Carathéodory Numbers and Shape Reconstruction

The Multi-Dimensional Truncated Moment Problem

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Based on joint work with Mario Kummer and Konrad Schmüdgen

Introduction

- $\mathcal{A}$ - finite dimensional space of measurable functions on measurable space $\mathcal{X}$
  - Example: $\mathcal{A} = \mathbb{R}[x_1, \ldots, x_n]_{\leq d}$ with $\mathcal{X} = \mathbb{R}^n$
- $L : \mathcal{A} \to \mathbb{R}$ - linear functional
  - L moment functional iff $L(a) = \int_{\mathcal{X}} a(x) \, \mathrm{d}\mu(x)$ for all $a \in \mathcal{A}$
  - μ - representing measure
  - Example: $l_x(a) := a(x)$ point evaluation at $x \in \mathcal{X}$
  - Example: $L = \sum_{i=1}^{k} c_i \cdot l_{x_i}$ with $c_i > 0$

Richter's Theorem

Theorem (Richter 1957[1])

Let $\mathcal{A}$ be a finite dimensional space of measurable functions on a measurable space $\mathcal{X}$. Then every moment functional $L : \mathcal{A} \to \mathbb{R}$ has a k-atomic representing measure:

$$L = \sum_{i=1}^{k} c_i \cdot l_{x_i} \qquad (c_i > 0)$$

with l_{x_i} point evaluation at $x_i \in \mathcal{X}$ and $k \leq \dim \mathcal{A}$.

Previous/parallel works: Wald,[2] Rosenbloom,[3] Tchakaloff,[4] and Rogosinski[5]

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[1] H. Richter: *Parameterfreie Abschätzung und Realisierung von Erwartungswerten*, Bl. Dtsch. Ges. Versmath. **3** (1957), 147–161

[2] A. Wald: *Limits of distribution function determined by absolute moment and inequalities satisfied by absolute moments*, Trans. Amer. Math. Soc. **46** (1939), 280–306

[3] P. C. Rosenbloom: *Quelque classes de problème extrémaux. II*, Bull. Soc. Math. France **80** (1952), 183–215

[4] M. V. Tchakaloff: *Formules de cubatures méchaniques a coefficients non négatifs*, Bull. Sci. Math. **81** (1957), 123–134

[5] W. W. Rogosinski: *Moments of non-negative mass*, Proc. Roy. Soc. Lond. A **245** (1958)

Setting Records Straight

Richter's Theorem was known before 2006

- H. Richter: *Parameterfreie Abschätzung und Realisierung von Erwartungswerten*, Bl. Dtsch. Ges. Versmath. **3** (1957), 147–161
- J. H. B. Kemperman: *The General Moment Problem, a Geometric Approach*, Ann. Math. Stat. **39** (1968), 93–122
- J. H. B. Kemperman: *Moment problems with convexity conditions I*, Optimizing Methods in Statistics (J. S. Rustagi, ed.), Acad. Press, 1971, pp. 115–178
- J. H. B. Kemperman: *Geometry of the moment problem*, Proc. Sym. Appl. Math. **37** (1987), 16–53

THEOREM 1 Let $f_1, \dots, f_N$ be given real-valued Borel measurable functions on a measurable space Ω (such as $g_1, \dots, g_n$ and h on X). Let μ be a probability measure on Ω such that each f_i is integrable with respect to μ. Then there exists a probability measure μ' of finite support on Ω (i.e., having nonzero mass only at a finite number of points) satisfying

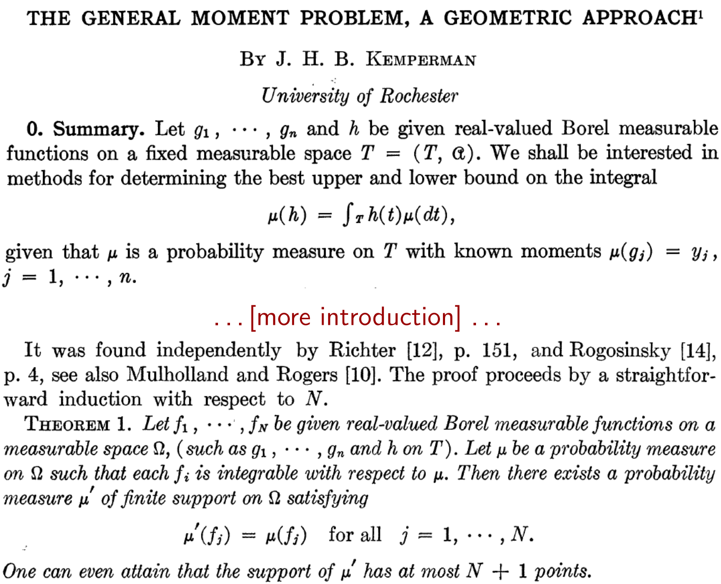
$$\int_\Omega f_i(t)\,\mu(dt) = \int_\Omega f_i(t)\,\mu'(dt),$$

