On truncated discrete moment problems

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Discrete truncated moment problem

This talk focus on

K discrete subset of $\mathbb{R}^d$ for $d = 1;\ n \in \mathbb{N}$ or $d \geq 2, n = 2$

mainly $K = \mathbb{N}_0$ or $K = \mathbb{Z}^d$.

$d-$dimensional truncated $K-$moment problem of degree n

Given $m := \left(m^{(0)}, \ldots, m^{(n)}\right)$ with $m^{(k)}$ a tuple

$$\left(m^{(k)}_{j_1,\ldots,j_d}\right)_{j_r \in \mathbb{N}_0; \sum_{r=1}^d j_r = k}$$

with $m^{(k)}_{j_1,\ldots,j_d} \in \mathbb{R}$.
Find a nonnegative Radon measure μ supported in K s.t.

$$m^{(k)}_{j_1,\ldots,j_d} = \int_K x_1^{j_1} \ldots x_d^{j_d} \mu(dx), \quad \forall\, k; j_r \in \mathbb{N}_0 \text{ with } \sum_r j_r = n$$

W.l.o.g. we can assume $m_0 = 1$ and μ is a probability measure on K.
We can use that the set is discrete

$$m^{(k)}_{j_1,\ldots,j_d} = \sum_{x \in K} x_1^{j_1} \ldots x_d^{j_d} \mu(\{x\}),$$

Motivation for the discrete TMP

Main motivation (for me)

- Moment problem for point processes
- Complex systems, Material science, Statistical mechanics

Point processes

Let R be a Riemannian manifold.

$$K := \left\{ \sum_{i \in I} \delta_{r_i} \in \mathcal{D}'(R) : \ I \text{ countable and } r_i \in R \in \right\} \subset \mathcal{D}'(R)$$

A measure μ on K is called a point process.

- K is infinite dimensional $d = \infty$.
- all element of K are Radon measures.
- Interpretation: μ is probability to find point configuration η.

Relation to $\mathbb{N}_0^d$-TMP

For $\eta = \sum_{i \in I} \delta_{r_i} \in K$, define

$$N_A(\eta) := \eta(A) = \text{number of points in } \eta \text{ which are in } A$$

By definition $N_A : K \to \mathbb{N}_0$.

Finite dimensional distribution of μ

- One-dimensional distributions μ_A:

$$\mu_A(C) := \mu(\{\eta \ : \ N_A(\eta) \in C\})$$

  Push-forward of μ w.r.t. N_A.
- Two-dimensional distribution μ_{A_1,A_2} given by

$$\mu_{A_1,A_2}(C_1 \times C_2) := \mu(\{\eta \ : \ N_{A_i}(\eta) \in C_i\})$$

- and so on

- Support of μ_A is $\mathbb{N}_0$.
- Support of μ_{A_1,A_2} is $\mathbb{N}_0 \times \mathbb{N}_0$.
- and so on

General convex analysis

Generalized Tchakaloff Thm (Richter-Bayer-Teichmann)

$\swarrow$

m has a $\mathbb{N}_0-$representing measure

$\Updownarrow$

$\exists\, N \in \mathbb{N}$ s.t. m has a $\{0, 1, \ldots, N\}-$representing measure

criterion to solve $\{0, 1, \ldots, N\}$-TMP

$\searrow \quad \swarrow$

criterion to solve $\mathbb{N}_0$-TMP depending on (unbeknown) N

Solving $\{0, 1, \dots, N\}$-TMP

Fix $n, N \in \mathbb{N}$ s.t. $N \geq n$.

Aim:

Characterize the set S_N of all $n-$tuple admitting $m = (m_1, \dots, m_n) \in \mathbb{R}^n$ a $\{0, 1, \dots, N\}$-representing probability measures.

