Universität Konstanz
Fachbereich Mathematik und Statistik
Dr. Maria Infusino

TOPOLOGICAL VECTOR SPACES–WS 2015/16
Exercise Sheet 5

You do not need to hand in solutions for these exercises, but please try to solve as many questions as you can. This sheet aims to self-assess your progress and to explicitly work out more details of some of the results proposed in the lectures. If you have any problem in solving it, please come to see me on Tuesday at 3 pm in room F408.

1) Prove the following statements.

a) If M is a linear dense subspace of a t.v.s X, then the quotient topology on X/M is the trivial topology.

b) If X is a Hausdorff t.v.s., then any linear subspace M of X endowed with the correspondent subspace topology is itself a Hausdorff t.v.s..

c) Give an example of a t.v.s X which is not Hausdorff and of a linear subspace $M \neq \{0\}$ of X such that M endowed with the subspace topology is instead a Hausdorff t.v.s..

2) Let X be the Cartesian product of a family $\{X_i : i \in I\}$ of t.v.s. endowed with the correspondent product topology. Show that:

a) X is a t.v.s.

b) X is Hausdorff if and only if each X_i is Hausdorff.

Does b) hold for any Cartesian product of a family of topological spaces (not necessarily t.v.s.) endowed with the product topology? Justify your answer.

3) Let M be a linear subspace of a t.v.s. X. Another linear subspace N of X is called an *algebraic supplementary* (or algebraic complement) of M in X if the mapping:

$$S : M \times N \to X;\ (m, n) \mapsto m + n$$

is an algebraic isomorphism between $M \times N$ and X. In this case, X is called *algebraic direct sum* of M and N. N is called a *topological supplementary* (or topological complement) of M in X if S is a topological isomorphism between $M \times N$ and X. In this case, X is called the *topological direct sum* of M and N.

Prove the equivalence of the following two properties:

a) N is a topological supplementary of M in X

b) the restriction to N of the canonical mapping $\phi : X \to X/M$ is a topological isomorphism between N and X/M.

Prove also that M has at least one topological supplementary in X if and only if there is a continuous linear map p of X onto M such that $p \circ p = p$ (then $p(x) = x$ for all $x \in M$).