all $i = 1, \dots, N$. □

One can even achieve that the support of μ' has at most $N+1$ points. So from now on we can talk only about finitely supported probability measures.

- C. F. Floudas, P. M. Pardalos (eds.): *Encyclopedia of optimization*, vol. 1, Kluwer Academic Publishers, Dordrecht, 2001, pp. 198–199.

J. H. B. Kemperman: *The General Moment Problem, a Geometric Approach*, Ann. Math. Stat. **39** (1968), 93–122:



THE GENERAL MOMENT PROBLEM, A GEOMETRIC APPROACH[1]

BY J. H. B. KEMPERMAN

University of Rochester

0. Summary. Let $g_1, \cdots, g_n$ and h be given real-valued Borel measurable functions on a fixed measurable space $T = (T, \mathfrak{A})$. We shall be interested in methods for determining the best upper and lower bound on the integral

$$\mu(h) = \int_T h(t)\mu(dt),$$

given that μ is a probability measure on T with known moments $\mu(g_j) = y_j$, $j = 1, \cdots, n$.

…[more introduction]…

It was found independently by Richter [12], p. 151, and Rogosinsky [14], p. 4, see also Mulholland and Rogers [10]. The proof proceeds by a straightforward induction with respect to N.

THEOREM 1. *Let $f_1, \cdots, f_N$ be given real-valued Borel measurable functions on a measurable space Ω, (such as $g_1, \cdots, g_n$ and h on T). Let μ be a probability measure on Ω such that each f_i is integrable with respect to μ. Then there exists a probability measure μ' of finite support on Ω satisfying*

$$\mu'(f_j) = \mu(f_j) \quad \text{for all} \quad j = 1, \cdots, N.$$

One can even attain that the support of μ' has at most $N + 1$ points.

More on the early history of Richter's Theorem

PdD + K. Schmüdgen: *The truncated moment problem: The moment cone*, arXiv1809.00584

Set of Atoms = Core Variety

- Core Variety introduced by L. Fialkow[6]
- Set of Atoms introduced by K. Schmüdgen[7,8]
  - Set of Atoms = Core Variety:

    $$L : \mathcal{A} \to \mathbb{R} \text{ moment functional} \Rightarrow \text{core variety} = \text{set of atoms } (\neq \emptyset)$$
  - intense studies of set of atoms already presented in Marsaille (Oct. 2015) and Oberwolfach (March 2017) by K. Schmüdgen in talks
- G. Blekherman + L. Fialkow:[9] for Hausdorff (topological) space

  Set of Atoms = Core Variety
- PdD + K. Schmüdgen:[10] Equivalence for measurable spaces
  - from geometric perspective by Karlin, Shapley (1953) and Kemperman (1968)

[6] L. Fialkow: *The core variety of a multisequence in the truncated moment problem*, J. Math. Anal. Appl. **456** (2017) 946–969

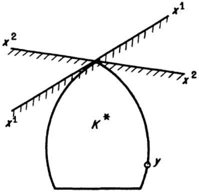
[7] PdD, K. Schmüdgen: *The truncated moment problem: Atoms, determinacy, and core variety*, J. Funct. Anal. **274** (2018), 3124–3148

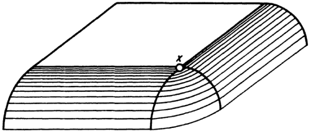
[8] K. Schmüdgen: *The Moment Problem*, Springer, 2018

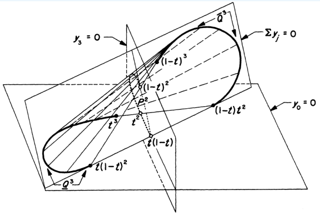
[9] GB + LF: *The core variety and representing measures in the truncated moment problem*, arXiv1804.0427

