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TOPOLOGICAL VECTOR SPACES–WS 2015/16
Exercise Sheet 3

You do not need to hand in solutions for these exercises, but please try to solve as many questions as you can. This sheet aims to self-assess your progress and to explicitly work out more details of some of the results proposed in the lectures. If you have any problem in solving it, please come to see me on Tuesday at 3 pm in room F408.

1) Show the following statements using just the definition of t.v.s.

a) Every normed vector space endowed with the topology given by the metric induced by the norm is a t.v.s.. (Hint: use the open ball $B_r(x_0) := \{x \in X : x - x_0\| < r\}$ as a base of the topology).

b) Consider the real vector space $\mathbb{R}$ endowed with the lower limit topology τ generated by the base $B = \{[a, b) : a < b\}$. Show that $(\mathbb{R}, \tau)$ is not a t.v.s..

c) Let us consider on $\mathbb{R}$ the metric:

$$d_h(x, y) := |h(x) - h(y)|, \forall x, y \in \mathbb{R}$$

where h is the following function on $\mathbb{R}$:

$$h(x) := \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \\ x & \text{otherwise} \end{cases}$$

Then the metric real vector space $(\mathbb{R}, d_h)$ is not a t.v.s. (Here the field of scalars is also $\mathbb{R}$ but endowed with the usual topology given by the modulus $|\cdot|$).

2) Prove the following statements.

a) The filter $\mathcal{F}(x)$ of neighbourhoods of the point x in a t.v.s. X coincides with the family of the sets $O + x$ for all $O \in \mathcal{F}(o)$, where $\mathcal{F}(o)$ is the filter of neighbourhoods of the origin o (i.e. the neutral element of the vector addition).

b) If B is a balanced subset of a t.v.s. X then so is $\bar{B}$.

c) If B is a balanced subset of a t.v.s. X and $o \in \mathring{B}$ then $\mathring{B}$ is balanced.

3) Prove the following statements.

a) Every t.v.s. has always a base of closed neighbourhoods of zero.

b) Every t.v.s. has always a base of balanced absorbing neighbourhoods of zero. In particular, it has always a base of closed balanced absorbing neighbourhoods of zero.

c) Proper subspaces of a t.v.s. are never absorbing. In particular, if M is an open subspace of a t.v.s. X then $M = X$.