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TOPOLOGICAL VECTOR SPACES–WS 2015/16

Exercise Sheet 1

You do not need to hand in solutions for these exercises, but please try to solve as many questions as you can. This sheet aims to self-assess your progress and to explicitly work out more details of some of the results proposed in the previous lecture. If you have any problem in solving it, please come to see me on Tuesday at 3 pm in room F408.

1) a) Prove the following characterization of basis of a topological space.

Proposition 1. *Let X be a set and let $\mathcal{B}$ be a collection of subsets of X. $\mathcal{B}$ is a basis for a topology τ on X iff*

i. $\mathcal{B}$ covers X, i.e. $\forall\, x \in X$, $\exists\, B \in \mathcal{B}$ s.t. $x \in B$. In other words, $X = \cup_{B \in \mathcal{B}} B$.

ii. If $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

b) Let $\mathcal{B}$ be the collection of all intervals (a, b) in $\mathbb{R}$ together with all the sets of the form $(a, b) - K$, where $K := \{\frac{1}{n} : n \in \mathbb{N}\}$. Prove that $\mathcal{B}$ is the basis for a topology on $\mathbb{R}$, which is usually called the *$K-$topology* on $\mathbb{R}$.

2) Show the following statements.

a) The family $\mathcal{G}$ of all subsets of a set X containing a fixed non-empty subset A is a filter and $\mathcal{B} = \{A\}$ is its base. $\mathcal{G}$ is known as the *principal filter* generated by A.

b) Given a topological space X and $x \in X$, the family $\mathcal{F}(x)$ of all neighbourhoods of x is a filter.

c) Let $S := \{x_n\}_{n \in \mathbb{N}}$ be a sequence of points in a set X. Then the family $\mathcal{F} := \{A \subset X : |S \backslash A| < \infty\}$ is a filter and it is known as the *filter associated* to the sequence S. For each $m \in \mathbb{N}$, set $S_m := \{x_n \in S : n \geq m\}$. Then $\mathcal{B} := \{S_m : m \in \mathbb{N}\}$ is a basis for $\mathcal{F}$.

3) Establish which of the following topologies on $\mathbb{R}$ are comparable and for each comparable pair say which one is finer.

- τ_1 :=standard topology, whose basis is $\mathcal{B}_1 := \{(a, b) : a, b \in \mathbb{R} \text{ with } a < b\}$
- $\tau_2 := K-$topology, whose basis $\mathcal{B}_2$ is the one defined in Exercise 1 b)
- $\tau_3 :=$ lower limit topology, whose basis is $\mathcal{B}_3 := \{[a, b) : a, b \in \mathbb{R} \text{ with } a < b\}$

4) Show the following statements.

a) Any continuous image of a separable space is separable.

b) If a topological space X is Hausdorff and $S \subset X$, then S is Hausdorff w.r.t. the induced topology.

c) The product of Hausdorff spaces endowed with the corresponding product topology is Hausdorff.

d) A topological space X is Hausdorff if and only if the diagonal $\Delta_X := \{(x, x) : x \in X\} \subset X \times X$ is closed w.r.t. the product topology.