- Every $\{0, 1, \dots, N\}$-representing probability for m is a convex combination of probabilities concentrated at $k = 0, 1, \dots, N$.
- Hence S_N is the convex hull of $A_N := \{(k, k^2, \dots, k^n) | k = 0, 1, \dots, N\}$
- Classical convex analysis yields, that S_N is the intersection of finitely many closed half-spaces H containing A_N whose

$$\text{bounding hyperplanes } \partial H \leftrightarrow \begin{pmatrix} \partial H \text{ contains at least } n \text{ points} \\ \text{from } A_N \end{pmatrix}$$

$$\text{bounding hyperplanes } \partial H \leftrightarrow \begin{pmatrix} \text{polynomials of degree } n \\ \text{with leading coefficient } \pm 1 \\ n \text{ distinct roots in } \{0, 1, \dots, N\} \\ \text{nonnegative on } \{0, 1, \dots, N\} \end{pmatrix}$$

Solving $\{0, 1, \ldots, N\}$-TMP

$$\mathcal{P}_{n,N} := \begin{pmatrix} \text{polynomials of degree } n \\ \text{with leading coefficient } +1 \\ n \text{ distinct roots in } \{0, 1, \ldots, N\} \\ \text{nonnegative on } \{0, 1, \ldots, N\} \end{pmatrix}$$

- If $n = 2j$ even, any $P \in \mathcal{P}_{n,N}$ is of the form:

$$P(x) = (x - k_1)(x - (k_1 + 1)) \ldots (x - k_j)(x - (k_j + 1))$$

with zeros $k_1 < k_1 + 1 < k_2 < k_2 + 1 < \ldots < k_j$ in $\{0, 1, \ldots, N\}$.

- If $n = 2j + 1$ odd, any $P \in \mathcal{P}_{n,N}$ is of the form:

$$P(x) = x(x - k_1)(x - (k_1 + 1)) \ldots (x - k_j)(x - (k_j + 1))$$

with zeros $0 < k_1 < k_1 + 1 < k_2 < k_2 + 1 < \ldots < k_j$ in $\{0, 1, \ldots, N\}$.

$$\mathcal{Q}_{n,N} := \begin{pmatrix} \text{polynomials of degree } n \\ \text{with leading coefficient } -1 \\ n \text{ distinct roots in } \{0, 1, \ldots, N\} \\ \text{nonnegative on } \{0, 1, \ldots, N\} \end{pmatrix} = \{P(x)(N - x) | P \in \mathcal{P}_{n-1,N-1}\}$$

From $\{0,1,\dots,N\}$-TMP to $\mathbb{N}_0$-TMP

Generalized Tchakaloff Thm (Richter-Bayer-Teichmann)

↙

m has a $\mathbb{N}_0-$representing probability
$$\Updownarrow$$
$\exists N \in \mathbb{N}$ s.t. m has a $\{0,1,\dots,N\}-$representing probability

criterion to solve $\{0,1,\dots,N\}$-TMP

m has a $\{0,1,\dots,N\}$-repr. prob.
$$\Updownarrow$$
$$L_m(p) \geq 0, \ \forall p \in \mathcal{P}_{n,N} \cup \mathcal{Q}_{n,N}$$

↘ ↙

first criterion to solve $\mathbb{N}_0$-TMP

m has a $\mathbb{N}_0-$representing probability
$$\Updownarrow$$
$\exists N \in \mathbb{N}$ s.t.
$$L_m(p) \geq 0 \text{ for all } p \in \mathcal{P}_{n,N} \cup \mathcal{Q}_{n,N}$$

Problem: How to identify or get rid of N?

N independent condition

- Note that $\mathcal{P}_{n,N} \subset \mathcal{P}_{n,N+1}$
- Define $\mathcal{P}_n := \bigcup_{N\in\mathbb{N}} \mathcal{P}_{n,N}$.

$$\Big(m \text{ has a } \mathbb{N}_0\text{-representing measure } \Big) \Rightarrow \Big(L_m(p) \geq 0 \quad \forall p \in \mathcal{P}_n. \Big)$$

- Recall

$$\big(\, m \text{ has a } \mathbb{N}_0\text{-repr. prob. } \big) \Leftrightarrow \left(\begin{array}{l} m \text{ has a } \{0,1,\dots,N\}\text{-repr. prob.} \\ \text{for some } N \text{ large enough} \end{array} \right)$$

