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TOPOLOGICAL VECTOR SPACES–WS 2015/16
Exercise Sheet 7

You do not need to hand in solutions for this sheet, but please try to solve it completely. If you have any problem, please come to see me on Tuesday at 3 pm in room F408.

In this sheet you are asked to explicitly prove some small parts of the proof of Theorem 2.5.15 on ***completion of t.v.s.*** *given in Lecture 7. Please, refer to the notes for the main steps of the proof.*

Let X be a Hausdorff t.v.s. and define $\hat{X}$ to be the quotient of the set of all Cauchy filters (c.f.) on X w.r.t. the equivalence relation (R) given by:

$$\mathcal{F} \sim_R \mathcal{G} \Leftrightarrow \forall\, U \text{ nbhood of the origin in } X,\ \exists\, A \in \mathcal{F},\ \exists\, B \in \mathcal{G} \text{ s.t. } A - B \subset U.$$

1) Prove that both of the operations introduced on the set $\hat{X}$ in *Step 2* of the main proof are well-defined.

2) Let U be an arbitrary nbhood of the origin in X. Define

$$\hat{U} := \{\hat{x} \in \hat{X} : U \in \mathcal{F} \text{ for some } \mathcal{F} \in \hat{x}\}$$

and consider the collection $\hat{\mathcal{B}} := \{\hat{U} : U \text{ nbhood of the origin in } X\}$. Prove that the filter generated by $\hat{\mathcal{B}}$ determines a unique topology on $\hat{X}$ compatible with the vector space structure defined in Step 2 (this proves *Step 3* in the main proof).

3) Show Lemma 2.5.16 in the lecture notes, i.e.

a) Two c.f. filters on X converging to the same point are equivalent w.r.t. (R)

b) If two c.f. filters $\mathcal{F}$ and $\mathcal{F}'$ on X are s.t. $\mathcal{F} \sim_R \mathcal{F}'$ and $\mathcal{F}' \to x \in X$ then also $\mathcal{F} \to x$.

4) Work out the missing details in *Step 6* of the main proof. Namely, show that $i : X \to \hat{X}$ defined by

$$\forall\, x \in X,\ \ i(x) := \{\mathcal{F} \text{ c.f. on } X : \mathcal{F} \sim_R \mathcal{F}(x)\}.$$

is a topological monomorphism (i.e. injective linear homeomorphism). Please, put particular attention to the proof that i is a homeomorphism.

5) Prove the completeness of $\hat{X}$ (i.e *Step 8* of the main proof) along the following scheme:

a) Take any Cauchy filter $\hat{\mathcal{F}}$ on $\hat{X}$ and consider

$$\hat{\mathcal{F}}' := \{\hat{G} \subset \hat{X} \ :\ \hat{M} + \hat{U} \subset \hat{G} \text{ for some } \hat{M} \in \hat{\mathcal{F}} \text{ and } \hat{U} \text{ nbhood of the origin in } \hat{X}\}.$$

Show that $\hat{\mathcal{F}}'$ is a filter contained in $\hat{\mathcal{F}}$. Actually $\hat{\mathcal{F}}'$ is a Cauchy filter on $\hat{X}$ as proved in Step 8.

b) Consider the family of subsets of $i(X)$ given by $\mathcal{F}' := \{\hat{A} \cap i(X) : \hat{A} \in \hat{F}'\}$ and prove that $\mathcal{F}'$ is a filter on $i(X)$ and actually a Cauchy filter.

c) Since i is a topological isomorphism between X and $i(X)$, we have that $i^{-1}(\mathcal{F}')$ is a Cauchy filter on X. Take $\hat{x} := \{\mathcal{F} \text{ c.f. on } X : \mathcal{F} \sim_R i^{-1}(\mathcal{F}')\}$. Prove that $\hat{F}$ converges to $\hat{x}$.

6) Work out all the details in *Step 9* and *Step 10* of the main proof. Namely, provide a detailed proof of both the universal property of the completion of a t.v.s. and its uniqueness up to isomorphisms.