorem [1957]

Let $\beta \equiv \beta^{(m)}$. For K compact, if L_β is K-positive, then β has a K-representing measure μ satisfying *card supp* $\mu \leq$ *dim* $\mathcal{P}_m$.

Note. *Concrete* criteria for the case when K is compact are known only in some special cases (e.g., $n = 1$ and $K = [a, b]$). For the closed disk, the problem is largely unsolved. For K non-compact, Riesz-Haviland is false.

Truncated Riesz-Haviland [CF,2009].

Let $\beta = \beta^{(2d)}$ or $\beta = \beta^{(2d+1)}$ and let K be a closed subset of $\mathbb{R}^n$. β has a K-representing measure if and only if β can be extended to a sequence $\widetilde{\beta} \equiv \beta^{(2d+2)}$ such that $L_{\widetilde{\beta}}$ is K-positive.

Low-rank Theorem [Blekherman, PAMS, 2015]

Given $\beta = \beta^{(2d)}$, if M_d is positive semidefinite and satisfies *rank* $M_d \leq 3d - 3$, then L_β is positive.

Note. In this case, $\beta^{(2d-1)}$ has a representing measure by TRH.

The core variety of a truncated moment functional
[J. Operator Theory, to appear]

The core variety provides an approach to establishing the existence of representing measures based on methods of convex analysis. For the polynomial case, this was introduced in [F.,JMAA, 2017], and some of the ideas go back to [F.-Nie, JFA, 2009]. The core variety has also been studied by P. di Dio and K. Schmüdgen [JFA, 2017] and in Schmüdgen's recent book [2017].

Let X be a T_1 space and let W be a finite dimensional subspace of $C(X)$ (or of $B(X)$). Consider a linear functional $L : W \mapsto \mathbb{R}$.

Problem. When does L have a finitely atomic representing measure, i.e., when does L belong to the convex cone generated by the point evaluations δ_x $(x \in X)$, where $\delta_x(f) = f(x)$ $(f \in W)$?

(Basic Example) Let $W = \mathcal{P}_m[x_1, \ldots, x_n]$. Given a multisequence $\beta \equiv \beta^{(m)}$, $\beta_0 > 0$, let $L = L_\beta$, the Riesz functional. Consider the case $m = 2d$ and suppose $p \in \mathcal{P}_d$ satisfies $M_d \widehat{p} = 0$. Then $L_\beta(p^2) = \langle M_d \widehat{p}, \widehat{p} \rangle = 0$. Note that if μ is any representing measure for L, then *supp* $\mu \subseteq \mathcal{Z}(p)$. For if there exists $x_0 \in$ *supp* μ with $p(x_0) \neq 0$, then $p^2 > 0$ in an open neighborhood of x_0, so $0 < \int p^2 d\mu = L(p^2) = 0$. Thus, the common zeros of such polynomials p contains the support of any representing measure; this set of common zeros is called the *variety* of M_d and it plays a significant role in the work of [C-F] in establishing representing measures via flat extensions of moment matrices. More generally, for $p \in \mathcal{P}_m$, if $p \in$ *ker* L and $p|\mathbb{R}^n \geq 0$, then *supp* $\mu \subseteq \mathcal{Z}(p)$.

Define $V_0 := X$ and for $i \geq 0$, define

$$V_{i+1} := \bigcap_{f \in ker \ L, \ f|V_i \geq 0} \mathcal{Z}(f).$$

We define the *core variety* of L by

$$\mathcal{CV}(L) := \bigcap_{i \geq 0} V_i.$$

Note. In the Basic Example (TMP), it is equivalent to start with $V_0 := V(M_d)$, the variety of M_d; this facilitates calculation of the core variety.

For the *continuous case*, when W is a finite dimensional subspace of $C(X)$, representing measures are always supported in the core variety:

Proposition

If μ is a representing measure for L, then *supp* $\mu \subseteq \mathcal{CV}(L)$.

