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TOPOLOGICAL VECTOR SPACES–WS 2015/16
Exercise Sheet 4

You do not need to hand in solutions for these exercises, but please try to solve as many questions as you can. This sheet aims to self-assess your progress and to explicitly work out more details of some of the results proposed in the lectures. If you have any problem in solving it, please come to see me on Tuesday at 3 pm in room F408.

1) Let X be a t.v.s.. Show that the following relations hold.

a) If $A \subset X$, then $\overline{A} = \bigcap_{V \in \mathcal{F}(o)} (A + V)$ where $\mathcal{F}(o)$ is the family of all neighbourhoods of the origin.

b) If $A \subset X$ and $B \subset X$, then $\overline{A} + \overline{B} \subseteq \overline{A + B}$.

c) The intersection of all neighborhoods of the origin o of X is a vector subspace of X, which is $\overline{\{o\}}$.

2) Show the following topological properties.

a) In a topological space which is (T1) all singletons are closed

b) Let X be a topological space and $\sim$ be any equivalence relation on X. Consider the quotient set $X/\sim$ endowed with the quotient topology. Then:

- the quotient map $\phi : X \to X/\sim$ is continuous.
- the quotient topology on $X/\sim$ is the finest topology on $X/\sim$ s.t. ϕ is continuous.

3) Let f, and g be two continuous mappings of a topological space X into a Hausdorff t.v.s. Y. Then:

a) The set A in which f and g coincide, i.e. $A := \{x \in X : f(x) = g(x)\}$ is closed in X.

b) If f and g are equal on a dense subset B of X , then they are equal everywhere in X.

4) Let X be a t.v.s. and L a linear functional on X, i.e. $L : X \to \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with the usual topology given by the modulus. Assume $L(x) \neq 0$ for some $x \in X$. Then the following are equivalent:

a) L is continuous.

b) The null space $Ker(L)$ is closed in X

c) $Ker(L)$ is not dense in X.

d) L is bounded in some neighbourhood of the origin in X.