Tractable semi-algebraic approximation using Christoffel-Darboux kernel

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Joint work with




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Consider the classical Burgers PDE

$$\frac{\partial y(t,x)}{\partial t} + \frac{1}{4}\frac{\partial y^2(t,x)}{\partial x} \;=\; 0, \;\; \forall t \in [0,1], \, \forall x \in [-1,1]$$

with the following discontinuous initial condition

$$y(0,x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

In the talk by Swann Marx, we saw that solving this nonlinear PDE amounts to solving a **linear** moment problem

Numerically we rely on the Lasserre moment-SOS hierarchy and **convex optimization**

The unique analytical solution is

$$y(t,x) = \begin{cases} 1 & \text{if } x < \frac{t}{4} \\ 0 & \text{if } x > \frac{t}{4} \end{cases}$$

i.e. the discontinuity propagates linearly

Solving the moment problem up to degree 12 with our interface GloptiPoly for Matlab and the semidefinite programming solver MOSEK, we end up with the following moments:

$$(\int y^k d\mu_{t,x}(y) dx\, dt)_{k=0,1,\ldots} = (1.0000, 0.6250, 0.6250, 0.6250, 0.6250, \ldots)$$

which correspond (up to numerical accuracy) exactly with the moments of the analytic solution $d\mu_{t,x} = \delta_{y(t,x)}$

How do we compute the graph from the moments ?

Let

$$\begin{array}{rcccl} f & : & X & \to & Y \\ & & \mathbf{x} & \mapsto & y \end{array}$$

be a bounded measurable function from a given compact set $X \subset \mathbb{R}^{p-1}$ to a given compact set $Y \subset \mathbb{R}$

Given a vector of polynomials $\mathbf{b}(\mathbf{x}, y)$ of total degree d, let

$$\int_X \mathbf{b}(\mathbf{x}, f(\mathbf{x}))\mathbf{b}(\mathbf{x}, f(\mathbf{x}))^\top d\mathbf{x}$$

be the moment matrix of order d supported on the graph

$$\{(\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in X\} \subset X \times Y$$

Problem [from moments to graph]: Given the moment matrix of order d, compute an approximation f_d of function f, with convergence guarantees when $d \to \infty$

The moment matrix just introduced can also be written

$$\mathbf{M}_{\mu,d} := \int \mathbf{b}(\mathbf{x}, y)\mathbf{b}(\mathbf{x}, y)^\top d\mu(\mathbf{x}, y)$$

for the measure

$$d\mu(\mathbf{x}, y) := \mathbb{I}_X(\mathbf{x})\, d\mathbf{x}\, \delta_{f(\mathbf{x})}(dy)$$

concentrated on the graph $\{(\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in X\}$ of function f, where $\mathbb{I}_X$ denotes the indicator function of X, and $\delta_{f(\mathbf{x})}$ denotes the Dirac measure at $f(\mathbf{x})$

If $\mathbf{M}_{\mu,d}$ is invertible, then it is known that the sublevel sets of the Christoffel-Darboux polynomial

$$q_{\mu,d}(\mathbf{x},y) := \mathbf{b}(\mathbf{x},y)^\top \mathbf{M}_{\mu,d}^{-1} \mathbf{b}(\mathbf{x},y)$$

approximate spt μ with convergence guarantees when $d \to \infty$, see

J. B. Lasserre, E. Pauwels. The empirical Christoffel function with applications in data analysis. Advances in Computational Mathematics, 45(3), 2019. arXiv:1701.02886

However, if μ is concentrated on a graph, $\mathbf{M}_{\mu,d}$ may be **singular**

Given a regularization parameter $\beta > 0$, define the **regularized Christoffel-Darboux polynomial**

$$q_{\mu+\beta\mu_0,d}(\mathbf{x}, y) := \mathbf{b}(\mathbf{x}, y)^\top (\mathbf{M}_{\mu,d} + \beta I)^{-1} \mathbf{b}(\mathbf{x}, y)$$

where reference measure μ_0 is absolutely continuous with respect to the Lebesgue measure with compact support, and basis $\mathbf{b}$ is orthonormal with respect to the bilinear form induced by μ_0

Polynomial $q_{\mu+\beta\mu_0,d}$ can be computed numerically by a **spectral decomposition** of the positive semi-definite matrix $\mathbf{M}_{\mu,d}$, and it can be expressed as a polynomial SOS

Instead of trying to approximate the (possibly discontinuous) function f with polynomials, we approximate it with a class of semi-algebraic functions

Define the **semi-algebraic approximant**

$$f_{d,\beta}(\mathbf{x}) := \min\{\text{argmin}_{y \in Y}\ q_{\mu+\beta\mu_0,d}(\mathbf{x}, y)\}$$

as the minimum of the argument of the minimum of the regularized Christoffel-Darboux polynomial, always well defined since this polynomial is SOS

