Proof. Let $f : E \to F$ be a bounded linear map. Suppose that f is not continuous. Then there exists a neighbourhood V of the origin in F whose preimage $f^{-1}(V)$ is not a neighbourhood of the origin in E. W.l.o.g. we can always assume that V is balanced. As E is metrizable, we can take a countable basis $\{U_n\}_{n\in\mathbb{N}}$ of neighbourhood of the origin in E s.t. $U_n \supseteq U_{n+1}$ for all $n \in \mathbb{N}$. Then for all $m \in \mathbb{N}$ we have $\frac{1}{m}U_m \not\subseteq f^{-1}(V)$ i.e.

$$\forall\, m \in \mathbb{N},\, \exists\, x_m \in \frac{1}{m}U_m \text{ s.t. } f(x_m) \notin V. \tag{2.3}$$

As for all $m \in \mathbb{N}$ we have $mx_m \in U_m$ we get that the sequence $\{mx_m\}_{m\in\mathbb{N}}$ converges to the origin o in E. In fact, for any neighbourhood U of the origin o in E there exists $\bar{n} \in \mathbb{N}$ s.t. $U_{\bar{n}} \subseteq U$. Then for all $n \geq \bar{n}$ we have $nx_n \in U_n \subseteq U_{\bar{n}} \subseteq U$, i.e. $\{mx_m\}_{m\in\mathbb{N}}$ converges to o.

Hence, Proposition 2.2.7 implies that $\{mx_m\}_{m\in\mathbb{N}_0}$ is bounded in E and so, since f is bounded, also $\{mf(x_m)\}_{m\in\mathbb{N}_0}$ is bounded in F. This means that there exists $\rho > 0$ s.t. $\{mf(x_m)\}_{m\in\mathbb{N}_0} \subseteq \rho V$. Then for all $n \in \mathbb{N}$ with $n \geq \rho$ we have $f(x_n) \in \frac{\rho}{n}V \subseteq V$ which contradicts (2.3).

□

To show that the previous proposition also hold for LF-spaces, we need to introduce the following characterization of bounded sets in LF-spaces.

Proposition 2.3.5.
Let (E, τ_{ind}) be an LF-space with defining sequence $\{(E_n, \tau_n)\}_{n\in\mathbb{N}}$. A subset B of E is bounded in E if and only if there exists $n \in \mathbb{N}$ s.t. B is contained in E_n and B is bounded in E_n.

To prove this result we will need the following refined version of Lemma 1.3.4.

Lemma 2.3.6. *Let Y be a locally convex space, Y_0 a closed linear subspace of Y equipped with the subspace topology, U a convex neighbourhood of the origin in Y_0, and $x_0 \in Y$ with $x_0 \notin U$. Then there exists a convex neighbourhood V of the origin in Y such that $x_0 \notin V$ and $V \cap Y_0 = U$.*

Proof.
By Lemma 1.3.4 we have that there exists a convex neighbourhood W of the origin in Y such that $W \cap Y_0 = U$. Now we need to distinguish two cases:
-If $x_0 \in Y_0$ then necessarily $x_0 \notin W$ since by assumption $x_0 \notin U$. Hence, we are done by taking $V := W$.
-If $x_0 \notin Y_0$, then let us consider the quotient Y/Y_0 and the canonical map $\phi : Y \to Y/Y_0$. As Y_0 is a closed linear subspace of Y and Y is locally convex,

we have that Y/Y_0 is Hausdorff and locally convex. Then, since $\phi(x_0) \neq o$, there exists a convex neighbourhood N of the origin o in Y/Y_0 such that $\phi(x_0) \notin N$. Set $\Omega := \phi^{-1}(N)$. Then Ω is a convex neighbourhood of the origin in Y such that $x_0 \notin \Omega$ and clearly $Y_0 \subseteq \Omega$ (as $\phi(Y_0) = o \in N$). Therefore, if we consider $V := \Omega \cap W$ then we have that: V is a convex neighbourhood of the origin in Y, $V \cap Y_0 = \Omega \cap W \cap Y_0 = W \cap Y_0 = U$ and $x_0 \notin V$ since $x_0 \notin \Omega$. □

Proof. of Proposition 2.3.5

Suppose first that B is contained and bounded in some E_n. Let U be an arbitrary neighbourhood of the origin in E. Then by Proposition 1.3.5 we have that $U_n := U \cap E_n$ is a neighbourhood of the origin in E_n. Since B is bounded in E_n, there is a number $\lambda > 0$ such that $B \subseteq \lambda U_n \subseteq \lambda U$, i.e. B is bounded in E.

