Bogoliubov generating functionals for interacting particle systems in the continuum

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Framework

The space of (locally finite) configurations:

$$\Gamma := \{\gamma \subset \mathbb{R}^d : |\gamma \cap \Lambda| < \infty, \forall \text{ compact } \Lambda \subset \mathbb{R}^d\}$$

Each $\gamma \in \Gamma$ is identified with a Radon measure:

$$\Gamma \ni \gamma \mapsto \sum_{x \in \gamma} \delta_x \quad \text{(configuration)}$$

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Interpretation:

- ✔ Mathematical Physics: particles
- ✔ Ecology: individuals of a population
- ✔ Biology: cells
- ✔ Economics: agents

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Stochastic Dynamics - Randomly particles may...

- ✔ ... appear (or born)
- ✔ ... disappear (or die)
- ✔ ... move to a free site (continuously or hopping)

Example: Birth-and-Death models

$$(LF)(\gamma) = \sum_{x\in\gamma} d(x,\gamma\setminus x)\big(F(\gamma\setminus x) - F(\gamma)\big) + \int_{\mathbb{R}^d} b(x,\gamma)\big(F(\gamma\cup x) - F(\gamma)\big)\,dx$$

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- **Glauber dynamics:**

$$d \equiv 1, \quad b(x,\gamma) = z\exp\Big(-\sum_{y\in\gamma}\phi(x-y)\Big)$$

Example: Hopping particle systems (Conservative Models)

$$(LF)(\gamma) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} c(x, y, \gamma) \left(F(\gamma \setminus x \cup y) - F(\gamma)\right) dy,$$

- **Kawasaki dynamics:**

$$c(x, y, \gamma) = a(x - y) \exp\Big(-\sum_{y\in\gamma} \phi(x - y)\Big)$$

Stochastic Dynamics

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- **Fokker-Planck equation:**

$$\boxed{\frac{d}{dt}\mu_t = L^*\mu_t}$$

$$\left(\frac{d}{dt}\int_\Gamma F(\gamma)\,d\mu_t(\gamma) = \int_\Gamma (LF)(\gamma)d\mu_t(\gamma)\right)$$

An alternative approach

Assume that for each $t \geq 0$ there is a family

$$k_t^{(n)} : (\mathbb{R}^d)^n \to \mathbb{R}_0^+, \quad n \in \mathbb{N}$$

such that

$$\int_\Gamma \sum_{\{x_1,\ldots,x_n\}\subset\gamma} G(x_1,\ldots,x_n)\, d\mu_t(\gamma) = \frac{1}{n!}\int_{(\mathbb{R}^d)^n} G(x_1,\ldots,x_n) k_t^{(n)}(x_1,\ldots,x_n)\, dx_1 \ldots dx_n$$

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(**Correlation functions or n-factorial moments of μ_t**)

Case $n = 1$:

$$\int_\Gamma |\gamma \cap \Lambda|\, d\mu_t(\gamma) = \int_\Gamma \sum_{x\in\gamma} 1\!\!1_\Lambda(x)\, d\mu_t(\gamma) = \int_\Lambda k_t^{(1)}(x)\, dx$$

Now take $G(x_1,\ldots,x_n)=\theta(x_1)\ldots\theta(x_n), n\in\mathbb{N}$, and sum n:

$$\int_\Gamma \underbrace{\sum_{n=0}^{\infty}\sum_{\{x_1,\ldots,x_n\}\subset\gamma} G(x_1,\ldots,x_n)}_{\prod_{x\in\gamma}(1+\theta(x))}\, d\mu_t(\gamma)$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\int_{(\mathbb{R}^d)^n} k_t^{(n)}(x_1,\ldots,x_n)\theta(x_1)\ldots\theta(x_n)\,dx_1\ldots dx_n$$

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Bogoliubov Generating Functional **(corresponding to** μ**)**:

$$\boxed{B_\mu(\theta):=\int_\Gamma\prod_{x\in\gamma}(1+\theta(x))d\mu(\gamma)}$$

Analyticity

- ✔ Assume that B is an entire functional on $L^1(\sigma)$ $(\sigma = dx)$

$$\boxed{B(\theta_0+\theta)=\sum_{n=0}^{\infty}\frac{1}{n!}d^nB(\theta_0;\theta)}$$

$$d^nB(\theta_0;\theta_1,\ldots,\theta_n):=\frac{\partial^n}{\partial z_1\ldots\partial z_n}B\left(\theta_0+\sum_{i=1}^{n}z_i\theta_i\right)\Big|_{\substack{z_i=0\\1\leq i\leq n}}$$

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$$\underline{\underline{\int_{(\mathbb{R}^d)^n} \frac{\delta^n B(\theta_0)}{\delta\theta_0(x_1)\ldots\delta\theta_0(x_n)} \prod_{i=1}^{n} \theta_i(x_i)\, d\sigma^{\otimes n}(x_1,..,x_n)}}$$

(The n-th variational derivative of B at the point θ_0)

