Chapter 1

Special classes of topological vector spaces

In these notes we consider vector spaces over the field $\mathbb{K}$ of real or complex numbers given the usual euclidean topology defined by means of the modulus.

1.1 Metrizable topological vector spaces

Definition 1.1.1. *A t.v.s. X is said to be* metrizable *if there exists a metric d which defines the topology of X.*

We recall that a metric d on a set X is a mapping $d : X \times X \to \mathbb{R}^+$ with the following properties:

1. $d(x, y) = 0$ if and only if $x = y$ (identity of indiscernibles);
2. $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetry);
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (triangular inequality).

To say that the topology of a t.v.s. X is defined by a metric d means that for any $x \in X$ the sets of all open (or equivalently closed) balls:

$$B_r(x) := \{y \in X : d(x, y) < r\}, \quad \forall r > 0$$

forms a basis of neighbourhoods of x w.r.t. to the original topology on X.

There exists a completely general characterization of metrizable t.v.s..

Theorem 1.1.2. *A t.v.s. X is metrizable if and only if X is Hausdorff and has a countable basis of neighbourhoods of the origin.*

One direction is quite straightforward. Indeed, suppose that X is a metrizable t.v.s. and that d is a metric defining the topology of X, then the collection of all $B_{\frac{1}{n}}(o)$ with $n \in \mathbb{N}$ is a countable basis of neighbourhoods of the origin o in X. Moreover, the intersection of all these balls is just the singleton $\{o\}$, which proves that the t.v.s. X is also Hausdorff (see Corollary 2.2.4 in TVS-I).

The other direction requires more work and we are not going to prove it in full generality but only for locally convex (l.c.) t.v.s., since this class of t.v.s. is anyway the most commonly used in applications. Before doing it, let us make another general observation:

Proposition 1.1.3. *In any metrizable t.v.s. X, there exists a translation invariant metric which defines the topology of X.*

Recall that a metric d on X is said to be *translation invariant* if

$$d(x+z, y+z) = d(x,y), \qquad \forall\, x,y,z \in X.$$

It is important to highlight that the converse of Proposition 1.1.3 does not hold in general. Indeed, the topology τ_d defined on a vector space X by a translation invariant metric d is a translation invariant topology and also the addition is always continuous w.r.t. τ_d. However, the multiplication by scalars might be not continuous w.r.t. τ_d and so (X, τ_d) is not necessarily a t.v.s.. For example, the discrete metric on any non-trivial vector space X is translation invariant but the discrete topology on X is not compatible with the multiplication by scalars (see Interactive Sheet 1).

Proof. (of Theorem 1.1.2 and Proposition 1.1.3 for l.c. t.v.s.)
Let X be a l.c. t.v.s.. Suppose that X is Hausdorff and has a countable basis $\{U_n, n \in \mathbb{N}\}$ of neighbourhoods of the origin. Since X is a l.c. t.v.s., we can assume that such a countable basis of neighbourhoods of the origin consists of barrels, i.e. closed, convex, absorbing and balanced sets (see Proposition 4.1.13 in TVS-I) and that satisfies the following property (see Theorem 4.1.14 in TVS-I):

$$\forall j \in \mathbb{N}, \forall \rho > 0, \exists\, n \in \mathbb{N} : U_n \subset \rho U_j.$$

We may then take

$$V_n = U_1 \cap \cdots \cap U_n, \quad \forall n \in \mathbb{N}$$

as a basis of neighbourhoods of the origin in X. Each V_n is a still barrel, $V_{n+1} \subseteq V_n$ for any $n \in \mathbb{N}$ and:

$$\forall j \in \mathbb{N}, \forall \rho > 0, \exists\, n \in \mathbb{N} : V_n \subset \rho V_j. \tag{1.1}$$

By Lemma 4.2.7 in TVS-I we know that for any $n \in \mathbb{N}$ we have $V_n \subseteq U_{p_{V_n}}$, where $p_{V_n} := \{\lambda > 0 : x \in \lambda V_n\}$ is the Minkowski functional associated to V_n and $U_{p_{V_n}} := \{x \in X : p_{V_n}(x) \leq 1\}$. Also, if $x \in U_{p_{V_n}}$ then there exists a sequence $(\lambda_j)_{j \in \mathbb{N}}$ such that $\lambda_j > 0$ and $x \in \lambda_j V_n$ for each $j \in \mathbb{N}$, and $\lambda_j \to 1$

as $j \to \infty$. This implies that $\frac{x}{\lambda_j} \to x$ as $j \to \infty$ and so $x \in V_n$ since V_n is closed. Hence, we have just showed that for any $n \in \mathbb{N}$ there is a seminorm p_n (i.e. $p_n := p_{V_n}$) on X such that $V_n = \{x \in X : p_n(x) \leq 1\}$. Then clearly we have that $(p_n)_{n\in\mathbb{N}}$ is a countable family of seminorms generating the topology of X and such that $p_n \leq p_{n+1}$ for all $n \in \mathbb{N}$.

Let us now fix a sequence of real positive numbers $\{a_j\}_{j\in\mathbb{N}}$ such that $\sum_{j=1}^{\infty} a_j < \infty$ and define the mapping d on $X \times X$ as follows:

$$d(x,y) := \sum_{j=1}^{\infty} a_j \frac{p_j(x-y)}{1+p_j(x-y)}, \qquad \forall, x, y \in X.$$

We want to show that this is a metric which defines the topology of X.