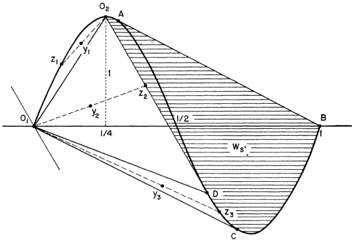
[10] PdD, K. Schmüdgen: *The multidimensional truncated moment problem: The moment cone*, arXiv1809.00584, Prop. 29 + Thm. 30

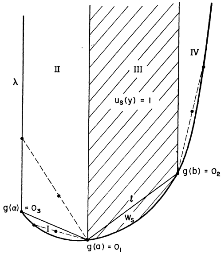
Karlin/Shapley's[11] + Kemperman[12] Geometric Approach

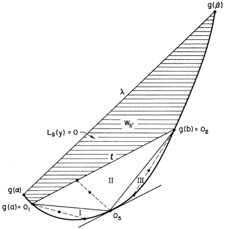












[11] S. Karlin and L. S. Shapley, *Geometry of moment spaces*, Mem. Amer. Math. Soc. **12** (1953)

[12] J. H. B. Kemperman: *The General Moment Problem, a Geometric Approach*, Ann. Math. Stat. **39** (1968), 93–122

History Lesson is over!

Back to New Stuff

Carathéodory Number: Definition and Bounds

Richter '57: Every moment functional $L : \mathcal{A} \to \mathbb{R}$ is of the form $\sum_{i=1}^{k} c_i \cdot l_{x_i}$

- Carathéodory number $\mathcal{C}_{\mathcal{A}}(L)$ of L = minimal k

Richter '57: $k \leq \dim \mathcal{A}$

- Carathéodory number $\mathcal{C}_{\mathcal{A}} = \max_L \mathcal{C}_{\mathcal{A}}(L)$

Bounds:

- Richter '57: $1 \leq \mathcal{C}_{\mathcal{A}}(L) \leq \mathcal{C}_{\mathcal{A}} \leq m$
- Thm:[13] $\mathcal{A} = \mathbb{R}[x_1, \dots, x_n]_{\leq d}$, $\mathcal{X} = \mathbb{R}^n$, then $\left\lceil \frac{1}{n+1}\binom{n+d}{n} \right\rceil \leq \mathcal{C}_{\mathcal{A}}$.
- Thm:[14] $\mathcal{X}$ and $\mathcal{A}$ "nice", then $\mathcal{C}_{\mathcal{A}} \leq \dim \mathcal{A} - 1$.
- Thm:[13] If $\mathcal{X}$ countable, then $\mathcal{C}_{\mathsf{A}} = \dim \mathcal{A}$.
- Thm:[13] If $a \geq 0$ with $\mathcal{Z}(a)$ finite, then $\dim \operatorname{span}\{l_x \mid x \in \mathcal{Z}(a)\} \leq \mathcal{C}_{\mathcal{A}}$.

[13] PdD, K. Schmüdgen: *The multidimensional truncated moment problem: Carathéodory numbers*, J. Math. Anal. Appl. **461** (2018) 1606–1638.

[14] PdD: *The multidimensional truncated moment problem: Gaussian and log-normal mixtures, their Carathéodory numbers, and set of atoms*, Proc. Amer. Math. Soc. **147** (2019) 3021–3038

Theorem (PdD + K. Schmüdgen[13])

If $a \geq 0$ with finite zero set $\mathcal{Z}(a)$, then $\dim \operatorname{span}\{l_x \,|\, x \in \mathcal{Z}(a)\} \;\leq\; \mathcal{C}_{\mathcal{A}}$.

Proof: $\operatorname{span}\{l_x \,|\, x \in \mathcal{Z}(a)\}$ = Polyhedral Cone. □

PdD + K. Schmüdgen:[13] special polynomials on $\mathbb{R}^2$ resp. $\mathbb{P}^2$:

- Motzkin polynomial: $\deg = 4$, $\#\mathcal{Z} = 6$ all lin. independent, i.e. $\mathcal{C} \geq 6$
- Robinson polynomial:[15] $\deg = 6$, $\#\mathcal{Z} = 10$ all lin. independent, i.e. $\mathcal{C} \geq 10$
- Harris polynomial:[16] $\deg = 10$, $\#\mathcal{Z} = 30$ all lin. independent, i.e. $\mathcal{C} \geq 30$

[15] simplified proof of result by Curto + Fialkow; $\mathcal{C} = 11$ by Kunert (Diss. 2014 Konstanz)
[16] Kuhlmann ⇒ Reznick + Schmüdgen ⇒ PdD

Theorem (PdD + K. Schmüdgen[13])

If $a \geq 0$ with finite zero set $\mathcal{Z}(a)$, then $\dim \operatorname{span}\{l_x \mid x \in \mathcal{Z}(a)\} \leq \mathcal{C}_{\mathcal{A}}$.