Conversely, assume that B is bounded in E. Suppose that B is not contained in any of the E_n's, i.e. $\forall\, n \in \mathbb{N}$, $\exists\, x_n \in B$ s.t. $x_n \notin E_n$. We will show that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is not bounded in E and so a fortiori B cannot be bounded in E.

Since $x_1 \notin E_1$ but $x_1 \in B \subseteq E$ and E_1 is a closed linear subspace of (E, τ_{ind}), given an arbitrary convex neighbourhood U_1 of the origin in E_1 we can apply Lemma 2.3.6 and get that there exists V_2 convex neighbourhood of the origin in E s.t. $x_1 \notin V_2$ and $V_2 \cap E_1 = U_1$. As $\tau_{ind} \restriction E_2 = \tau_2$, we have that $U_2 := V_2 \cap E_2$ is a convex neighbourhood of the origin in E_2 s.t. $x_1 \notin U_2$ and $U_2 \cap E_1 = V_2 \cap E_2 \cap E_1 = V_2 \cap E_1 = U_1$.

Since $x_1 \notin U_2$, we can once again apply Lemma 2.3.6 and proceed as above to get that there exists U_3' convex neighbourhood of the origin in E_3 s.t. $x_1 \notin U_3'$ and $U_3' \cap E_2 = U_2$. Since $x_2 \notin E_2$ we also have that $\frac{1}{2}x_2 \notin E_2$ and so $\frac{1}{2}x_2 \notin U_2$. By applying again Lemma 2.3.6 and proceeding as above, we get that there exists U_3'' convex neighbourhood of the origin in E_3 s.t. $\frac{1}{2}x_2 \notin U_3''$ and $U_3'' \cap E_2 = U_2$. Taking $U_3 := U_3' \cap U_3''$ we have that $U_3 \cap E_2 = U_2$ and $x_1, \frac{1}{2}x_2 \notin U_2$.

By induction on n, we get a sequence $\{U_n\}_{n\in\mathbb{N}}$ such that for any $n \in \mathbb{N}$:

- U_n is a convex neighbourhood of the origin in E_n
- $U_n = U_{n+1} \cap E_n$ (and so $U_n \subseteq U_{n+1}$)
- $x_1, \frac{1}{2}x_2, \ldots, \frac{1}{n}x_n \notin U_{n+1}$.

Note that:

$$U_n = U_{n+1} \cap E_n = U_{n+2} \cap E_{n+1} \cap E_n = U_{n+2} \cap E_n = \cdots = U_{n+k} \cap E_n, \quad \forall k \in \mathbb{N}.$$

Consider $U := \bigcup_{j=1}^{\infty} U_j$, then for each $n \in \mathbb{N}$ we have

$$U \cap E_n = \left(\bigcup_{j=1}^{n} U_j \cap E_n \right) \cup \left(\bigcup_{j=n+1}^{\infty} U_j \cap E_n \right) = U_n \cup \left(\bigcup_{k=1}^{\infty} U_{n+k} \cap E_n \right) = U_n,$$

i.e. U is a neighbourhood of the origin in (E, τ_{ind}).

Suppose that $\{x_j\}_{j\in\mathbb{N}}$ is bounded in E and take a balanced neighbourhood V of the origin in E s.t. $V \subseteq U$. Then there exists $\lambda > 0$ s.t. $\{x_j\}_{j\in\mathbb{N}} \subseteq \lambda V$ and so $\{x_j\}_{j\in\mathbb{N}} \subseteq nV$ for all $n \in \mathbb{N}$ with $n \geq \lambda$. In particular, we have $x_n \in nV$ and so $\frac{1n}{x\ n} \in V \subseteq U$, which contradicts the third property of the U_j's (i.e. $1nx_n \notin = \bigcup_{j=1}^{\infty} U_{n+j}.\bigcup_{j=n+1}^{\infty} U_j = U$ since $U_j \subseteq U_{j+1}$ for all $j \in \mathbb{N}$). Hence, $\{x_j\}_{j\in\mathbb{N}}$ is not bounded in E and so B is not bounded in E. This contradicts our original assumption and so proves that $B \subseteq E_n$ for some $n \in \mathbb{N}$.

It remains to show that B is bounded in E_n. Let W_n be a neighbourhood of the origin in E_n. By Proposition 1.3.5, there exists a neighbourhood W of the origin in E such that $W \cap E_n = W_n$. Since B is bounded in E, there exists $\mu > 0$ s.t. $B \subseteq \mu W$ and hence

$$B = B \cap E_n \subseteq \mu W \cap E_n = \mu(W \cap E_n) = \mu W_n.$$

□

Corollary 2.3.7. *A bounded linear map from an LF- space into an arbitrary t.v.s. is always continuous.*

Proof. (Exercise Sheet 5) □

Chapter 3

Topologies on the dual space of a t.v.s.