Note: In the polynomial case, if μ is a representing measure, then $rank\ M_d(\beta) \le card\ supp\ \mu \le card\ \mathcal{CV}(L_\beta) \le card\ \mathcal{V}^{(i)}$. Thus, if $card\ V_i < rank\ M_d$ for some i, then β has no representing measure.

Example with $M_3 \succ 0$, L_β positive, but no measure [F, 2017].

With $n = 2$, consider $M_3(\beta)$ defined as follows:

$$\begin{pmatrix} 8 & 0 & 0 & 6 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 6 & 0 & 4 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 4 & 0 & 6 \\ 6 & 0 & 0 & 6 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 4 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 8 & 0 & 6 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 4 & 0 & 0 & 0 & 0 & 6 & 0 & 6 & 0 \\ 0 & 0 & 6 & 0 & 4 & 0 & 0 & 6 & 0 & 8 \end{pmatrix}.$$

M_3 is positive definite, and results of [F-Nie, JFA, 2012] imply that L_β is positive. The de-homogenized Robinson polynomial $r(x, y)$ is in $ker\ L_\beta$. Since r is nonnegative and has exactly 8 affine zeros, it follows that $card\ V_1 \le 8 < 10 = rank\ M_3$, so β has no representing measure. Further, $f(x, y) := 2 - x^2 - y^2$ and $g(x, y) := \frac{3}{2}x^2y^2 - x^2y^4$ are in $ker\ L_\beta$, are nonnegative on the zeros of r, and have no common zeros on this set, so $\mathcal{CV}(L_\beta) = V_2 = \emptyset$.

Nonempty core variety and representing measures

We now assume there is some $\rho \in W$ such that $\rho|X > 0$ and $L(\rho) > 0$; in the classical polynomial case, we have $\rho \equiv 1$.

Core Variety Theorem

$L \in W^*$ has a representing measure if and only if its core variety $\mathcal{V} \equiv \mathcal{CV}(L)$ is nonempty. In this case, the union of the supports of all f.a.r.m. for L is precisely $\mathcal{V}$, as is the union of the supports of all representing measures in the $C(X)$ case. Further, there exists $k \leq dim\ W$ such that $\mathcal{CV}(L) = V_k$, and if some V_j is finite, then $\mathcal{CV}(L) = V_j$ or $\mathcal{CV}(L) = V_{j+1}$.

Instead of discussing the proof of this result, we will consider a more general version.

The Core Set problem

Let V be a finite dimensional real vector space with norm $||\cdot||$ and dual space V'. Let $\mathcal{F} \equiv \mathcal{F}_0$ be a proper subset of V, $0 \notin \mathcal{F}$, and let $C \equiv C_0$ denote the conical hull of $\mathcal{F}$. For a nonzero element $L \in V$, we refer to an element $F_0 \in \mathcal{F}$ as a *support point* for L if L admits a representation

$$L = \sum_{j=0}^{m} a_j F_j, \tag{1}$$

where each $a_j > 0$ and each $F_j \in \mathcal{F}$. The *core set* of L, $\mathcal{CS}(L)$, is the set of all support points of L, so $L \in C$ if and only if $\mathcal{CS}(L) \neq \emptyset$.

In the Basic Example (TMP), let $V = W'$, let $\mathcal{F} = \{\delta_x : \ x \in \mathbb{R}^n\}$, and let $L = L_\beta$. Thus, the core set of L is nonempty if and only if L admits a finitely atomic representing measure.

Problem. Describe a procedure for computing $\mathcal{CS}(L)$.

Observe that if $F \equiv F_0$ is a support point for L, then F belongs to the set

$$\mathcal{F}_1 := \{F \in \mathcal{F}_0 : \; Q(F) = 0 \,\forall\, Q \in V' \;\ni\; Q|\mathcal{F}_0 \geq 0 \text{ and } Q(L) = 0\}. \tag{2}$$

To see this, suppose to the contrary that $F_0 \in \mathcal{F} \setminus \mathcal{F}_1$. Then there exists $Q \in V'$ such that $Q|\mathcal{F} \geq 0$ and $Q(L) = 0$, but $Q(F_0) > 0$.