Example: sign function as SOS partial minimum

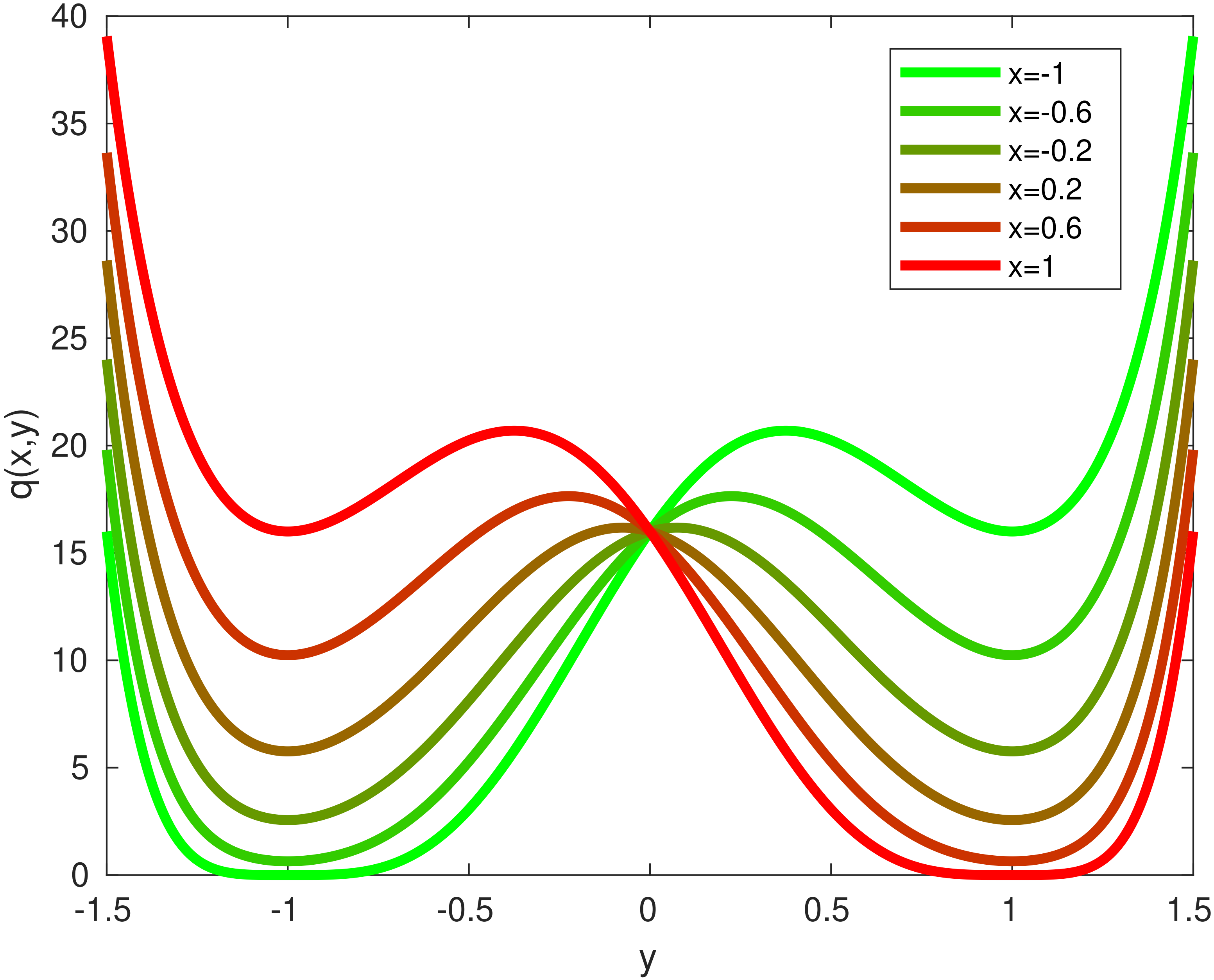
The polynomial

$$p(\mathbf{x}, y) = 4 - 3\mathbf{x}y - 4y^2 + \mathbf{x}y^3 + 2y^4$$

is such that

$$\mathrm{argmin}_{y \in Y}\, p^2(\mathbf{x}, y) = \mathrm{sign}(\mathbf{x})$$

for all $\mathbf{x} \in X := [-1, 1]$ and $Y \subset \mathbb{R}$


x=-1
x=-0.6
x=-0.2
x=0.2
x=0.6
x=1
q(x,y)
y
40
35
30
25
20
15
10
5
0
-1.5
-1
-0.5
0
0.5
1
1.5

Recall that we propose to approximate the function $f(\mathbf{x})$ with the semi-algebraic function

$$f_{d,\beta}(\mathbf{x}) := \min\{\mathrm{argmin}_{y\in Y}\ q_{\mu+\beta\mu_0,d}(\mathbf{x},y)\}$$