- The condition

$$\Big(L_m(p) \geq 0 \quad \forall p \in \mathcal{Q}_{n,M} \Big) \Leftrightarrow \Big(L_m((M-x)p) \geq 0 M L_m(p) \geq L_m(xp) L_m(p) \geq \frac{1}{M}$$

which implies that

$$\boxed{L_m(p) \geq 0, \forall p \in \mathcal{P}_{n-1}} \quad \text{and} \quad \boxed{\text{if } L_m(p) = 0 \text{ for some } p \in \mathcal{P}_{n-1}, \text{ then } L_m(xp) = 0}$$

Necessary conditions

$$m \text{ has a } \mathbb{N}_0-\text{repr. prob.} \Rightarrow \begin{array}{l} L_m(p) \geq 0, \ \forall p \in \mathcal{P}_n \cup \mathcal{P}_{n-1} \\ \text{if } L_m(p) = 0 \text{ for some } p \in \mathcal{P}_{n-1} \text{ then } L_m(xp) = 0 \end{array}$$

Theorem (Infusino, K., Lebowitz, Speer, 2017)

m has a $\mathbb{N}_0$*−repr. prob.* $\Leftrightarrow$ $\begin{array}{l} L_m(p) \geq 0, \ \forall p \in \mathcal{P}_n \cup \mathcal{P}_{n-1} \\ \text{if } L_m(p) = 0 \text{ for some } p \in \mathcal{P}_{n-1} \text{ then } L_m(xp) = 0 \end{array}$

Moreover, non of the conditions can be dropped.

Proof of $\Leftarrow$: One need to derive an a priori bound on N using only the above conditions not realizability.

Previous results:

- **Karlin and Studden 1966** on $K = \mathbb{N}_0 \cup \{\infty\}$.
  Solvability condition depending on an unknown parameter
- The best one could *hope* to obtain using *Semi-algebraic techniques* is conditions

$$L_m\big((x-k)(x-(k+1))p^2\big) \geq 0 \qquad \forall p \text{ polynomial and } \forall k \in \mathbb{N}_0$$

Challenge:
Can one reduce the conditions further by making them m dependent?

m dependent conditions: what was known

Case $n = 1$:

$$\Big(m = (m_1) \text{ is realizable}\Big) \Leftrightarrow \Big(m_1 \geq 0\Big)$$

Case $n = 2$: (Yamada 1961)

$$\Big(m = (m_1, m_2) \text{ is realizable}\Big) \Rightarrow \big(\; m_2 - (m_1)^2 \geq \lfloor m_1 \rfloor \lceil m_1 \rceil \;\big)$$

Case $n = 2$: (K., Lebowitz, Speer 2009)

$$\Big(m = (m_1, m_2) \text{ is realizable}\Big) \Leftrightarrow \left(\begin{array}{l} m_1 > 0;\; m_2 - (m_1)^2 \geq \lfloor m_1 \rfloor \lceil m_1 \rceil \\ \text{or } m_1 = 0 \text{ and } m_2 = 0 \end{array} \right)$$

Kwerel 1975, Prekopa et al. 1986: $K = \{0, 1, \ldots, N\}$

some explicit (necessary) conditions for $n = 2, 3$
but no explicit conditions for $n \geq 4$.

m dependent conditions

We partition the set of all $m := (m_1, \ldots, m_n) \in \mathbb{R}^n$ realizable on $\mathbb{N}_0$ into:

(i) $m := (m_1, \ldots, m_n)$ is **B-realisable** if

$$\exists p \in \bigcup_{k=1}^{n} \mathcal{P}_k \text{ with } L_m(p) = 0$$

(ii) otherwise m is **I -realisable**, i.e.

$$\forall p \in \bigcup_{k=1}^{n} \mathcal{P}_k \text{ one has } L_m(p) > 0$$

Main Theorem (Infusino, K., Lebowitz, Speer, 2017)

Let $m := (m_1, \ldots, m_n) \in \mathbb{R}^n$.

If $(m_1, \ldots, m_{n-1})$ is I-realisable, then $\exists\, p_m^{(n)} \in \mathcal{P}_n$ s.t.

$$L_m(q) \geq L_m(p_m^{(n)}), \quad \forall\, q \in \mathcal{P}_n$$

We call such a $p_m^{(n)}$ a **minimizing polynomial** for m.