Then (1) implies $0 = Q(L) = \sum_{j=0}^{m} a_j Q(F_j) > 0$, a contradiction.

For each $i \geq 0$, we now iteratively define

$$\mathcal{F}_{i+1} = \{F \in \mathcal{F}_i : \; Q(F) = 0 \,\forall\, Q \in V' \;\ni\; Q|\mathcal{F}_i \geq 0 \text{ and } Q(L) = 0\}. \tag{3}$$

We also set $\mathcal{F}_\infty := \bigcap_{i \geq 0} \mathcal{F}_i$. An induction based on the above argument shows $\mathcal{CS}(L) \subseteq \mathcal{F}_\infty$, so if $\mathcal{F}_\infty = \emptyset$, then $L \notin C$.

Our main result is the following converse.

Theorem

Let L be a nonzero element of V. Then there exists $k \leq dim\ V$ such that $\mathcal{F}_\infty = \mathcal{F}_k$ and thus $\mathcal{CS}(L) = \mathcal{F}_k$. Therefore, L belongs to C if and only if $\mathcal{F}_k$ is nonempty, in which case the support points of L are precisely the elements of $\mathcal{F}_k$.

In the case of the truncated moment problem for β, we see that β has a finitely atomic representing measure if and only if the core set of L_β is nonempty, in which case the core set is the set $\{\delta_x : x \text{ is a support point for some f.a.r.m. for } \beta\}$. In the core variety paper, we also showed that if β has *any* representing measure, then the core variety is nonempty, so β has a finitely atomic representing measure (Bayer-Teichmann Theorem, now attributed to Richter (1957)).

Outline of the proof

We have already verified the following:

Lemma (inclusion)

$\mathcal{CS}(L) \subseteq \mathcal{F}_\infty$.

For $i \geq 1, let C_i := conhull(\mathcal{F}_i)$ and $V_i := lin.span(\mathcal{F}_i)$; clearly $\mathcal{F}_{i+1} \subseteq \mathcal{F}_i$ and $V_{i+1} \subseteq V_i$. An application of the definitions shows that if $\mathcal{F}_{i+1}$ is a proper subset of $\mathcal{F}_i$, then $dim\ V_{i+1} < dim\ V_i$.

Lemma (stability)

There exists $k \leq dim\ V$ such that we have *stability*, i.e., $\mathcal{F}_{k+1} = \mathcal{F}_k$, whence $\mathcal{F}_\infty = \mathcal{F}_k$.

From the inclusion lemma we see that if $\mathcal{F}_\infty = \emptyset$, then $\mathcal{CS}(L) = \emptyset$, so $L \notin C$. In view of the stability lemma, we seek to show that if $\mathcal{F}_k \neq \emptyset$, then $\mathcal{CS}(L) = \mathcal{F}_k$ and, in particular, L is an element of C. We first recall the notion of strict positivity.

Strict positivity

Let W denote a finite dimensional real vector space with norm $||\cdot||$, and let W' denote the dual space of W. For $x \in W$, let $\widehat{x} \in W''$ be defined by $\widehat{x}(Q) = Q(x)$ $(Q \in W')$; the map $x \mapsto \widehat{x}$ is an isometric linear isomorphism. For a subset S in W, S^* denotes the *dual cone*, i.e., $S^*; = \{Q \in W' : \ Q(x) \geq 0 \ \forall x \in S\}$. We say that $Q \in S^*$ is *strictly S-positive* (or *strictly positive with respect to S*) if $Q(x) > 0$ for each nonzero $x \in S$.

Proposition (strict positivity)

If S is a closed set in W, then $Q \in W'$ is strictly S-positive if and only if $Q \in int(S^*)$.

Returing to the proof of the theorem, we now assume that there exists a functional $\rho \in V'$ such that $\rho|\mathcal{F} > 0$ and $\rho(L) > 0$. This assumption is natural, becasue in the classical TMMP for multisequence β, we have $1 \in W \equiv \mathcal{P}_m$, so we can let $\rho = \widehat{1} \in V'(= \mathrm{W}'')$; then for each $F \equiv \delta_x$ in $\mathcal{F}$, $\rho(\delta_x) = \delta_x(1) = 1$, and also $\rho(L_\beta) = L_\beta(1) = \beta_0 > 0$. Using ρ, we establish a property for L consistent with membership of L in C_k.