Theorem ($\mathscr{L}_1$ convergence): letting $\beta_d := 2^{3-\sqrt{d}}$, if the set $S \subset X$ of continuity points of f is such that $X \setminus S$ has Lebesgue measure zero, then

$$\lim_{d\to\infty} f_{d,\beta_d}(\mathbf{x}) = f(\mathbf{x})$$

for almost all $\mathbf{x} \in X$, and

$$\lim_{d\to\infty} \|f - f_{d,\beta_d}\|_{\mathscr{L}^1(X)} = 0$$

Theorem (convergence rate for Lipschitz functions): if f is L-Lipschitz on X for some $L > 0$, then for any $r > p$ it holds

$$\begin{aligned}\|f - f_{d,\beta_d}\|_{\mathscr{L}^1(X)} \;\le\; & \mathrm{vol}(X)\tfrac{\delta_0}{\sqrt{d}-1}(1+L) \\ & + \mathrm{diam}(Y)\tfrac{8(m+m_0)(3r)^{2r}e^{\frac{p^2}{d}}}{p^p e^{2r-p} d^{r-p}}\end{aligned}$$

where

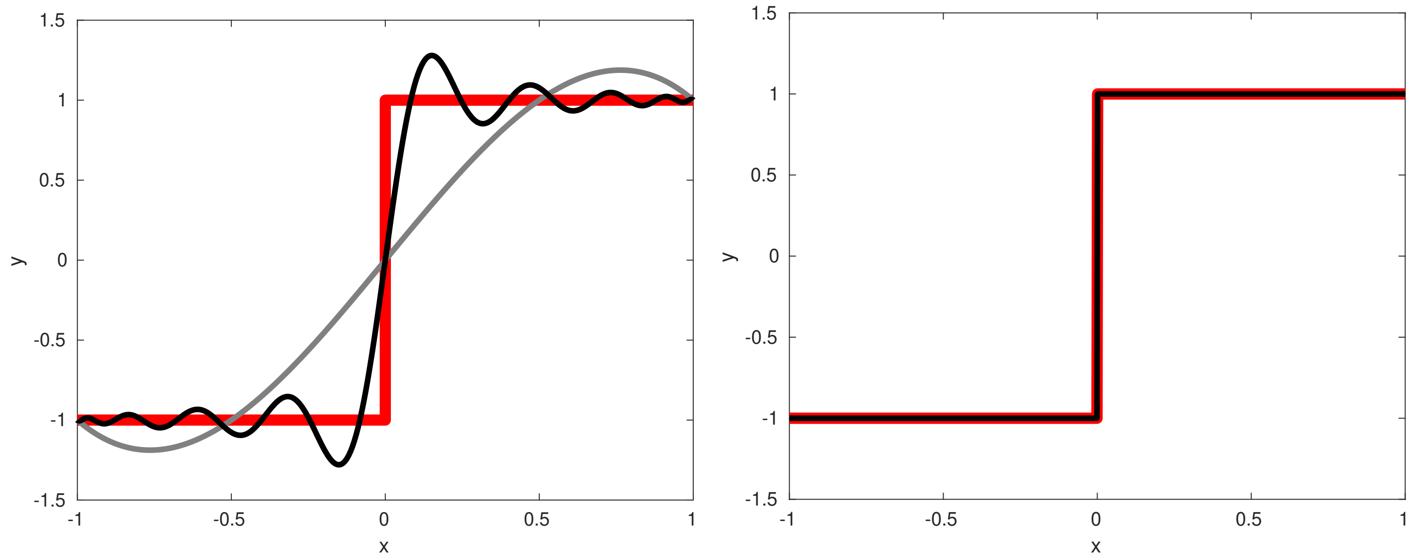
$$\delta_0 := \mathrm{diam}(\mathrm{spt}(\mu+\mu_0)), \quad m := \mu(\mathbb{R}^p), \quad m_0 := \mu_0(\mathbb{R}^p)$$