- C. Riener + M. Schweighofer:[17] grid $G = \{1, \ldots, d\}^2 = \mathcal{Z}(p_1^2 + p_2^2)$ with

$$p_i(x_1, x_2) = (x_i - 1) \cdots (x_i - d)$$

Result: $\{l_x \mid x \in G\}$ lin. ind. on $\mathbb{R}[x_1, x_2]_{\leq 2d} \quad \Rightarrow \quad \mathcal{C}_{\mathsf{A}} \geq d^2$.

- PdD + K. Schmüdgen:[18] extension to higher dimensions (calculations)

- PdD + M. Kummer:[19] coordinate ring $\mathbb{R}[G] \cong \mathbb{R}[x_1, \ldots, x_n]/(p_1, \ldots, p_n)$, homogenization $R_n = \mathbb{R}[x_0, \ldots, x_n]/(p_1^*, \ldots, p_n^*)$ and its Hilbert function

$$HF_{R_n}(k) = \sum_{i=0}^{n} (-1)^i \cdot \binom{n}{i} \cdot HF_{\mathbb{P}^n}(k - id)$$

[17] *Optimization approaches to quadrature: New characterizations of Gaussian quadrature on the line and quadrature with few nodes on plane algebraic curves, on the plane and in higher dimensions*, J. Complexity **45** (2018), 22–54

[18] *The multidimensional truncated Moment Problem: The Moment Cone*, arXiv:1804.00584

[19] *The multidimensional truncated moment problem: Carathéodory numbers from Hilbert Functions*, arXiv1903.00598

Carathéodory Numbers from Hilbert Functions

Theorem (PdD + M. Kummer[20])

$\mathcal{X} \subseteq \mathbb{R}^n$ with non-empty interior. For $\mathcal{A} = \mathbb{R}[x_1, \dots, x_n]_{\leq 2d}$ on $\mathcal{X}$ we have

$$\mathcal{C}_{\mathcal{A}} \quad \geq \quad \binom{n+2d}{n} - n \cdot \binom{n+d}{n} + \binom{n}{2},$$

and for $\mathcal{A} = \mathbb{R}[x_1, \dots, x_n]_{\leq 2d+1}$ on $\mathcal{X}$ we have

$$\mathcal{C}_{\mathcal{A}} \quad \geq \quad \binom{n+2d+1}{n} - n \cdot \binom{n+d+1}{n} + 3 \cdot \binom{n+1}{3}.$$

$$\liminf_{d\to\infty} \frac{\mathcal{C}_{\mathsf{A}_{n,d}}}{|\mathsf{A}_{n,d}|} \quad \geq \quad 1 - \frac{n}{2^n} \qquad \text{and} \qquad \lim_{n\to\infty} \frac{\mathcal{C}_{\mathsf{A}_{n,d}}}{|\mathsf{A}_{n,d}|} \quad = \quad 1$$

For every $\varepsilon > 0$ and $d \in \mathbb{N}$ there exist $n \in \mathbb{N}$: $\mathcal{C}_{n,d} \geq (1-\varepsilon) \cdot \binom{n+d}{n}$.

[20] PdD, M. Kummer: *The multidimensional truncated moment problem: Carathéodory Numbers from Hilbert Functions*, arXiv1903.00598

Impact on Flat Extension

Theorem (PdD + M. Kummer)

1. *For every moment functional* $L : \mathbb{R}[x_1, \dots, x_n]_{\leq 2d} \to \mathbb{R}$ *there is a* $D \leq 2d$ *and an extension to a moment functional* $L_0 : \mathbb{R}[x_1, \dots, x_n]_{\leq 2D} \to \mathbb{R}$ *that admits a flat extension* $L_\infty : \mathbb{R}[x_1, \dots, x_n] \to \mathbb{R}$.
2. *For every* $d \in \mathbb{N}$ *there is an* $N \in \mathbb{N}$ *such that for every* $n \geq N$ *there is a moment functional* L *on* $\mathbb{R}[x_1, \dots, x_n]_{\leq 2d}$ *such that* $D = 2d$ *in (1) is required.*

Examples: Worst case attained for $L : \mathbb{R}[x_1, \dots, x_n]_{\leq 2d} \to \mathbb{R}$ with

$$(n, 2d) = (9,4), (7,6), (6,8), (6,10), (n', 12) \quad (n' \geq 6)$$

Below 6 variables we found no worst case.

