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Fachbereich Mathematik und Statistik
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TOPOLOGICAL VECTOR SPACES–WS 2015/16

Exercise Sheet 2

You do not need to hand in solutions for these exercises, but please try to solve as many questions as you can. This sheet aims to self-assess your progress and to explicitly work out more details of some of the results proposed in the previous lecture. If you have any problem in solving it, please come to see me on Tuesday at 3 pm in room F408.

1) Given a topological space X and a subset $A \subset X$, prove that the following hold.

- **a)** A point x is a *closure point* of A, i.e. $x \in \bar{A}$, if and only if each neighborhood of x has a nonempty intersection with A.
- **b)** A point x is an *interior point* of A, i.e. $x \in \mathring{A}$, if and only if there exists a neighborhood of x which entirely lies in A.
- **c)** A is closed in X iff $A = \bar{A}$.
- **d)** A is open in X iff $A = \mathring{A}$.

2)
- **a)** Let X be a set endowed with the discrete topology. Then the only convergent sequences in X are the ones that are eventually constant, that is, sequences $\{q_i\}_{i\in\mathbb{N}}$ of points in X such that $q_i = q$ for all $i \geq N$ for some $N \in \mathbb{N}$.
- **b)** Let Y be a set endowed with the trivial topology. Then every sequence in Y converges to every point of Y.

3) Let $X_1, \dots, X_n$ be n topological spaces and let A_i be a subset of X_i for each i. Show that:

- **a)** If A_i is closed in X_i for each i, then $A_1 \times \dots \times A_n$ is closed in $X_1 \times \dots \times X_n$ w.r.t. the product topology.
- **b)** $\overline{A_1 \times \dots \times A_n} = \bar{A}_1 \times \dots \times \bar{A}_n$.

4) Show the following properties of continuous mappings.

- **a)** Let $f : X \to Y$ be a continuous map between the topological spaces (X, τ_X) and (Y, τ_Y). Let $\mathcal{B}$ be a basis for τ_X and consider the following collection $f(\mathcal{B}) := \{f(B) : B \in \mathcal{B}\}$ of subsets of Y. If f is surjective and open, then $f(\mathcal{B})$ is a basis for τ_Y.
- **b)** Continuous maps preserve the convergence of sequences. That is, if $f : X \to Y$ is a continuous map between two topological spaces (X, τ_X) and (Y, τ_Y) and if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of points in X convergent to a point $x \in X$, then $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $f(x) \in Y$.

5)
- **a)** Suppose X is a topological space, and for every $p \in X$ there exists a continuous functional $f : X \to \mathbb{R}$ such that $f^{-1}(\{0\}) = \{p\}$. Then X is Hausdorff.
- **b)** Let X be a set endowed with the trivial topology and Y be any topological space. If Y is Hausdorff, then the only continuous maps $h : X \to Y$ are constant maps.