Letting $r := p + 1/2$ yields an $O(d^{-1/2})$ convergence rate

We observe in practice a much faster convergence

Examples

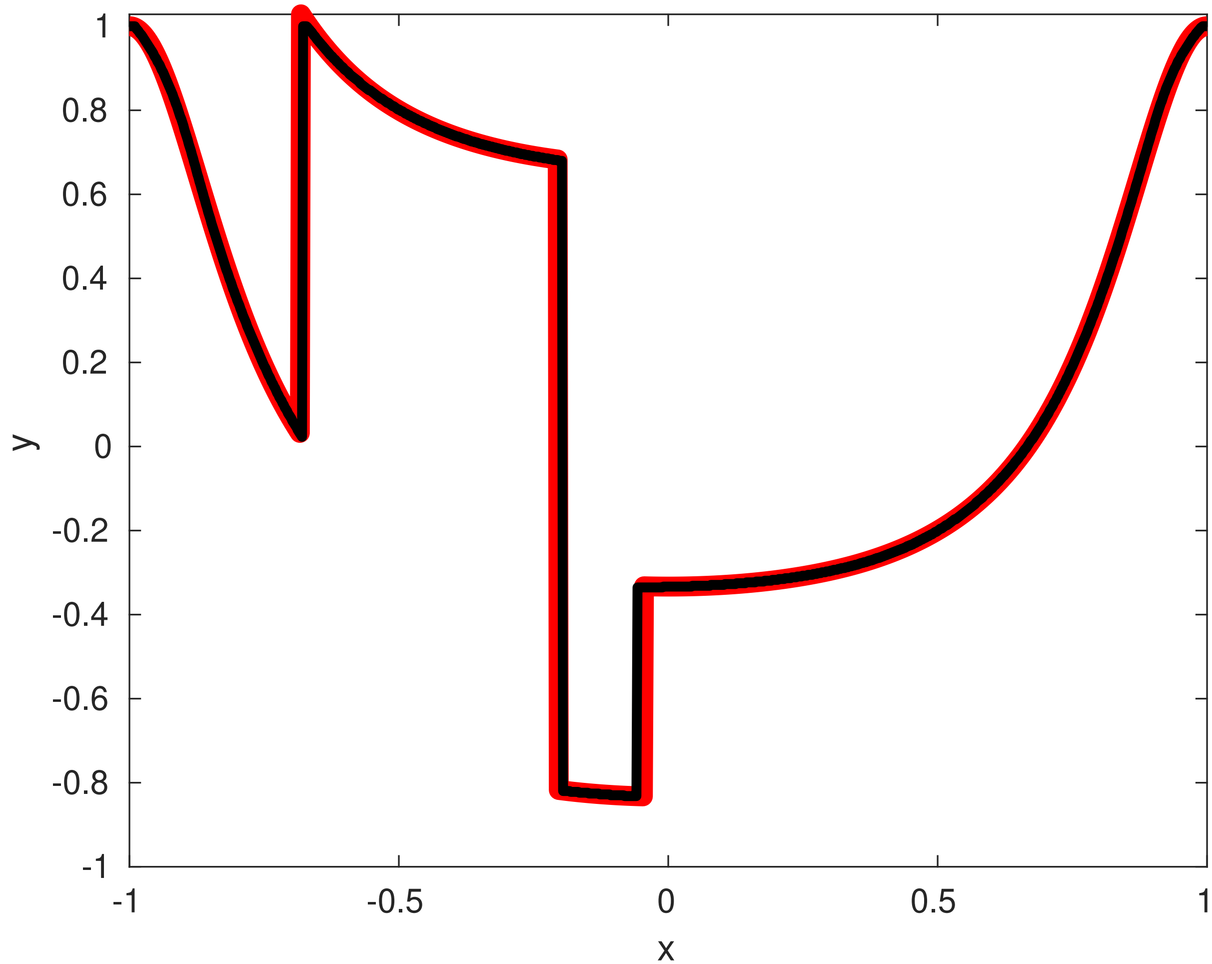
Let us approximate $\mathbf{x} \mapsto f(\mathbf{x}) := \mathrm{sign}(\mathbf{x})$



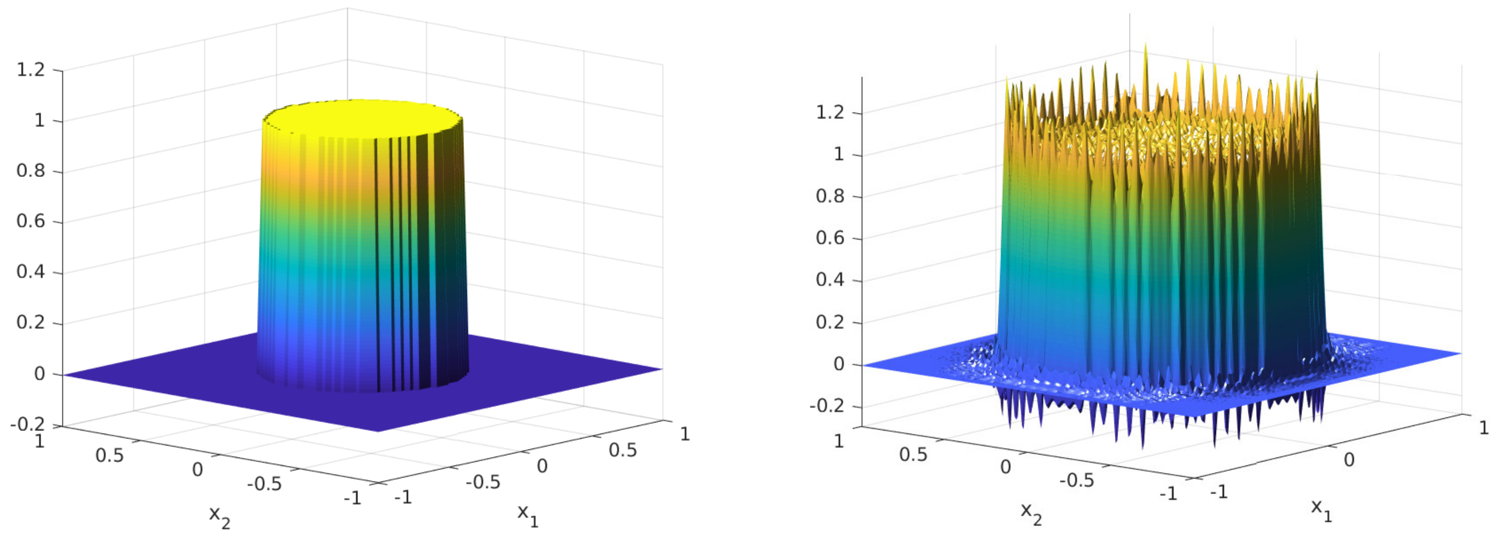
The Chebsyhev polynomial interpolants of degrees 4 (left gray) and 20 (left black) illustrate the typical Gibbs phenomenon