Analyticity

$$B_\mu(\theta) = \int_\Gamma \prod_{x\in\gamma} (1+\theta(x))\, d\mu(\gamma)$$

Assume that B_μ is entire on $L^1(\sigma)$. Then, k_μ exists and for $\theta_0 = 0$,

$$d^n B_\mu(\theta_0;\theta_1,\ldots,\theta_n) := \frac{\partial^n}{\partial z_1 \ldots \partial z_n} B_\mu\left(\theta_0 + \sum_{i=1}^n z_i\theta_i\right)\Big|_{\substack{z_i=0\\1\le i\le n}} =$$

$$\int_{(\mathbb{R}^d)^n} \underbrace{\frac{\delta^n B_\mu(\theta_0)}{\delta\theta_0(x_1)\ldots\delta\theta_0(x_n)}}_{k_\mu^{(n)}(x_1,\ldots,x_n)} \prod_{i=1}^n \theta_i(x_i) d\sigma^{\otimes n}(x_1,..,x_n)$$

$$\int_\Gamma \sum_{\{x_1,\ldots,x_n\}\subset\gamma} G(x_1,\ldots,x_n)\,d\mu_t(\gamma)$$
$$= \frac{1}{n!}\int_{(\mathbb{R}^d)^n} G(x_1,\ldots,x_n)k_t^{(n)}(x_1,\ldots,x_n)\,dx_1\ldots dx_n$$

(**Correlation functions or n-factorial moments of μ_t**)

Summing over n:

$$\int_\Gamma \underbrace{\sum_{n=0}^{\infty} \sum_{\{x_1,\ldots,x_n\}\subset\gamma} G(x_1,\ldots,x_n)}_{:=(KG)(\gamma)} \, d\mu_t(\gamma)$$

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$$= \int_{\Gamma_0} G(\eta) k_t(\eta)\, d\lambda(\eta),$$

where

$$\Gamma_0 := \bigsqcup_{n=0}^{\infty} \underbrace{\{\gamma \in \Gamma : |\gamma| = n\}}_{\{x_1,\dots,x_n\}}, \quad \lambda := \sum_{n=0}^{\infty} \frac{1}{n!} dx^{\otimes n}$$

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$$= \int_{\Gamma_0} G(\eta) k_t(\eta)\, d\lambda(\eta).$$

As a result

$$\int_\Gamma (KG)(\gamma)\, d\mu_t(\gamma) = \int_{\Gamma_0} G(\eta) k_t(\eta)\, \lambda(\eta)$$

$$\frac{d}{dt}\int_{\Gamma} F(\gamma)\, d\mu_t(\gamma) = \int_{\Gamma} (LF)(\gamma)\, d\mu_t(\gamma)$$

$$\frac{d}{dt}\int_\Gamma \underbrace{F(\gamma)}_{(KG)}\,d\mu_t(\gamma) = \int_\Gamma (LF)(\gamma)\,d\mu_t(\gamma)$$

$$\Downarrow$$

$$\frac{d}{dt}\int_\Gamma (KG)(\gamma)\,d\mu_t(\gamma) = \int_\Gamma (KK^{-1}LKG)(\gamma)\,d\mu_t(\gamma)$$

Stochastic Dynamics

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$$\Downarrow$$

$$\frac{d}{dt}\underbrace{\int_\Gamma (KG)(\gamma)\, d\mu_t(\gamma)}_{\int_{\Gamma_0} G(\eta)k_t(\eta)\,\lambda(\eta)} = \underbrace{\int_\Gamma (KK^{-1}LKG)(\gamma)d\mu_t(\gamma)}_{\int_{\Gamma_0} (K^{-1}LKG)(\eta)k_t(\eta)\,\lambda(\eta)}$$

$$\boxed{\int_\Gamma (KG)(\gamma)\, d\mu_t(\gamma) = \int_{\Gamma_0} G(\eta)k_t(\eta)\,\lambda(\eta)}$$

Consequences of

$$\underbrace{\frac{d}{dt}\int_{\Gamma}(KG)(\gamma)\,d\mu_t(\gamma)}_{\int_{\Gamma_0} G(\eta)k_t(\eta)\,\lambda(\eta)} = \underbrace{\int_{\Gamma}(KK^{-1}LKG)(\gamma)d\mu_t(\gamma)}_{\int_{\Gamma_0}(K^{-1}LKG)(\eta)k_t(\eta)\,\lambda(\eta)}$$

For $\hat{L} := K^{-1}LK$:

- ✔ Correlation functions $\frac{\partial}{\partial t}k_t = \hat{L}^* k_t$

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For $\hat{L} := K^{-1}LK$:

- ✔ Correlation functions $\frac{\partial}{\partial t}k_t = \hat{L}^* k_t$
- ✔ Bogoliubov functionals $(G(\eta) = \prod_{x\in\eta}\theta(x) =: e_\lambda(\theta,\eta))$

$$\boxed{\frac{\partial}{\partial t}B_t(\theta) = \int_{\Gamma_0} (\hat{L}e_\lambda(\theta))(\eta)k_t(\eta)\,d\lambda(\eta) =: (\tilde{L}B_t)(\theta)}$$

Example: Glauber Dynamics

$$\frac{\partial}{\partial t} B_t = \tilde{L} B_t$$

$$(\tilde{L}B)(\theta) = -\int_{\mathbb{R}^d} dx\, \theta(x) \left(\frac{\delta B(\theta)}{\delta \theta(x)} - zB\big((1+\theta)\big(e^{-\phi(x-\cdot)} - 1\big) + \theta\big) \right)$$

References

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