In this chapter we are going to describe a general method to construct a whole class of topologies on the topological dual of a t.v.s. using the notion of polar of a subset. Among these topologies, usually called polar topologies, there are: the weak topology, the topology of compact convergence and the strong topology. In this chapter we will denote by:

- E a t.v.s. over the field $\mathbb{K}$ of real or complex numbers.
- E^* the algebraic dual of E, i.e. the vector space of all linear functionals on E.
- E' its topological dual of E, i.e. the vector space of all continuous linear functionals on E.

Moreover, given $x' \in E'$, we denote by $\langle x', x\rangle$ its value at the point x of E, i.e. $\langle x', x\rangle = x'(x)$. The bracket $\langle \cdot, \cdot \rangle$ is often called *pairing* between E and E'.

3.1 The polar of a subset of a t.v.s.

Definition 3.1.1. *Let A be a subset of E. We define the* polar of A *to be the subset A° of E' given by:*

$$A^\circ := \left\{ x' \in E' : \sup_{x \in A} |\langle x', x\rangle| \leq 1 \right\}.$$

Let us list some properties of polars:

a) The polar A° of a subset A of E is a convex balanced subset of E'.
b) If $A \subseteq B \subseteq E$, then $B^\circ \subseteq A^\circ$.
c) $(\rho A)^\circ = (\frac{1}{\rho})A^\circ$, $\forall\, \rho > 0, \forall\, A \subseteq E$.
d) $(A \cup B)^\circ = A^\circ \cap B^\circ$, $\forall\, A, B \subseteq E$.
e) If A is a cone in E, then $A^\circ \equiv \{x' \in E' : \langle x', x\rangle = 0, \forall\, x \in A\}$ and A° is a linear subspace of E'. In particular, this property holds when A is a linear subspace of E and, in this case, A° is called the *orthogonal of A*.

Proof.
Let us show just property e) while the proof of a), b), c) and d) is left as an exercise for the reader. Suppose that A is a cone, i.e. $\forall\,\lambda > 0$, $\forall x \in A, \lambda x \in A$. Then $x' \in A^\circ$ implies that $|\langle x', x\rangle| \leq 1$ for all $x \in A$. Since A is a cone, we must also have $|\langle x', \lambda x\rangle| \leq 1$ for all $x \in A$ and all $\lambda > 0$. This means that $|\langle x', x\rangle| \leq \frac{1}{\lambda}$ for all $x \in A$ and all $\lambda > 0$, which clearly gives $\langle x', x\rangle = 0$ for all $x \in A$. Hence, $A^\circ \subseteq \{x' \in E' : \langle x', x\rangle = 0, \forall\, x \in A\}$. The other inclusion is trivial. In this case, it is easy to see that A° is a linear subspace of E'. Indeed: $\forall\, x', y' \in A^\circ$, $\forall\, x \in A$, $\forall\, \lambda, \mu \in \mathbb{K}$ we have

$$\langle \lambda x' + \mu y', x\rangle = \lambda\langle x', x\rangle + \mu\langle y', x\rangle = \lambda \cdot 0 + \mu \cdot 0 = 0.$$

□

Proposition 3.1.2. *Let E be a t.v.s.. If B is a bounded subset of E, then the polar B° of B is an absorbing subset of E'.*

Proof.
Let $x' \in E'$. As B is bounded in E, Corollary 2.2.10 guarantees that any continuous linear functional x' on E is bounded on B, i.e. there exists a constant $M(x') > 0$ such that $\sup_{x\in B} |\langle x', x\rangle| \leq M(x')$. This implies that for any $\lambda \in \mathbb{K}$ with $|\lambda| \leq \frac{1}{M(x')}$ we have $\lambda x' \in B^\circ$, since

$$\sup_{x\in B} |\langle \lambda x', x\rangle| = |\lambda| \sup_{x\in B} |\langle x', x\rangle| \leq \frac{1}{M(x')} \cdot M(x') = 1.$$

□

3.2 Polar topologies on the topological dual of a t.v.s.

We are ready to define an entire class of topologies on the dual E' of E, called *polar topologies*. Consider a family Σ of bounded subsets of E with the following two properties:
(P1) If $A, B \in \Sigma$, then $\exists\, C \in \Sigma$ s.t. $A \cup B \subseteq C$.
(P2) If $A \in \Sigma$ and $\lambda \in \mathbb{K}$, then $\exists\, B \in \Sigma$ s.t. $\lambda A \subseteq B$.
Let us denote by Σ° the family of the polars of the sets belonging to Σ, i.e.

$$\Sigma^\circ := \{A^\circ : \; A \in \Sigma\}.$$

Claim: Σ° is a basis of neighbourhoods of the origin for a locally convex topology on E' compatible with the linear structure.

