Pair correlation estimates for the zeros of the zeta function via semidefinite programming

Andrés Chirre (IMPA)
Felipe Gonçalves (Universität Bonn)
David de Laat (TU Delft)

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Simple zeros of the zeta function

- The Riemann zeta function is the analytic continuation to $\mathbb{C} \setminus \{1\}$ of

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for} \quad \mathrm{Re}(s) > 1$$

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First goal

Find small $c \geq 1$ for which we can prove (under RH or GRH):

$$N^*(T) \leq (c + o(1))N(T)$$

Results for N^*

$$N^*(T) \le (c + o(1))N(T)$$

	c assuming RH	c assuming GRH
Montgomery	1.3333	
Cheer, Goldston	1.3275	
Goldston, Gonek, Özlük, Snyder		1.3262
New	1.3208	1.3155

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Source of improvements:
Optimizing over Schwartz functions instead of bandlimited functions

Cohn-Elkies bound

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Proof of the inequality when we restrict to Lattice packings:

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$$C = \frac{1}{\det(\Lambda)} \sum_{x \in \Lambda^*} \hat{f}(x) \geq \frac{1}{\det(\Lambda)}. \quad \square$$

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- Note 2: The above C really is the following double sum:

$$C = \lim_{r \to \infty} \frac{1}{\mathrm{vol}(B_r)} \sum_{x,y \in \Lambda \cap B_r} f(x-y)$$

LP bound for N^*

Lemma: Under RH we have

$$N^*(T) \le (c + o(1))N(T),$$

with

$$c = \inf\left\{\mathcal{Z}(f) : f \in S(\mathbb{R}),\ \hat{f}(0) = 1,\ \hat{f} \ge 0,\ f(x) \le 0 \text{ for } |x| \ge 1\right\},$$

where

$$\mathcal{Z}(f) = f(0) + 2\int_0^1 f(x)x\,dx$$

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Proof: Consider the double sum

$$C = \sum_{0<\gamma,\gamma'\leq T} w(\gamma-\gamma')\hat{f}\left(\frac{\log(T)}{2\pi}(\gamma-\gamma')\right), \quad w(u) = \frac{4}{4+u^2}$$

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Then, $C \geq N^*(T)$. By Fourier inversion we have

$$C = N(T)\int_{-\infty}^{\infty} f(x)F(x,Y)\,dx$$

with Montgomery's function

$$F(x,T) = \frac{1}{N(T)}\sum_{0<\gamma,\gamma'\leq T} T^{ix(\gamma-\gamma')}w(\gamma-\gamma')$$

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(Here we use the identity $T^{ix(\gamma-\gamma')} = e^{2\pi i \frac{\log(T)}{2\pi}(\gamma-\gamma')}$.)
We know that $F(x,T) \geq 0$, so

$$C \leq N(T)\int_{-1}^{1} f(x)F(x,Y)\,dx$$

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Under RH we have the information [Goldston-Montgomery 1987]:

$$F(x,T) = (T^{-2|x|}\log(T) + |x|)(1 + o(1)) \quad \text{uniformly for} \quad |x| \leq 1$$

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For large T, $T^{-2|x|}\log(T)$ becomes the Dirac delta at 0, so

$$\begin{aligned} C &\le N(T)\int_{-1}^{1} f(x)F(x,Y)\,dx \\ &\le N(T)\left(f(0) + 2\int_0^1 f(x)x\,dx + o(1)\right) \quad \square \end{aligned}$$

Optimization

- Cohn and Elkies restrict to radial Schwartz functions of the form

$$f(x) = p(\|x\|)e^{-\pi\|x\|^2} \quad \text{with} \quad p(u) = \sum_{k=0}^{d} p_k u^{2k}$$

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- The Fourier transform can be computed in terms of Legendre polys:

$$\hat{f}(x) = \sum_{k=0}^{d} p_k \frac{\pi^k}{k!} L_k^{n/2-1}(\pi\|x\|^2)e^{-\pi\|x\|^2}$$

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- Instead we use semidefinite programming to optimize over f as was also done for binary sphere packing [Vallentin-Oliveira-dL 2014]

Optimization

If $f(x)$ is of the form $p(\|x\|)e^{-\pi\|x\|^2}$, then

$$\hat{f} \geq 0 \quad \Leftrightarrow \quad q(u) := \sum_{k=0}^{d} p_k \frac{\pi^k}{k!} L_k^{n/2-1}(\pi u) \geq 0 \text{ for } u \geq 0$$

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$$\Leftrightarrow \quad q(u) = s_1(u) + u s_2(u), \text{where } s_1, s_2 \text{ are SOS polys}$$

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with $v(u)$ a vector whose ith entry is a polynomial of degree i

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We use the following identity to model the objective:

$$\int x^m e^{-\pi x^2}\, dx = -\frac{1}{2\pi^{m/2+1/2}} \Gamma\Big(\frac{m+1}{2}, \pi x^2\Big),$$

and use Arb to verify the results using ball arithmetic

Pair correlation

Montgomery's pair correlation conjecture:

$$N(x,T) := \sum_{\substack{0<\gamma,\gamma'\leq T \\ 0<\gamma'-\gamma\leq \frac{2\pi x}{\log T}}} 1 \quad \sim \quad N(T)\int_0^x \left(1-\frac{\sin(\pi y)^2}{(\pi y)^2}\right)\,dy$$

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Second goal

Find small $c > 0$ for which we can prove $N(T) = O(N(c,T))$
assuming RH or GRH and $N(T) \sim N^*(T)$

Pair correlation

Lemma: Suppose RH holds and $N(t) \sim N^*(T)$. Suppose $\varepsilon > 0$ and $f \in S(\mathbb{R})$ with $\hat{f}(0) = 0$, $\hat{f} \geq 0$, and

$$r(f) := \inf\{\lambda : f(x) \leq 0 \text{ for } |x| > \lambda\} < \infty$$

Then, $N(T) = O(N(\mathcal{P}(f) + \varepsilon, T))$, where

$$\mathcal{P}(f) = \inf\{\lambda > 0 : p_f(\lambda) > 0\},$$

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$$p_f(\lambda) = -1 + \frac{\lambda}{r(f)} + \frac{2r(f)}{\lambda} \int_0^{\frac{\lambda}{r(f)}} \hat{f}(x) x \, dx$$

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The optimization approach is similar to the approach mentioned earlier, with the addition of Brent's method and binary search to find the optimal sign changes

Pair correlation

Assuming RH (or GRH) and $N(T) \sim N^*(T)$ we have

$$N(T) = O(N(c,T))$$

	RH	GRH
Montgomery	0.68	
Goldston, Gonek, Özlük, Snyder	0.6072	0.5781
Carneiro, Chandee, Littmann, Milinovich	0.6068	
New	0.6039	0.5769

Thank you!