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TOPOLOGICAL VECTOR SPACES–WS 2015/16
Exercise Sheet 9

You do not need to hand in solutions for these exercises, but please try to solve as many questions as you can. This sheet aims to self-assess your progress and to explicitly work out more details of some of the results proposed in the lectures. If you have any problem in solving it, please come to see me on Tuesday at 3 pm in room F408.

1) Let S, T be arbitrary subsets of a vector space X. Show that the following hold.

a) $conv(S)$ is convex
b) $S \subseteq conv(S)$
c) A set is convex if and only if it is equal to its own convex hull.
d) If $S \subseteq T$ then $conv(S) \subseteq conv(T)$
e) $conv(conv(S)) = conv(S)$.
f) $conv(S + T) = conv(S) + conv(T)$.
g) The convex hull of S is the smallest convex set containing S, i.e. $conv(S)$ is the intersection of all convex sets containing S.
h) The convex hull of a balanced set is balanced

2) Prove the following characterization of locally convex t.v.s (i.e. Theorem 4.1.14 in the lecture notes)

Theorem 1. *If X is a l.c. t.v.s. then there exists a basis $\mathcal{B}$ of neighbourhoods of the origin consisting of absorbing absolutely convex subsets s.t.*

a) $\forall U, V \in \mathcal{B}$, $\exists W \in \mathcal{B}$ s.t. $W \subseteq U \cap V$
b) $\forall U \in \mathcal{B}$, $\forall \rho > 0$, $\exists W \in \mathcal{B}$ s.t. $W \subseteq \rho U$

Conversely, if $\mathcal{B}$ is a collection of absorbing absolutely convex subsets of a vector space X s.t. a) and b) hold, then there exists a unique topology compatible with the linear structure of X s.t. $\mathcal{B}$ is a basis of neighbourhoods of the origin in X for this topology (which is necessarily locally convex).

3) Let $\mathcal{C}(\mathbb{R})$ be the vector space of all real valued continuous functions on the real line. For any bounded interval $[a, b]$ with $a < b$ and any $p > 0$, we define:

$$q_p(f) := \left(\int_a^b |f(t)|^p dt \right)^{\frac{1}{p}}, \ \forall f \in \mathcal{C}(\mathbb{R}).$$

Show that for any $1 \leq p < \infty$ the function q_p is a seminorm but that if $0 < p < 1$ then q_p is not a seminorm.

4) Let $0 < p < 1$ and consider the vector space

$$\ell_p := \{(x_i)_{i \in \mathbb{N}} : \forall i \in \mathbb{N}, x_i \in \mathbb{R} \text{ and } \sum_{i=1}^{\infty} |x_i|^p < \infty\}.$$

For any $x, y \in \ell_p$ define $d(x, y) := |x - y|_p$ where for any $z := (x_i)_{i \in \mathbb{N}} \in \ell_p$ we set $|z|_p := \sum_{i=1}^{\infty} |z_i|^p$. Show that the t.v.s. given by ℓ_p endowed with the topology induced by d is not locally convex.