A factorization problem related to the convolution of positive definite functions.

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Convolution equations

- Let G be a locally compact abelian (l.c.a.) group.
- A set $S \subset G$ is called symmetric if

$$0 \in S \quad \text{and} \quad x \in S \Longleftrightarrow -x \in S.$$

- A function $f : G \mapsto \mathbb{C}$ is positive definite (p.d.) if for any $x_1, \ldots, x_m \in G$ and any $\xi_1, \ldots, \xi_m \in \mathbb{C}$, we have

$$\sum_{i,j=1}^{m} f(x_i - x_j)\, \xi_i\, \overline{\xi_j} \geq 0.$$

- Note that if $f \neq 0$, then $f(0) > 0$ and $f(-x) = \overline{f(x)}$, so the support of f is symmetric.

- By Bochner's theorem, any continuous p.d. function on G has an integral representation in the form

$$f(x) = \int_{\hat{G}} \xi(x)\, d\mu(\xi), \quad x \in G,$$

where μ is a bounded, positive Borel measure on the dual group $\hat{G}$.

- Let us consider first the case of a finite group G. Then, up to a group isomorphism,

$$G = \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_r\mathbb{Z},$$

for certain integers $m_1, \ldots m_r \geq 2$.

- The characters of G are given by functions $\chi : G \to \mathbb{T}$ of the form

$$\chi(x) = e^{2\pi i x_1 a_1/m_1} \ldots e^{2\pi i x_r a_r/m_r}, \quad x = (x_1, \ldots, x_r) \in G,$$

where $a = (a_1, \ldots, a_r) \in G$.

- If $f : G \to \mathbb{C}$ is a function, its Fourier transform is the function $\hat{f} : \hat{G} \to \mathbb{C}$ defined by

$$\hat{f}(\chi) = \mathcal{F}f(\chi) = \sum_{x \in G} f(x)\,\overline{\chi(x)}.$$

- The convolution of two functions f and g on G is the function $f * g$ on G defined by

$$(f * g)(x) = \sum_{y \in G} f(x-y)\,g(y), \quad x \in G.$$

- We have the usual "exchange" formula

$$\mathcal{F}(f * g)(\chi) = \mathcal{F}f(\chi)\,\mathcal{F}g(\chi), \quad \chi \in \hat{G}.$$

- Note that a function $f : g \to \mathbb{C}$ is p.d. if and only if $\hat{f} \geq 0$.

- For $S \subset G$ symmetric, we will denote by $\mathcal{PD}(S)$, the set of positive definite functions which vanish outside of S.
- If $S \subset G$ is a symmetric set, we associate with it the symmetric set S^* consisting of the points in G which are not in S together with 0, i.e. $S^* = (G \setminus S) \cup \{0\}$.

Theorem

Suppose that G is a finite abelian group. Let $f : G \to \mathbb{C}$ be positive definite and let $S \subset G$ be a symmetric set. Then, there exist $g \in \mathcal{PD}(S)$ and $h \in \mathcal{PD}(S^*)$ such that $f = g * h$.

- Note that the method of the proof is based on the Lagrange multipliers method and is non-constructive.
- Also, the functions g and h need not be unique.

- The previous result to extended to other l.c.a groups using standard "approximation" arguments such as periodization, weak-* compactness,etc...
- The exact statement of the result will depend on the group G.
- For example, the statement for the group $\mathbb{Z}^d$ reads as follows.

Theorem

Let $S \subset \mathbb{Z}^d$ be a finite symmetric set and let $S^* = (\mathbb{Z}^d \setminus S) \cup \{0\}$. Then, given any $f \in \mathcal{PD}(\mathbb{Z}^d)$, there exists $g \in \mathcal{PD}(S)$ and $h \in \mathcal{PD}(S^*)$ such that $f = g * h$ on $\mathbb{Z}^d$.

- Note that the convolution product makes sense as g has finite support. Alternatively, $\hat{g}$ is a trigonometric polynomial so the product $\hat{g}\,\hat{h}$ is well defined.