$p_m^{(n)}$ does not depend on m_n

Challenge: How to find $p_m^{(n)}$

Finding $p_m^{(2)}$: case $n = 2$

Let $m = (m_1, m_2) \in \mathbb{R}^2$ be such that m_1 is I-realisable, i.e. $m_1 > 0$.

$$\mathcal{P}_2 := \left\{ t_k(x) := (x-k)(x-k-1) | k \in \mathbb{N}_0 \right\}$$

Case $n = 2$, $m_1 > 0$

$$\Big((m_1, m_2) \text{ realisable on } \mathbb{N}_0 \Big) \Leftrightarrow m_2 - (m_1)^2 \geq \lfloor m_1 \rfloor \lceil m_1 \rceil$$

Case: n=2

$$P_m^{(2)}(x) = \big(x-k\big)\big(x-(k+1)\big) \text{ for } k = \lfloor m_1 \rfloor$$

corresponds to condition

$$m_2 - (m_1)^2 \geq \lfloor m_1 \rfloor \lceil m_1 \rceil .$$

Connection to Stieltjes TMP

Case $n = 2$, $m_1 > 0$

$$\Big((m_1, m_2) \text{ realisable on } [0, +\infty) \Big) \Leftrightarrow \begin{vmatrix} 1 & m_1 \\ m_1 & m_2 \end{vmatrix} \geq 0 \Leftrightarrow m_2 - m_1^2 \geq 0$$

Connection between $\mathbb{N}_0$−TMP & $[0,+\infty)$−TMP

Let $m = (m_1, \ldots, m_{n-1}, m_n) \in \mathbb{R}^n$ s.t. $(m_1, \ldots, m_{n-1})$ is I-realizable on $\mathbb{N}_0$

$\Downarrow$

$(m_1, \ldots, m_{n-1})$ is I-realizable on $[0, +\infty)$

Take the smallest $\hat{m}_n \in \mathbb{R}$ s.t. $\hat{m} := (m_1, \ldots, m_{n-1}, \hat{m}_n)$ is realizable on $[0,+\infty)$

Curto-Fialkow 1991

$\Downarrow$

- $\hat{m}$ is B-realizable on $[0, +\infty)$
- $\hat{m}$ has a unique $[0, +\infty)$−representing probability ν
- the support of ν is given by the zeros of a polynomial determined only by $(m_1, \ldots, m_{n-1})$.

$$n = 2: \quad supp(\nu) = \{m_1\}$$
$$n = 3: \quad supp(\nu) = \{0, m_2/m_1\}$$

$$n = 2: \quad supp(\nu) = \{m_1\}, \qquad \text{zeros of } p_m^{(2)} = \{\lfloor m_1 \rfloor, \lfloor m_1 \rfloor + 1\};$$
$$n = 3: \quad supp(\nu) = \{0, m_2/m_1\}, \quad \text{zeros of } p_m^{(2)} = \{0, \lfloor m_2/m_1 \rfloor, \lfloor m_2/m_1 \rfloor + 1\}.$$

Conjecture

The zeros of $p_m^{(n)}$ are the nearest integers to the points in $supp(\nu)$

Finding $p_m^{(n)}$: case $n \geq 4$

Let $m = (m_1, \ldots, m_{n-1}, m_n) \in \mathbb{R}^n$ s.t. $(m_1, \ldots, m_{n-1})$ is I-realizable on $\mathbb{N}_0$.

Theorem*

At least one pair of zeros of $p_m^{(2)}$ consists of the nearest integers to a point $y_i \in supp(\nu)$, i.e. $\exists y_i \in supp(\nu)$ s.t. $p_m^{(n)}(\lfloor y_i \rfloor) = 0 = p_m^{(n)}(\lceil y_i \rceil)$.

Notation Take the smallest $\tilde{m}_n \in \mathbb{R}$ s.t. $\tilde{m} := (m_1, \ldots, m_{n-1}, \tilde{m}_n)$ is realizable on $\mathbb{N}_0$.