Proof. of Claim.
By Property a) of polars and by Proposition 3.1.2, all elements of Σ° are convex balanced absorbing susbsets of E'. Also:

1. $\forall\, A^\circ, B^\circ \in \Sigma^\circ$, $\exists\, C^\circ \in \Sigma^\circ$ s.t. $C^\circ \subseteq A^\circ \cap B^\circ$.
   Indeed, if A° and B° in Σ° are respectively the polars of A and B in Σ, then by (P1) there exists $C \in \Sigma$ s.t. $A \cup B \subseteq C$ and so, by properties b) and d) of polars, we get: $C^\circ \subseteq (A \cup B)^\circ = A^\circ \cap B^\circ$.
2. $\forall\, A^\circ \in \Sigma^\circ$, $\forall\, \rho > 0$, $\exists\, B^\circ \in \Sigma^\circ$ s.t. $B^\circ \subseteq \rho A^\circ$.
   Indeed, if A° in Σ° is the polar of A, then by (P2) there exists $B \in \Sigma$ s.t. $\frac{1}{\rho} A \subseteq B$ and so, by properties b) and c) of polars, we get that $B^\circ \subseteq \left(\frac{1}{\rho} A\right)^\circ = \rho A^\circ$.

By Theorem 4.1.14 in TVS-I, there exists a unique locally convex topology on E' compatible with the linear structure and having Σ° as a basis of neighborhoods of the origin. □

Definition 3.2.1. *Given a family Σ of bounded subsets of a t.v.s. E s.t. (P1) and (P2) hold, we call* $\Sigma-$topology *on E' the locally convex topology defined by taking, as a basis of neighborhoods of the origin in E', the family Σ° of the polars of the subsets that belong to Σ. We denote by E'_Σ the space E' endowed with the Σ-topology.*

It is easy to see from the definition that (Exercise Sheet 6):

- The $\Sigma-$topology on E' is generated by the following family of seminorms:

$$\{p_A : A \in \Sigma\}, \text{ where } p_A(x') := \sup_{x \in A} |\langle x', x\rangle|, \forall\, x' \in E'. \tag{3.1}$$

- Define for any $A \in \Sigma$ and $\varepsilon > 0$ the following subset of E':

$$W_\varepsilon(A) := \left\{ x' \in E' : \sup_{x \in A} |\langle x', x\rangle| \leq \varepsilon \right\}.$$

  The family $\mathcal{B} := \{W_\varepsilon(A) : A \in \Sigma, \varepsilon > 0\}$ is a basis of neighbourhoods of the origin for the $\Sigma-$topology on E'.

Let us introduce now some important examples of polar topologies.

The weak topology on E'

The *weak topology on E'* is the $\Sigma-$topology corresponding to the family Σ of all finite subsets of E and it is usually denoted by $\sigma(E', E)$ (this topology is often also referred with the name of *weak*-topology* or *weak dual topology*). We denote by E'_σ the space E' endowed with the topology $\sigma(E', E)$.

A basis of neighborhoods of $\sigma(E', E)$ is given by the family

$$\mathcal{B}_\sigma := \{W_\varepsilon(x_1, \ldots, x_r) : r \in \mathbb{N}, x_1, \ldots, x_r \in E, \varepsilon > 0\}$$

where

$$W_\varepsilon(x_1, \ldots, x_r) := \left\{x' \in E' : |\langle x', x_j \rangle| \leq \varepsilon, \, j = 1, \ldots, r\right\}. \tag{3.2}$$

The topology of compact convergence on E'

The *topology of compact convergence on* E' is the $\Sigma-$topology corresponding to the family Σ of all compact subsets of E and it is usually denoted by $c(E', E)$. We denote by E'_c the space E' endowed with the topology $c(E', E)$.

The strong topology on E'

The *strong topology on* E' is the $\Sigma-$topology corresponding to the family Σ of all bounded subsets of E and it is usually denoted by $b(E', E)$. As a filter in E' converges to the origin in the strong topology if and only if it converges to the origin uniformly on every bounded subset of E (see Proposition **??**), the strong topology on E' is sometimes also referred as *the topology of bounded convergence*. When E' carries the strong topology, it is usually called the *strong dual of* E and denoted by E'_b.