Gaussian Mixtures in (algebraic) Statistics

- Thm:[21] A moment functional is a linear combination of Gaussian (or log-normal or more general) measures iff it is in the interior of the moment cone (set of all moment functionals).

- Thm:[22] If $a \geq 0$ and $\mathcal{Z}(a)$ finite and $k = \dim \operatorname{lin}\{l_x \mid x \in \mathcal{Z}(a)\}$, then there is a moment functional $L : \mathcal{A} \to \mathbb{R}$ which is a conic combination of k general measures (Gaussian, log-normal, ...) but not less.

Corollary (PdD[22])

For every $d \in \mathbb{N}$ and $\varepsilon > 0$ there is a $n \in \mathbb{N}$ and a moment functional

$$L : \mathbb{R}[x_1, \dots, x_n]_{\leq 2d} \to \mathbb{R}$$

such that L is a conic combination of $(1-\varepsilon) \cdot \binom{n+2d}{n}$ Gaussians but not less.

[21] PdD: *The multidimensional truncated moment problem: Gaussian and log-normal mixtures, their Carathéodory numbers, and set of atoms*, Proc. Amer. Math. Soc. **147** (2019) 3021–3038

[22] PdD: *The multidimensional truncated Moment Problem: Shape and Gaussian Mixture Reconstruction from Derivatives of Moments*, arXiv:1907.00790

Take Away for Carathéodory numbers

As $n \to \infty$ with $\varepsilon > 0$, for $L : \mathbb{R}[x_1, \ldots, x_n]_{\leq d} \to \mathbb{R}$ we need

$$(1 - \varepsilon) \cdot \binom{n+d}{n} \leq \mathcal{C}_{\mathcal{A}}$$

point evaluations, or Gaussian distributions, log-normal distribution, . . . !

Motivation - Polytope Reconstruction

- $P \subset \mathbb{R}^n$ polytope with vertices $v_1, \dots, v_k$
- directional moments ($r \in \mathbb{R}^n$):

$$s_i(r) := \int_P \langle x, r \rangle^i \, \mathrm{d}\lambda^n(x) \overset{\text{Fubini}}{=} \int_{\mathbb{R}} y^i \cdot \Theta_{P,r}(y) \, \mathrm{d}\lambda(y)$$

- $\Theta_{P,r}(y)$: $n-1$-dim. area function of $P \cap \{x \in \mathbb{R}^n : \langle x, r \rangle = y\}$
- Idea ($n = 2$):
  - $\Theta_{P,r}$ is piece-wise linear, kinks exactly at $\xi_i = \langle v_i, r \rangle$
  - $\Theta'_{P,r}$ is piece-wise constant, leaps exactly at $\xi_i = \langle v_i, r \rangle$
  - $\Theta''_{P,r}$ is k-atomic measure (distribution), Dirac deltas exactly at $\xi_i = \langle v_i, r \rangle$
- Solution: Derivatives of moments![23]
- special attention: Gaussian mixtures (linear combinations of Gaussian distributions)

[23] PdD: *The multidimensional truncated moment problem: Shape and Gaussian mixture reconstruction from derivatives of moments*, arXiv1907.00790

Derivatives of Moments and Measures

$\partial^\alpha \mathcal{A} \subseteq \mathcal{A}$, $\alpha \in \mathbb{N}_0^n$.

- Derivative of (moment) functional:

$$\partial^\alpha L := (-1)^{|\alpha|} \cdot L \circ \partial^\alpha$$

- Example: $\mathcal{A} = \mathbb{R}[x]$. $s_i = L(x^i) \quad \Rightarrow \quad \partial s_i = -L(i \cdot x^{i-1}) = -i \cdot s_{i-1}$
  - $s = (s_0, s_1, s_2, s_3, s_4, \dots)$
  - $\partial s = (0, -s_0, -2s_1, -3s_2, -4s_3, \dots)$
  - $\partial^2 s = (0, 0, 2s_0, 6s_1, 12s_2, \dots)$
  - $\dots$

- Derivative of measure: μ measure. Distributional derivative of μ by

$$\partial^\alpha \mu := (-1)^{|\alpha|} \cdot \mu(\partial^\alpha f)$$

with $\mu(f) = \int f \, \mathrm{d}\mu$ if $\partial^\alpha \mu$ is a measure again.