The Christoffel-Darboux approximant f_2 (right black) with 15 moments of degree 4 cannot be distinguished from f (red)

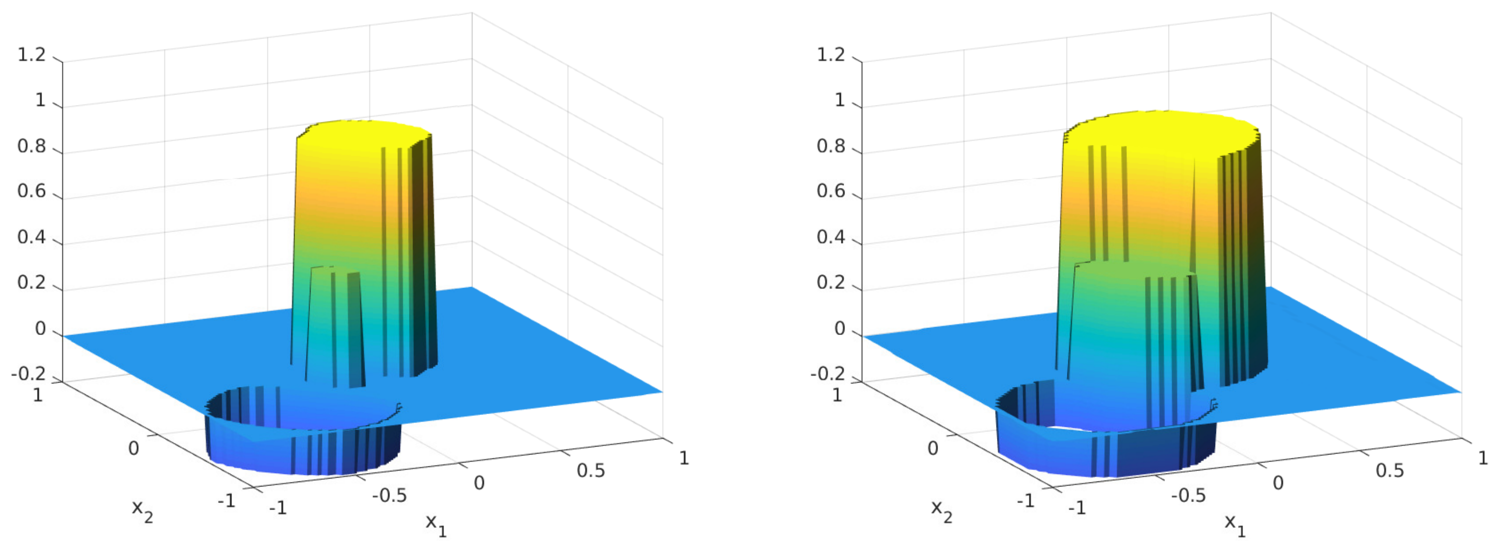
Degree 10 approximation (black) of discontinuous function (red)



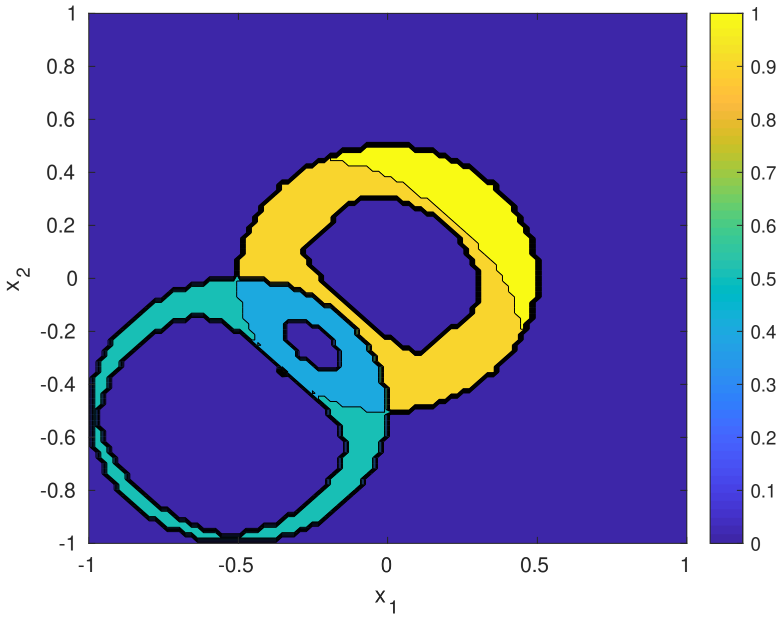
Degree 4 Christoffel-Darboux approximation (left) and
high degree Chebyshev polynomial approximation (right)
of the indicator function of a disk

