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TOPOLOGICAL VECTOR SPACES–WS 2015/16
Exercise Sheet 10

You do not need to hand in solutions for these exercises, but please try to solve as many questions as you can. This sheet aims to self-assess your progress and to explicitly work out more details of some of the results proposed in the lectures. If you have any problem in solving it, please come to see me on Tuesday at 3 pm in room F408.

1) Consider the following theorem (Theorem 4.2.12 in the lecture notes) about the comparison of locally convex topologies

Theorem. *Let $\mathcal{P} = \{p_i\}_{i\in I}$ and $\mathcal{Q} = \{q_j\}_{j\in J}$ be two families of seminorms on the vector space X inducing respectively the topologies $\tau_{\mathcal{P}}$ and $\tau_{\mathcal{Q}}$, which both make X into a locally convex t.v.s.. Then $\tau_{\mathcal{P}}$ is finer than $\tau_{\mathcal{Q}}$ (i.e. $\tau_{\mathcal{Q}} \subseteq \tau_{\mathcal{P}}$) iff*

$$\forall q \in \mathcal{Q}\ \exists n \in \mathbb{N},\, i_1, \dots, i_n \in I,\, C > 0 \text{ s.t. } Cq(x) \leq \max_{k=1,\dots,n} p_{i_k}(x),\ \forall x \in X. \tag{1}$$

a) Give an alternative proof of this result without using Proposition 4.2.11 in the lecture notes.

b) Show that the theorem still holds if we replace (1) with:

$$\forall q \in \mathcal{Q}\ \exists n \in \mathbb{N},\, i_1, \dots, i_n \in I,\, C > 0 \text{ s.t. } Cq(x) \leq \sum_{k=1}^{n} p_{i_k}(x),\ \forall x \in X.$$

c) Consider the space of infinitely differentiable functions $\mathcal{C}^\infty(\mathbb{R}^d)$ and the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ endowed respectively with the locally convex topologies $\tau_{\mathcal{P}}$ and $\tau_{\mathcal{Q}}$ defined in Examples 4.3.5. Since $\mathcal{S}(\mathbb{R}^d)$ is a linear subspace of $\mathcal{C}^\infty(\mathbb{R}^d)$, we can also endow $\mathcal{S}(\mathbb{R}^d)$ with the subspace topology $\tau_{\mathcal{P}}^S$ induced on it by $\tau_{\mathcal{P}}$. Use the theorem above to show that $\tau_{\mathcal{Q}}$ is finer than $\tau_{\mathcal{P}}^S$.

2) Let X be a locally convex t.v.s. whose topology is induced by a family of directed family of seminorms $\mathcal{P}$. Show that a basis of neighbourhoods of the origin in X for such a topology is given by:

$$\mathcal{B}_d := \{r\mathring{U}_p : p \in \mathcal{P}, r > 0\},$$

where $\mathring{U}_p := \{x \in X : p(x) < 1\}$.

3) Keeping in mind the definition of finite topology on a countable dimensional vector space (see Definition 4.5.1 in the lecture notes), prove the following statements.

a) Let X, Y be two infinite dimensional vector spaces of countable dimension each endowed with the corresponding finite topology. Then the finite topology on the product $X \times Y$ coincides with the product topology.

b) Let X be an infinite dimensional vector space with basis $\{x_n\}_{n\in\mathbb{N}}$ endowed with the finite topology τ_f and (Y, τ) any other topological space. For any $i \in \mathbb{N}$ set $X_i := span\{x_1, \dots, x_i\}$ so that $X = \bigcup_{i=1}^{\infty} X_i$. A map $f : X \to Y$ is continuous (w.r.t. τ_f and τ) if and only if for each $i \in \mathbb{N}$ the restriction $f|_{X_i}$ of f to X_i is continuous (w.r.t. the euclidean topology and τ).

c) Any countable dimensional vector space endowed with the finite topology is a t.v.s..
(*Hint*: use the properties (a) and (b))