Theorem (PdD)

μ measure of L and $\partial^\alpha \mu$ is measure again, then $\partial^\alpha \mu$ is measure of $\partial^\alpha L$.

Brion–Lawrence–Khovanskii–Pukhlikov–Barvinok Formulas

BBaKLP formulas:

Let P be a polytope in $\mathbb{R}^n$ with vertices $v_1, \dots, v_k$ $(k \geq n+1)$, then

$$0 \quad = \quad \sum_{i=1}^{k} \langle v_i, r\rangle^j \tilde{D}_{v_i}(r) \qquad (j = 0, \dots, n-1)$$

$$\int_P \langle x, r\rangle^j \, \mathrm{d}\lambda^n(x) \quad =: \quad s_j(r) \quad = \quad \frac{j!(-1)^n}{(j+n)!} \sum_{i=1}^{k} \langle v_i, r\rangle^{j+n} \tilde{D}_{v_i}(r), \qquad (j \geq n)$$

where $\tilde{D}_{v_i}(r)$ is a rational function on $r \in \mathbb{R}^n$, i.e., r can be chosen in general position such that $\tilde{D}_{v_i}(\,\cdot\,)$ has no zero or pole at r.

Lemma

$$\partial^n \Theta_{P,r} \quad = \quad \sum_{i=1}^{k} \tilde{D}_{v_i}(r) \cdot \delta_{\langle r, v_i\rangle}$$

Reconstruction of Polytopes

Corollary (Main Theorem[24])

Let

$$s_j(r) := \int_P \langle x, r\rangle^j \,\mathrm{d}\lambda^n(x),$$

$j = 0, \dots, k$, $k \geq n+1$, *be the directional moments of a polytope* P *with vertices* $v_1, \dots, v_k$, *and* $r \in \mathbb{R}^n$ *in general position. Then* $\partial^n s$ *is represented by the signed* k*-atomic measure*

$$\partial^n \Theta_{P,r} \quad = \quad \sum_{i=1}^{k} \tilde{D}_{v_i}(r) \cdot \delta_{\langle r, v_i\rangle}.$$

Proof: $s = (s_0, \dots, s_k)$ represented by $\Theta_{P,r} \Rightarrow \partial^n s$ represented by $\partial^n \Theta_{P,r}$. □

Advantage:

- $\partial^n L$ and $\int \dots \,\mathrm{d}\mu$ linear (in L resp. μ)
- Corollary extends to linear combinations of polytopes (one line proof)

[24] N. Gravin, J. Lasserre, D. V. Pasechnik, and S. Robins, *The inverse moment problem for convex polytopes*, Discrete Comput. Geom. **48** (2012), 596–621.

Reconstruction of Measures on Semi-algebraic Sets[25]

- $G \subset \mathbb{R}^n$ semi-algebraic, $\partial G \subseteq \mathcal{Z}(g)$ for some $g \in \mathbb{R}[x_1,\dots,x_n]$
- $p(x) = \sum_\alpha c_\alpha x^\alpha \in \mathbb{R}[x_1,\dots,x_n]$
- $f(x) = \exp(p(x)) \cdot \chi_G$
- $\partial_1 f(x) = \partial_1 p(x) \cdot \exp(p(x)) \cdot \chi_G + \exp(p(x)) \cdot \partial_i \chi_G$
- $\partial_i \chi_G$: distribution with $\operatorname{supp} \partial_i \chi_G \subseteq \partial G \subseteq \mathcal{Z}(g)$, i.e.

$$g(x) \cdot \partial_i p(x) \cdot \exp(p(x)) \cdot \chi_G = \sum_\alpha \alpha_i \cdot c_\alpha \cdot x^{\alpha - e_i} \cdot g(x) \cdot f(x) \qquad (*)$$

- $g(M) \cdot \partial_i L$ is represented by $(*)$:

$$g(M) \cdot \partial_i L = \sum_\alpha \alpha_i \cdot c_\alpha \cdot g(M) L$$

[25] F. Bréhard, M. Joldes, and J.-B. Lasserre, *On a moment problem with holonomic functions*, 2019, https://hal.archives-ouvertes.fr/hal-02006645.