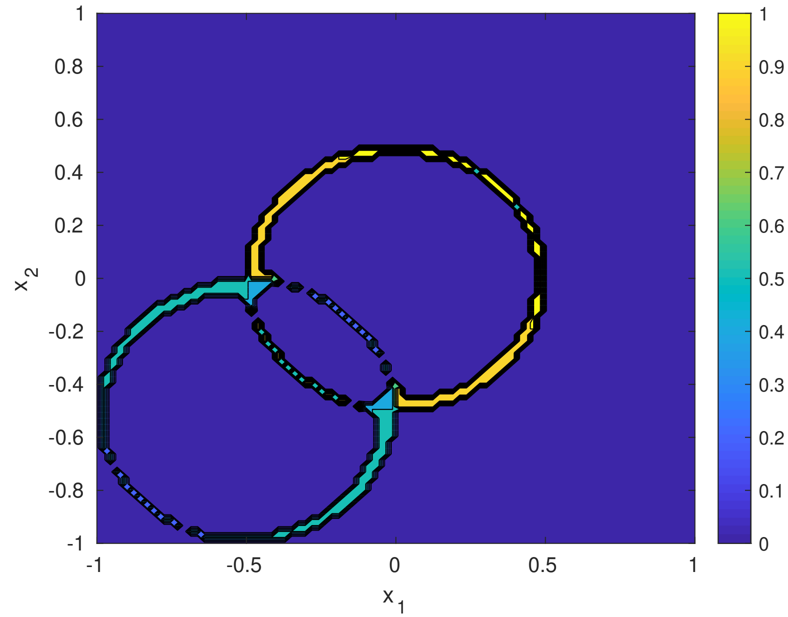


Degree 8 (left) and degree 16 (right) approximations of the superposition of signed indicator functions of two disks