- We consider next the case of $\mathbb{R}^d$.
- In this setting, we will actually consider two different set-ups, the first one dealing with a symmetric open set $U \subset \mathbb{R}^d$ with $|U| < \infty$, where $|.|$ denotes the Lebesgue measure and the other involving a symmetric open set $U \subset \mathbb{R}^d$ whose complement is compact.
- Note that now it makes no sense to consider continuous p.d. functions supported on $(\mathbb{R}^d \setminus U) \cup \{0\}$ since this set does not contain a neighborhood of 0.
- Instead one has to consider positive definite distributions on $\mathbb{R}^d$ equal to a multiple of the Dirac mass δ_0 on the open set U.
- It turns out that these are tempered distributions and their Fourier transforms are unbounded tempered measure by the Bochner-Schwartz theorem. In fact such a measure μ is translation-bounded i.e. there exists $C > 0$ such that

$$\mu(x + [0,1]^d) \leq C, \quad x \in \mathbb{R}^d. \tag{1}$$

Theorem

Let $U \subset \mathbb{R}^d$ be a symmetric open set with $|U| < \infty$ and let $K \subset \mathbb{R}^d$ be the closed set defined by $K = \left(\mathbb{R}^d \setminus U\right) \cup \{0\}$. Then, given any continuous positive definite function f on $\mathbb{R}^d$, there exists a continuous positive definite function g on $\mathbb{R}^d$ such that $g = 0$ on $\mathbb{R}^d \setminus \overline{U}$ and a positive definite distribution h supported on K with $h = \delta_0$ on U and with a Fourier transform $\mu = \mathcal{F}(T)$ which is a translation-bounded measure such that

$$f = g * h \text{ on } \mathbb{R}^d.$$

Furthermore, $g = 0$ a.e. on ∂U and if $U = \mathrm{int}(\overline{U})$, the function g actually vanishes on $\mathbb{R}^d \setminus U$.

- Note that the convolution product can be defined as $\mathcal{F}^{-1}(\hat{g}\,\hat{h})$. Since $g \in L^1(\mathbb{R}^d)$, $\hat{g}$ is continuous and so $\hat{g}\,\hat{h}$ is well-defined.

- The second version is as follows.

Theorem

Let $U \subset \mathbb{R}^d$ be a symmetric open set such that the set $K \subset \mathbb{R}^d$ defined by $K = \left(\mathbb{R}^d \setminus U\right) \cup \{0\}$ is compact. Then, given any continuous positive definite function f on $\mathbb{R}^d$, there exists a continuous positive definite function g on $\mathbb{R}^d$ such that $g = 0$ on $\mathbb{R}^d \setminus \overline{U}$ and a positive definite distribution h supported on K with $h = \delta_0$ on U and with a Fourier transform $\hat{h}$ which is a continuous bounded function such that

$$f = g * h \text{ on } \mathbb{R}^d.$$

Furthermore, the function g constructed above is zero a.e. on ∂U and if $U = \text{int}(\overline{U})$, the function g actually vanishes on $\mathbb{R}^d \setminus U$.

Connection with the truncated trigonometric moment problem

- We consider the case $G = \mathbb{Z}^d$ for simplicity.
- Suppose that V is a finite subset of $\mathbb{Z}^d$. Then the set $U := V - V = \{v - v' : \; v, v' \in V\}$ is symmetric.
- If f is p.d. on $\mathbb{Z}^d$ and we write $f = g * h$ with $g \in \mathcal{PD}(U)$ and $h \in \mathcal{PD}(U^*)$ with $h = \delta_0$ on U.
- Thus h is a solution of the truncated trigonometric moment problem which consists in extending the data corresponding to the identity operator on $\ell^2(V)$, i.e. we have

$$\sum_{k,l \in V} h(k-l)\, x(k)\, \overline{x(l)} = \sum_{k \in V} |x(k)|^2 = \int_{\mathbb{T}^d} |\hat{x}|^2 \, d\mu, \quad x \in \ell^2(V)$$

and $\mu = \mathcal{F}(h)$ is a positive Borel measure on $\mathbb{T}^d = \hat{G}$, called a representing measure.