Let us immediately observe that the positive homogeneity of the seminorms p_j gives that d is a symmetric function. Also, since X is a Hausdorff t.v.s., we get that $\{o\} \subseteq \cap_{n=1}^{\infty} Ker(p_n) \subseteq \cap_{n=1}^{\infty} V_n = \{o\}$, i.e. $\cap_{n=1}^{\infty} Ker(p_n) = \{o\}$. This provides that $d(x,y) = 0$ if and only if $x = y$. We must therefore check the triangular inequality for d. This will follow by applying, for any fixed $j \in \mathbb{N}$ and $x, y, z \in X$, Lemma 1.1.4 below to $a := p_j(x-y)$, $b := p_j(y-z)$ and $c := p_j(x-z)$. In fact, since each p_j is a seminorm on X, we have that the above defined a, b, c are all non-negative real numbers such that: $c = p_j(x-z) = p_j(x-y+y-z) \leq p_j(x-y) + p_j(y-z) = a + b$. Hence, the assumption of Lemma 1.1.4 are fulfilled for such a choice of a, b and c and we get that for each $j \in \mathbb{N}$:

$$\frac{p_j(x-z)}{1+p_j(x-z)} \leq \frac{p_j(x-y)}{1+p_j(x-y)} + \frac{p_j(y-z)}{1+p_j(y-z)}, \qquad \forall x, y, z \in X.$$

Since the a_j's are all positive, this implies that $d(x,z) \leq d(x,y) + d(y,z)$, $\forall x, y, z \in X$. We have then proved that d is indeed a metric and from its definition it is clear that it is also translation invariant.

To complete the proof, we need to show that the topology defined by this metric d coincides with the topology initially given on X. By Hausdorff criterion (see Theorem 1.1.17 in TVS-I), we therefore need to prove that for any $x \in X$ both the following hold:

1. $\forall r > 0, \exists n \in \mathbb{N} : x + V_n \subseteq B_r(x)$
2. $\forall n \in \mathbb{N}, \exists r > 0 : B_r(x) \subseteq x + V_n$

Because of the translation invariance of both topologies, we can consider just the case $x = o$.

Let us fix $r > 0$. As $\sum_{j=1}^{\infty} a_j < \infty$, we can find $j(r) \in \mathbb{N}$ such that

$$\sum_{j=j(r)+1}^{\infty} a_j < \frac{r}{2}. \tag{1.2}$$

Using that $p_n \leq p_{n+1}$ for all $n \in \mathbb{N}$ and denoting by A the sum of the series of the a_j's, we get:

$$\sum_{j=1}^{j(r)} a_j \frac{p_j(x)}{1+p_j(x)} \leq p_{j(r)}(x) \sum_{j=1}^{j(r)} a_j \leq p_{j(r)}(x) \sum_{j=1}^{\infty} a_j = A p_{j(r)}(x). \tag{1.3}$$

Combining (1.2) and (1.3), we get that if $x \in \frac{r}{2A} V_{j(r)}$, i.e. if $p_{j(r)}(x) \leq \frac{r}{2A}$, then:

$$d(x,o) = \sum_{j=1}^{j(r)} a_j \frac{p_j(x)}{1+p_j(x)} + \sum_{j=j(r)+1}^{\infty} a_j \frac{p_j(x)}{1+p_j(x)} < A p_{j(r)}(x) + \frac{r}{2} \leq r.$$

This proves that $\frac{r}{2A} V_{j(r)} \subseteq B_r(o)$. By (1.1), there always exists $n \in \mathbb{N}$ s.t. $V_n \subseteq \frac{r}{2A} V_{j(r)}$ and so 1 holds. To prove 2, let us fix $j \in \mathbb{N}$. Then clearly

$$a_j \frac{p_j(x)}{1+p_j(x)} \leq d(x,o), \qquad \forall x \in X.$$

As the a_j's are all positive, the latter implies that:

$$p_j(x) \leq a_j^{-1}(1+p_j(x)) d(x,o), \qquad \forall x \in X.$$

Therefore, if $x \in B_{\frac{a_j}{2}}(o)$ then $d(x,o) \leq \frac{a_j}{2}$ and so $p_j(x) \leq \frac{(1+p_j(x))}{2}$, which gives $p_j(x) \leq 1$. Hence, $B_{\frac{a_j}{2}}(o) \subseteq V_j$ which proves 2. $\square$

Let us show now the small lemma used in the proof above:

Lemma 1.1.4. *Let $a, b, c \in \mathbb{R}^+$ such that $c \leq a+b$ then $\frac{c}{1+c} \leq \frac{a}{1+a} + \frac{b}{1+b}$.*

Proof. W.l.o.g. we can assume $c > 0$ and $a + b > 0$. (Indeed, if $c = 0$ or $a + b = 0$ then there is nothing to prove.)Then $c \leq a + b$ is equivalent to $\frac{1}{a+b} \leq \frac{1}{c}$. This implies that $\left(1 + \frac{1}{c}\right)^{-1} \leq \left(1 + \frac{1}{a+b}\right)^{-1}$ which is equivalent to:

$$\frac{c}{1+c} \leq \frac{a+b}{1+a+b} = \frac{a}{1+a+b} + \frac{b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b}.$$

$\square$

We have therefore the following characterization of l.c. metrizable t.v.s.:

Proposition 1.1.5. *A locally convex t.v.s. (X, τ) is metrizable if and only if τ can be generated by a countable separating family of seminorms.*