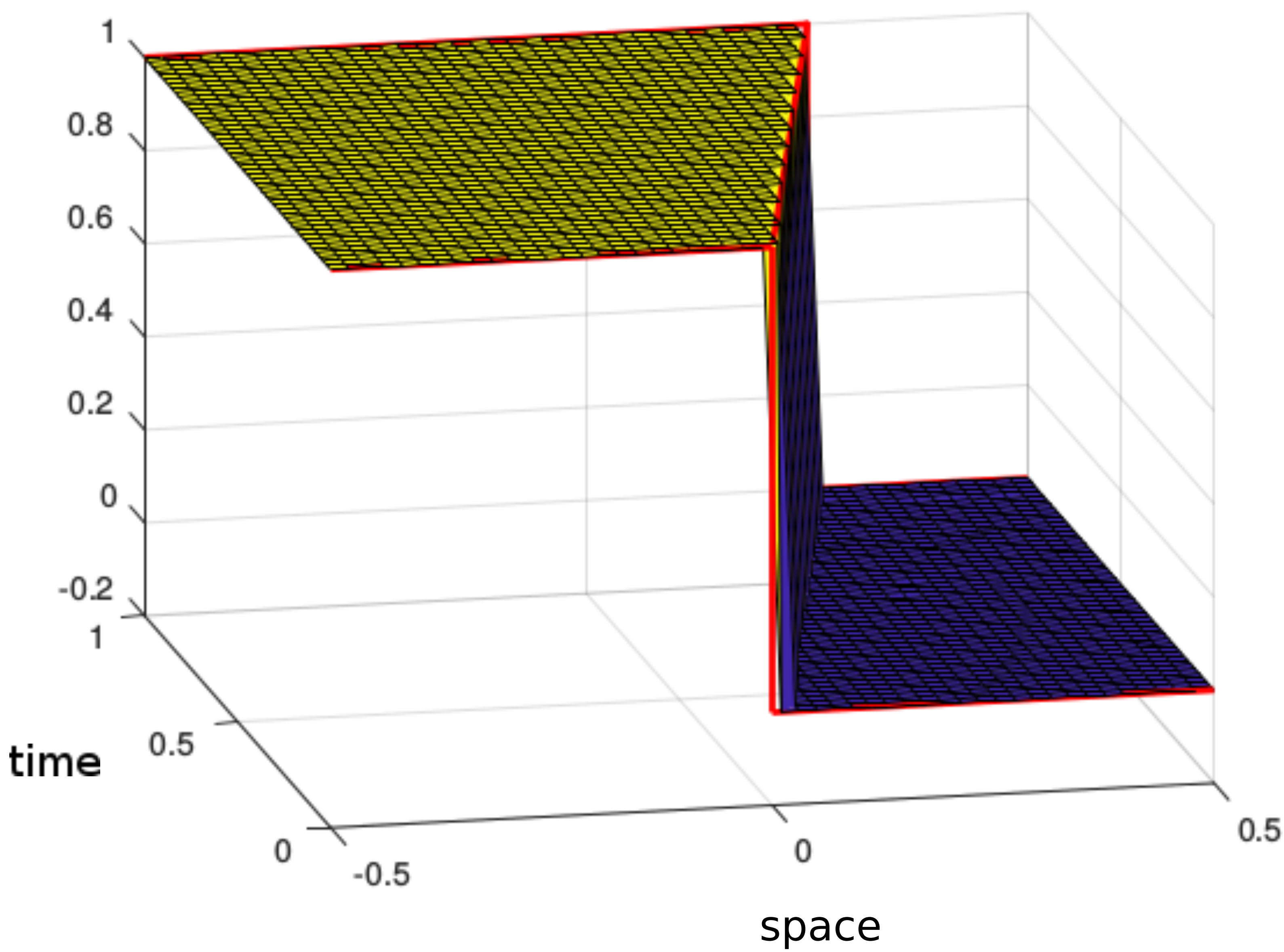


Contour plots of the error for the degree 8 (left) and degree 16 (right) approximations of the two disks

