Universität Konstanz
Fachbereich Mathematik und Statistik
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TOPOLOGICAL VECTOR SPACES–WS 2015/16

Christmas Assignment

This assignment is due by Wednesday the 13th of January. Your solutions will be collected during the Wednesday's lecture. If you cannot come, please send them via email to maria.infusino@uni-konstanz.de or hand them in on the 12th of January during my office hour (15:00-16:00, Room F408).

1) Let X be a Hausdorff t.v.s.. Assume that there exists a countable basis $\mathcal{B}$ of neighborhoods of the origin in X. Prove the following statements:

a) X is complete if and only if X is sequentially complete.

b) Suppose additionally that Y is another t.v.s. (not necessarily with a countable basis). A mapping $f : X \to Y$ (not necessarily linear) is continuous if and only if it is sequentially continuous.

Recall that a mapping f from a topological space X into a topological space Y is said to be *sequentially continuous* if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ convergent to a point $x \in X$ the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $f(x)$ in Y.

2) Let $\mathcal{C}(\mathbb{R})$ be the vector space of real valued functions defined and continuous on the real line, and $\mathcal{C}_c(\mathbb{R})$ the space of functions $f \in \mathcal{C}(\mathbb{R})$ whose support is a compact subset of $\mathbb{R}$. For any $\varepsilon > 0$ and any $n \in \mathbb{N}$, set

$$N_{\varepsilon,n} := \left\{ f \in \mathcal{C}(\mathbb{R}) : \sup_{|t|\leq n} |f(t)| \leq \varepsilon \right\}.$$

Prove that:

a) The collection of the sets $N_{\varepsilon,n}$ for all $\varepsilon \in \mathbb{R}^+$ and all $n \in \mathbb{N}$ is a basis of neighborhoods of the origin for a Hausdorff topology τ on $\mathcal{C}(\mathbb{R})$ which is compatible with the linear structure (given by the pointwise addition and scalar multiplication of functions in $\mathcal{C}(\mathbb{R})$).

b) The t.v.s. $(\mathcal{C}(\mathbb{R}), \tau)$ is complete [Hint: use Exercise 1].

c) The linear subspace $\mathcal{C}_c(\mathbb{R})$ is dense in $\mathcal{C}(\mathbb{R})$.

d) $\mathcal{C}(\mathbb{R})$ is topologically isomorphic to the completion of $\mathcal{C}_c(\mathbb{R})$.

3) Let $\mathcal{C}^1(\mathbb{R})$ be the vector space of real valued functions defined and once continuously differentiable on the real line. For $\varepsilon > 0$ and $n \in \mathbb{N}$, set

$$W_{\varepsilon,n} := \left\{ f \in \mathcal{C}^1(\mathbb{R}) : \sup_{|t| \leq n} \big(|f(t)| + |f'(t)| \big) \leq \varepsilon \right\}.$$

a) Show that the collection of the sets $W_{\varepsilon,n}$ for all $\varepsilon \in \mathbb{R}^+$ and all $n \in \mathbb{N}$ is a basis of neighborhoods of the origin for a Hausdorff topology on $\mathcal{C}^1(\mathbb{R})$ which is compatible with the linear structure.

b) Consider the t.v.s. $\mathcal{C}(\mathbb{R})$ defined in Exercise 2 and the mapping:

$$\begin{array}{rcl} D : \mathcal{C}^1(\mathbb{R}) & \to & \mathcal{C}(\mathbb{R}) \\ f & \mapsto & D(f) := f'. \end{array}$$

Prove that D is continuous.

4) Let X be a t.v.s. over $\mathbb{R}$ and X^* its algebraic dual. Provide X^* with the topology of pointwise convergence in X. A basis of neighborhoods of the origin in this topology is provided by the sets

$$W(S, \varepsilon) := \{\ell \in X^* : \sup_{x \in S} |\ell(x)| \leq \varepsilon\},$$

where S ranges over the family of finite subsets of X and $\varepsilon \in \mathbb{R}^+$. Prove that X^* is complete.