Lemma (consistency)

If $\mathcal{F}_k = \mathcal{F}_{k+1} \neq \emptyset$, then L is $\mathcal{F}_k$-*consistent*, i.e., if $Q \in V'$ satisfies $Q|\mathcal{F}_k \equiv 0$, then $Q(L) = 0$.

Restriction

Now let $W_k := \{Q|V_k : \ Q \in V'\}$. Define a linear functional $\widetilde{L} : W_k \longrightarrow \mathbb{R}$ by $\widetilde{L}(Q|V_k) := \widehat{L}(Q)$ $(= Q(L))$. Consistency implies that $\widetilde{L}$ is well-defined. Using ρ together with stability, we have the following:

Lemma (strict positivity of $\widetilde{L}$)

If $\mathcal{F}_k = \mathcal{F}_{k+1} \neq \emptyset$, then $\widetilde{L}$ is strictly positive with respect to $\mathcal{C}_k := \{Q|V_k \in W_k : \ Q|\mathcal{F}_k \geq 0\}$.

Since $\mathcal{C}_k$ is a closed cone in W_k, the previous result and the strict-positivity Proposition (above) imply that $\widetilde{L}$ belongs to the interior of the dual cone $\mathcal{C}_k^*$. We next present a concrete description of this dual cone.

For $G \in V_k$, we define $\check{G} \in W_k'$ by
$\check{G}(Q|V_k) := (Q|V_k)(G) = Q(G)$.

Proposition (dual space of W_k).

If $\mathcal{F}_k = \mathcal{F}_{k+1} \neq \emptyset$, then $W_k' = \{\check{G} : \ G \in V_k\}$.

Corollary

$C_k^* = \{\check{G} \in W_k' : \ Q \in V', \ Q|C_k \geq 0 \Longrightarrow Q(G) \geq 0\}$.

Let $\Gamma := \{ \ \check{G} : \ G \in C_k\}$, a convex cone. The preceding result and the Separation Theorem for closed convex cones now imply:

Proposition (approximation)

$\mathcal{C}_k^* \subseteq \overline{\Gamma}$.

Conclusion of the proof

Proof.

Suppose $\mathcal{CS}(L) \equiv \mathcal{F}_k \neq \emptyset$. We first show that $L \in C_k$. We have $\widetilde{L} \in int(\mathcal{C}_k^*) \subseteq int(\overline{\Gamma}) = int(\Gamma)$ (since Γ is convex). Thus, $\widetilde{L} \in \Gamma$, so there exists $G \in C_k$ such that $\widetilde{L} = \check{G}$. Now G is of the form $G = \sum a_i F_i$ with each $a_i > 0$ and each $F_i \in \mathcal{F}_k$. For each $Q \in V'$, $\widehat{L}(Q) = Q(L) = \widetilde{L}(Q|V_k) = \sum a_i \check{F}_i(Q|V_k) = \sum a_i Q(F_i)$ $= \sum a_i \widehat{F}_i(Q) = \widehat{\sum a_i F_i}(Q)$. Thus, $\widehat{L} = \widehat{\sum a_i F_i}$ in V'', so $L = \sum a_i F_i$ in V (with each $F_i \in \mathcal{F}_k$). This completes the proof that L is an element of C_k.

Now let $F \in \mathcal{F}_k$. Since $\widetilde{L} \in int(\mathcal{C}_k^*)$, there exists $\epsilon > 0$ such that $\widetilde{L} - \epsilon \check{F} \in int(\mathcal{C}_k^*)$. Then, exactly as above, we see that there are positive reals $b_1, \ldots, b_q$ and elements $H_1, \ldots, H_q$ in $\mathcal{F}_k$ such that $L - \epsilon F = \sum b_i H_i$. Thus $L = \epsilon F + \sum b_i H_i$, whence F is a support point for L. □