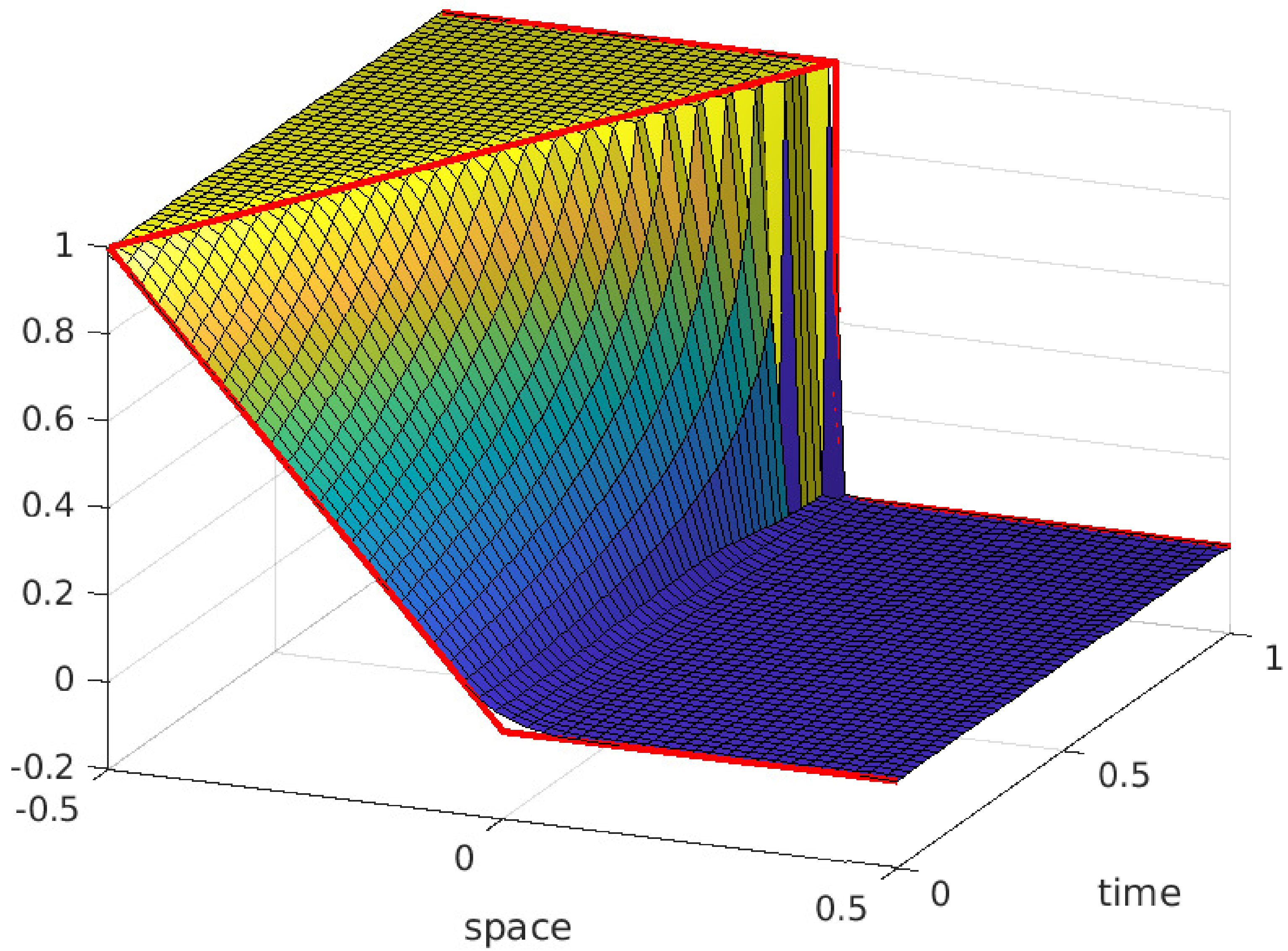


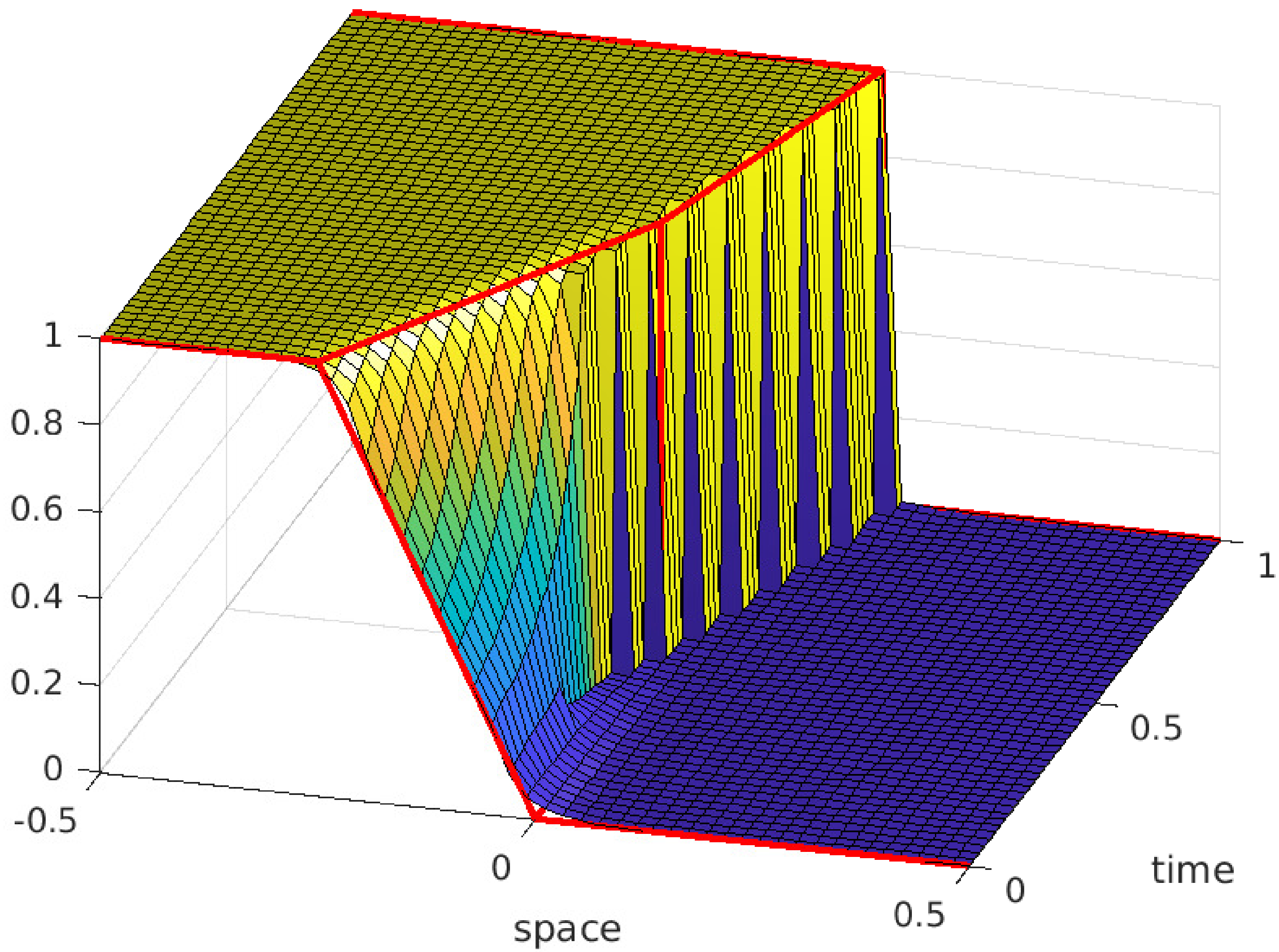


For our motivating application, namely recovering discontinuous solutions of non-linear PDEs from their approximate moments, here are the numerical results



1
0.8
0.6
0.4
0.2
0
-0.2
1
0.5
time
0
-0.5
0
0.5
space


1
0.8
0.6
0.4
0.2
0
-0.2
-0.5
0
0.5
space
0
0.5
1
time


1
0.8
0.6
0.4
0.2
0
-0.5
0
space
0.5
0
0.5
1
time

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