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TOPOLOGICAL VECTOR SPACES–WS 2015/16
Exercise Sheet 6

You do not need to hand in solutions for these exercises, but please try to solve as many questions as you can. If you have any problem in solving them, please come to see me on Tuesday at 3 pm in room F408.

This sheet aims to establish some properties of ***uniformly continuous functions*** *defined on a subset of a t.v.s. and valued in a t.v.s., which will be essential to prove the theorem on completion of t.v.s. in the next lecture. Here is the definition of such functions:*

Definition 1. *Let X and Y be two t.v.s. and let A be a subset of X. A mapping $f : A \to Y$ is said to be* uniformly continuous *if for every neighborhood V of the origin in Y, there exists a neighborhood U of the origin in X such that for all pairs of elements $x_1, x_2 \in A$*

$$x_1 - x_2 \in U \;\Rightarrow\; f(x_1) - f(x_2) \in V.$$

1) Prove the following statements.

- **a)** If X and Y are two t.v.s. whose topologies are given respectively by the translation invariant metrics d_1 and d_2, then Definition 1 reduces to the usual one of uniformly continuous function between metric spaces.
- **b)** Let X and Y be two t.v.s. and let A be a subset of X. Any uniformly continuous map $f : A \to Y$ is continuous at every point of A.
- **c)** Give an example of continuous map between t.v.s. which is not uniformly continuous.

2) Let X and Y be two t.v.s. and let A be a subset of X. Show that:

- **a)** If $f : A \to Y$ be uniformly continuous, then the image under f of a Cauchy filter on A is a Cauchy filter on Y.
- **b)** If A is a linear subspace of X, then every continuous linear map from A to Y is uniformly continuous.

3) Let X and Y be two Hausdorff t.v.s., A a dense subset of X, and $f : A \to Y$ a uniformly continuous mapping. Assume that Y is complete and show that the following hold.

- **a)** There exists a unique continuous mapping $\bar{f} : X \to Y$ which extends f, i.e. such that for all $x \in A$ we have $\bar{f}(x) = f(x)$.
- **b)** $\bar{f}$ is uniformly continuous.
- **c)** If we additionally assume that f is linear and A is a linear subspace of X, then $\bar{f}$ is linear.