Theorem (F. Bréhard, M. Joldes, and J.-B. Lasserre)

Let $G \subseteq \mathbb{R}$ be a semi-algebraic set, let $g \in \mathbb{R}[x_1, \dots, x_n]$ with $\gamma := \deg g$ and $\partial G \subseteq \mathcal{Z}(g)$, $p \in \mathbb{R}[x_1, \dots, x_n]$ with $d := \deg p$, and s_α the moments of $\exp(p) \cdot \chi_G$,

$$s_\alpha \quad := \quad \int_G x^\alpha \cdot \exp(p(x)) \, \mathrm{d}\lambda^n(x),$$

for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$ for some $k \geq 2d + 2\gamma - 2$. The following are equivalent:

1. $p = \sum_{\alpha \in \mathbb{N}_0 : |\alpha| \leq d} c_\alpha \cdot x^\alpha$.
2. *For each $i = 1, \dots, n$ let $\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(m)}$ with $m = \binom{n+d-1}{n}$ denote an enumeration of $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ with $|\alpha| \leq d$ and $\alpha_i \geq 1$. The kernel of*

$$(g(M)\partial_{x_i} s,\ g(M) M_{\alpha^{(1)} - e_i} s,\ \dots,\ g(M) M_{\alpha^{(m)} - e_i} s)_{k-d} \tag{1}$$

is spanned by $(1, -\alpha_i^{(1)} \cdot c_{\alpha^{(1)}}, \dots, -\alpha_i^{(m)} \cdot c_{\alpha^{(m)}})$ for every $i = 1, \dots, n$.

c_0 is determined by normalization. If $g \geq 0$ on G then $k \geq 2d + \gamma - 2$ is sufficient.

Gaussian Reconstruction: One Gaussian on $\mathbb{R}^n$

$g(x) = c \cdot \exp(-\frac{1}{2}a(x-b)^2)$, i.e., $g'(x) = (-ax+b) \cdot g(x)$ and

$$\partial L = -aM_x L + bL \qquad \Leftrightarrow \qquad 0 = \partial L - bL + aM_x L$$

Theorem (PdD)

Let $n \in \mathbb{N}$, $A = (a_1, \dots, a_n) = (a_{i,j})_{i,j=1}^n \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$, $c \neq 0$, and $k \in \mathbb{N}$ with $k \geq 2$. Set

$$g(x) := c \cdot \exp\left(-\frac{1}{2}(x-b)^T A(x-b)\right).$$

For a multi-indexed real sequence $s = (s_\alpha)_{\alpha\in\mathbb{N}_0^n:|\alpha|\leq k}$ the following are equivalent:

1. $s_\alpha = \int x^\alpha \cdot g(x)\, \mathrm{d}\lambda^n(x)$ *for all* $\alpha \in \mathbb{N}_0^n$ *with* $|\alpha| \leq k$.
2. *For* $i = 1, \dots, n$ *we have*

$$\ker(\partial_i s, s, M_{e_1} s, \dots M_{e_n} s)_{k-1} = (1, -\langle b, a_i\rangle, a_{i,1}, \dots, a_{i,n}) \cdot \mathbb{R}.$$

Gaussian Reconstruction: Gaussian Mixtures on $\mathbb{R}$

- same variance:

$$F(x) = \sum_{i=1}^{k} c_i \cdot \exp(-\frac{a}{2}(x-b_i)^2) \qquad (a>0)$$

- define $\Delta_a f(x) := \frac{1}{a}(\partial + ax)f(x)$ resp. $\Delta_a L := \frac{1}{a}(\partial + aM_1)L$:

$$(\Delta_a)^n F(x) = \sum_{i=1}^{k} c_i \cdot b_i^n \cdot \exp(-\frac{a}{2}(x-b_i)^2)$$

- recover $b_1,\dots,b_n$ from

$$\ker(s, \Delta_a s, \Delta_a^2 s, \dots, \Delta_a^k s)_{d-k} \quad = \quad (v_k, v_{k-1}, \dots, v_1, 1)\cdot\mathbb{R}.$$

and $\mathcal{Z}(p) = \{b_1,\dots,b_k\}$ for

$$p(x) \quad = \quad x^k + v_1 x^{k-1} + v_2 x^{k-2} + \dots + v_k.$$

Take Away for Derivatives of Moments

$$\partial^\alpha L$$

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