$\mathcal{S}_m := supp(\text{unique } \mathbb{N}_0\text{−representing probability for } \tilde{m}) \subseteq$ zero set of $p_m^{(n)}$

Sketch of algorithm to find $p_m^{(n)}$ for $n \geq 4$

1. use Curto-Fialkow '91 to compute $supp(\nu) = \begin{cases} (y_1, \ldots, y_{\lfloor \frac{n}{2} \rfloor}) & \text{if } n \text{ even} \\ (0, y_1, \ldots, y_{\lfloor \frac{n}{2} \rfloor}) & \text{if } n \text{ odd} \end{cases}$
2. For each y_j in $supp(\nu)$ construct $M_j(m)$ in a particular way such that $\mathcal{S}_m^{(n)} = \mathcal{S}_{M_j(m)}^{(n-2)} \sqcup \{\lfloor y_i \rfloor, \lfloor y_i \rfloor + 1\}$.
3. Construct inductively $\mathcal{S}_{M_j(m)}^{(n-2)}$.
4. Construct for each of the choices a polynomial Q
5. p is the Q such that $L_m(Q)$ is minimal.

we do not know *a priori* the right y_i, so in the worst case we need $\lfloor \frac{n}{2} \rfloor!$ stages.

Explicit formulas for $n = 4$

Suppose (m_1, m_2, m_3) is I-realizable, i.e. $\begin{cases} m_1 > 0 \\ m_2 - m_1^2 > \lfloor m_1 \rfloor \lceil m_1 \rceil \\ m_3 m_1 - m_2^2 \geq \left\lfloor \frac{m_2}{m_1} \right\rfloor \left\lceil \frac{m_2}{m_1} \right\rceil \ m_1^2 \end{cases}$

Curto-Fialkow 1991 $\Rightarrow supp(\nu) = \{y_1, y_2\}$ with y_1, y_2 solutions of:

$$\begin{vmatrix} 1 & m_1 \\ m_1 & m_2 \end{vmatrix} x^2 - \begin{vmatrix} 1 & m_1 \\ m_2 & m_3 \end{vmatrix} x + \begin{vmatrix} m_1 & m_2 \\ m_2 & m_3 \end{vmatrix} = 0$$

Define $Y_1 := \lfloor y_1 \rfloor, Y_2 := \lfloor y_2 \rfloor$ and

$$t_1 = \frac{m_3 - (2Y_2+1)m_2 + Y_2(Y_2+1)m_1}{m_2 - (2Y_2+1)m_1 + Y_2(Y_2+1)m_0}, \qquad T_1 = \lfloor t_1 \rfloor;$$

$$t_2 = \frac{m_3 - (2Y_1+1)m_2 + Y_1(Y_1+1)m_1}{m_2 - (2Y_1+1)m_1 + Y_1(Y_1+1)m_0}, \qquad T_2 = \lfloor t_2 \rfloor.$$

Take $p_m^{(4)}(x) = (x - T_1)(x - T_1 - 1)(x - T_2)(x - T_2 - 1)$,
and compute the associated condition

$$L_m(p_m^{(4)}) \geq 0$$

Final remarks and open problems

Further remarks

- our results can be easily adapted to solve the $\mathbb{M}-$TMP when $\mathbb{M} \subset \mathbb{R}$ is a general discrete set which is bounded below:

$$\begin{aligned} \lfloor y \rfloor & \rightsquigarrow \text{the largest element of } \mathbb{M} \text{ not greater than } y \\ \lfloor y \rfloor + 1 & \rightsquigarrow \text{the smallest element of } \mathbb{M} \text{ larger than } y \end{aligned}$$

- generalization to any unbounded discrete subset of $\mathbb{R}$, e.g. $\mathbb{Z}$
- $K = \mathbb{Z}$ can be treated in the same way
- and generalization as above

TMP for $K = \mathbb{Z}_0^d$ and $n = 2$

Three fundamental points:

Classify polynomials non-negative on $\mathbb{Z}_0^d$.

- All non-negative polynomials on $\mathbb{Z}^2$ of degree 2 are squares.
- We have a complete classification of these polynomials
- Done for $d = 2$. True for $d \leq 5$.
- Unclassified for $d > 5$: key words *L-polytopes, empty spheres* [Voronoi], [Delone], [Ryshkov], [Erdahl '92].

Identify minimal set of polynomials

- Additional spurious conditions appear.
- Done for $d = 2$. Seems doable for all n.

Identify p_m

- In $d = 2$ there exists an algorithm which will give p_m.
- Spurious solutions are the root of complications.
- Something radical new needed like distance to spurious solutions.

Thank you for you attention