On the Random Moment Problem

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based on joint work with Holger Dette and Dominik Tomecki

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What is the "Random Moment Problem"?

- Classical moment problem (on $\mathbb{R}$): Which sequences are sequences of moments of probability measures? Aim: Describe **all** moment sequences.
- Aim of random moment problem: Describe **typical** moment sequences.
- Idea: Consider probability distribution on "moment sequences" and study their (probabilistic) behavior!
- We consider moments of (probability) measures on $E = [0, 1]$ (Hausdorff-MP), $E = \mathbb{R}_+$ (Stieltjes-MP) and $E = \mathbb{R}$ (Hamburger-MP).

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- Approach via n-th moment spaces: n-th moment space for $E = [0,1], \mathbb{R}_+, \mathbb{R}$

$$\mathcal{M}_n(E) := \left\{ (m_1, \ldots, m_n) \, : \, m_j = \int x^j \mu(dx), \, 1 \leq j \leq n, \, \mu \in \mathcal{P}(E) \right\},$$

$\mathcal{P}(E)$ set of Borel probability measures on E with existing moments.

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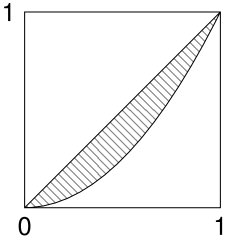
The moment space $\mathcal{M}_2([0,1])$



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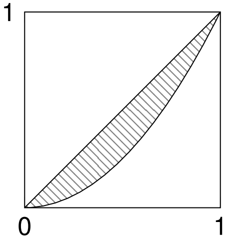

- Approach: Equip $\mathcal{M}_n(E)$ with probability distribution and study asymptotic ($n \to \infty$) behavior.

Uniform distribution on $\mathcal{M}_n([0,1])$

- Chang, Kemperman, Studden '93: If $(m_1^{(n)}, \ldots, m_n^{(n)}) \in \mathcal{M}_n([0,1])$ uniformly distributed, then as $n \to \infty$ for any fixed l

$$(m_1^{(n)}, \ldots, m_l^{(n)}) \to (m_1(\mu_{[0,1]}), \ldots, m_l(\mu_{[0,1]})),$$

in probability, where $m_j(\mu_{[0,1]})$ is the j-th moment of the measure

$$\mu_{[0,1]}(dx) := \frac{1}{\pi\sqrt{x(1-x)}} 1_{[0,1]}(x)dx. \quad \text{arcsine distribution}$$

- Fluctuations $\sqrt{n}\left((m_1^{(n)}, \ldots, m_l^{(n)}) - (m_1(\mu_{[0,1]}), \ldots, m_l(\mu_{[0,1]}))\right)$, are Gaussian: Strong dependence between coordinates $m_1, \ldots, m_n$.
- $\mathcal{M}_n([0,1])$ interesting convex body (far from being isotropic).

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Canonical Moments

- Proof of Chang,Kemperman,Studden uses canonical moments (Skibinsky): Parametrize moment space $\mathcal{M}_n([0,1])$ by canonical moments $y_1, \ldots, y_n$,

$$y_j := \frac{m_j - m_j^-}{m_j^+ - m_j^-},$$

  where $[m_j^-, m_j^+]$ is the moment range given $m_1, \ldots, m_{j-1}$.
- Canonical moments are relative positions in the moment space.
- Map

$$\mathbf{m}_n := (m_1, \ldots, m_n) \mapsto \mathbf{y}_n := (y_1, \ldots, y_n)$$

  is diffeomorphism from $\mathcal{M}_n([0,1])^\circ$ onto $(0,1)^n$.

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Properties of uniform distribution on $\mathcal{M}_n([0,1])$

- Jacobian of the map $\mathbf{y}_n := (y_1, \dots, y_n) \mapsto \mathbf{m}_n := (m_1, \dots, m_n)$ is

$$\left| \det \left[\frac{\partial \mathbf{m}_n(\mathbf{y}_n)}{\partial \mathbf{y}_n} \right] \right| = \prod_{j=1}^{n} (y_j(1-y_j))^{n-j} = e^{\sum_{j=1}^{n} (n-j) \log(y_j(1-y_j))}.$$

- Thus: If $\mathbf{m}_n^{(n)} = (m_1^{(n)}, \dots, m_n^{(n)})$ is uniformly distributed on $\mathcal{M}_n([0,1])$, then

  1. $y_1^{(n)}, \dots, y_l^{(n)}$ are (stochastically) independent, $l = 1, \dots, n$!
  2. $y_1^{(n)}, \dots, y_l^{(n)}$ are nearly identically distributed if $n \gg l$!
  3. $y_j^{(n)}$ is beta$(n-j+1, n-j+1)$-distributed.

- Question: Properties 1 and 2 meaningful. What if property 3 is dropped?

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Canonical Coordinates for $\mathcal{M}_n(\mathbb{R}_+)$ and $\mathcal{M}_n(\mathbb{R})$

- Cases $E = \mathbb{R}_+, \mathbb{R}$. Dette,Nagel'12 provide good parametrizations:
- For $\mathbf{m}_n \in \mathcal{M}_n(\mathbb{R}_+)$ define the canonical coordinates

$$y_j := \frac{m_j - m_j^-}{m_{j-1} - m_{j-1}^-}.$$

- Diffeom., product domain, Jacobian factorizes, (nearly) identical distr.
- For $\mathbf{m}_n \in \mathcal{M}_n(\mathbb{R})$ define the canonical coordinates

$$y_j := \begin{cases} \alpha_{(j+1)/2}, & j \text{ odd}, \\ \beta_{j/2}, & j \text{ even}, \end{cases}$$

$\alpha_j \in \mathbb{R}$, $\beta_j \in \mathbb{R}_+$ recurrence coeff. of orth. polynomials generated by $\mathbf{m}_n$.