The Turán problem

Definition

Let U be a symmetric open set in the l.c.a. group G. Then, we will denote by $\mathcal{T}_G(U)$ the supremum of the quantity $\int_U g(x)\,dx$, where g ranges over all positive definite functions with supp$(g) \Subset U$ and satisfying $g(0) = 1$ ($dx =$ Haar measure).

- The Turán problem, which asks to compute the value of $\mathcal{T}_G(U)$ was first proposed by Turán and Stechkin
- Many authors studied the problem for particular sets U mainly in $\mathbb{R}^d$ and $\mathbb{T}^d$.
- On $\mathbb{R}^d$, special attention has been given to convex symmetric sets (Siegel, Arestov and Berdysheva, Gorbachov,. . .) and products of symmetric intervals in $\mathbb{T}^d$ (Gorbachev and Manoshina,. . .).

- The problem has been studied recently in the general setting of l.c.a. groups by Kolountzakis and Révész.
- It turns out that the Turán problem is related in an essential way to the previous convolution identity where the p.d. function is the constant function $f(x) = 1$.
- We will discuss here the problem when G is a finite group and $G = \mathbb{R}^d$.
- If G is finite, any subset of G is open and if $S \subset G$ is symmetric, $\mathcal{T}_G(S)$ is the largest possible value of a sum $\sum_{k \in S} g(k)$ where g is p.d., supported on S and satisfies $g(0) = 1$.

Theorem

Let G be a finite abelian group and let $S \subset G$ be a symmetric set. If $g_0 \in \mathcal{PD}(S)$ and $h_0 \in \mathcal{PD}(S^*)$ satisfy $g_0(0) = 1 = h_0(0)$ as well as $g_0 * h_0 = 1$ on G (g_0 and h_0 exist by our previous result), then we have

$$\mathcal{T}_G(S) = \sum_{k \in S} g_0(k) \quad \text{and} \quad \mathcal{T}_G(S^*) = \sum_{k \in S^*} h_0(k).$$

In particular, we have the identity

$$\mathcal{T}_G(S)\,\mathcal{T}_G(S^*) = |G|.$$

- We call the Turán problem for S^* the dual Turán problem.

- The analogue of this result holds for $G = \mathbb{R}^d$, but we first have to find what to should replace $\mathcal{T}_G(S^*)$ in that case.
- If h is a positive-definite tempered distribution, we define the density of h to be the number

$$\mathcal{D}(h) := \lim_{\epsilon \to 0^+} \langle h(x), \epsilon^{d/2} \, e^{-\epsilon\pi|x|^2} \rangle$$

- Note that if $\mu = \mathcal{F}(h)$, we have $\mathcal{D}(h) = \mu(\{0\})$.
- If $U \subset \mathbb{R}^d$ is symmetric and $K = (\mathbb{R}^d \setminus U) \cup \{0\}$, the dual Turán problem consists in maximizing the quantity $\mathcal{D}(h)$ over all p.d. distributions supported on K and equal to δ_0 on U.
- We denote the supremum of these quantities by $\tilde{\mathcal{T}}_G(K)$.

Theorem

Let $U \subset \mathbb{R}^d$ be a symmetric open with $|U| < \infty$. set. Let g_0 be a continuous p.d. function supported on $\overline{U}$ with $g_0(0) = 1$ and let h_0 be a p.d. distribution supported on K with $h_0 = \delta_0$ on U such that $g_0 * h_0 = 1$ on $\mathbb{R}^d$ (as constructed in our previous result). Then,

$$\mathcal{T}_G(U) = \int_U g_0(x)\,dx \quad \text{and} \quad \tilde{\mathcal{T}}_G(K) = \mathcal{D}(h_0).$$

In particular, we have the identity

$$\mathcal{T}_G(U)\,\tilde{\mathcal{T}}_G(K) = 1.$$

THANK YOU!