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General distributions on $\mathcal{M}_n(E)$

- For $V_1, V_2 \in C^2(\mathbb{R})$ with super-logarithmic growth define distribution $\mathbb{P}_{n,E}$ on $\mathcal{M}_n(E)$ with density

$$P_{n,E}(\mathbf{m}_n) \propto \exp\Big[-n \sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} V_1(y_{2j-1}(\mathbf{m}_n)) - n \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} V_2(y_{2j}(\mathbf{m}_n)) \Big].$$

- Odd canonical coordinates $y_{2j-1}^{(n)}$ determined by V_1, even $y_{2j}^{(n)}$ by V_2.
- Under $\mathbb{P}_{n,E}$, canonical coordinates $y_j^{(n)}$ are independent, odd/even ones nearly identically distributed for $n \gg j$ (like uniform distribution on $\mathcal{M}_n([0,1])$).
- Question: How do the random ordinary moments $m_1^{(n)}, \ldots, m_l^{(n)}$ behave for $n \to \infty$, l fixed?

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Universality

Theorem (Dette-Tomecki-V., Electron. J. Probab. '18)

For $(m_1^{(n)}, \dots, m_n^{(n)}) \sim \mathbb{P}_{n,E}$*, l fixed and* generic V_1, V_2*, we have for* $n \to \infty$

$$(m_1^{(n)}, \dots, m_l^{(n)}) \xrightarrow{a.s.} (m_1(\mu_E), \dots, m_l(\mu_E)).$$

Limiting measure μ_E given by (a, b, y_1^*, y_2^* constants depending on E, V_1, V_2),

$$\begin{cases} \left(1 - \frac{y_1^*}{y_2^*}\right)_+ \delta_0 + \left(\frac{y_1^* + y_2^* - 1}{y_2^*}\right)_+ \delta_1 + \frac{\sqrt{(x-a)(b-x)}}{2\pi y_2^* x(1-x)} 1_{[a,b]}(x)dx & , \quad \text{if } E = [0,1], \\ \left(1 - \frac{y_1^*}{y_2^*}\right)_+ \delta_0 + \frac{1}{2\pi y_2^*} \frac{\sqrt{(x-a)(b-x)}}{x} 1_{[a,b]}(x)dx & , \quad \text{if } E = \mathbb{R}_+, \\ \frac{1}{2\pi y_2^*} \sqrt{(x-a)(b-x)} 1_{[a,b]}(x)dx & , \quad \text{if } E = \mathbb{R}. \end{cases}$$

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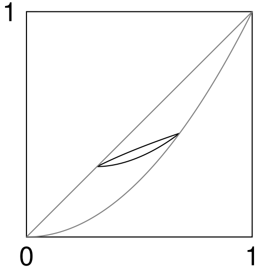
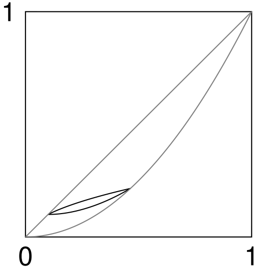


(a) Constraint $m_3 = 0.3125$



(b) Constraint $m_3 = 0.1$

Results for Constrained Random Moment Problems

- Consider probability distribution $\mathbb{P}_{n,E}$ on $\mathcal{M}_n^{\mathcal{C}}(E)$ (w.r.t. variables $\mathbf{m}_n^{\mathcal{C}} := (m_j, j = 1, \dots, n,\ j \neq i_1, \dots, i_k)$).
- Most interesting case: $i_k > k$, then some coordinates are dependent!

Theorem (Dette-Tomecki-Venker, Ann. Probab. '19+)

Given constraint $\mathcal{C}$, for generic V_1, V_2 with some growth conditions

$$(m_1^{(n)}, \dots, m_l^{(n)}) \xrightarrow{a.s.} (m_1(\mu_E^{\mathcal{C}}), \dots, m_l(\mu_E^{\mathcal{C}}))$$

where the universal measure $\mu_E^{\mathcal{C}}$ has the form

$$\frac{1}{p(x)}\mu_E(dx) + atoms. \qquad \textit{Bernstein-Szegő class}$$

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Special Case: Uniform Distribution on $\mathcal{M}_n^{\mathcal{C}}([0,1])$

$$\mu_{[0,1]}(dx) = \frac{1}{\pi\sqrt{x(1-x)}}dx \quad \text{arcsine distribution}$$

$$\mathcal{K}(\mu_{[0,1]} \mid \mu) := \begin{cases} \int \log \frac{d\mu_{[0,1]}}{d\mu} d\mu_{[0,1]} & \mu_{[0,1]} \ll \mu \\ \infty & \text{else} \end{cases}$$

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Conclusion

- If truncated random moment sequences are chosen such that the canonical coordinates are independent and (nearly) identically distributed, then typical moment sequences belong to the families of Kesten-McKay ($E = [0, 1]$), Marchenko-Pastur ($E = \mathbb{R}_+$) and semicircle ($E = \mathbb{R}$) distributions, respectively.
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Thank you very much for your attention!