A basis of neighborhoods of $\sigma(E', E)$ is given by the family

$$\mathcal{B}_\sigma := \{W_\varepsilon(x_1, \dots, x_r) : r \in \mathbb{N}, x_1, \dots, x_r \in E, \varepsilon > 0\}$$

where

$$W_\varepsilon(x_1, \dots, x_r) := \left\{x' \in E' : |\langle x', x_j\rangle| \leq \varepsilon, \, j = 1, \dots, r\right\}. \tag{3.2}$$

The topology of compact convergence on E'

The *topology of compact convergence on* E' is the $\Sigma-$topology corresponding to the family Σ of all compact subsets of E and it is usually denoted by $c(E', E)$. We denote by E'_c the space E' endowed with the topology $c(E', E)$.

The strong topology on E'

The *strong topology on* E' is the $\Sigma-$topology corresponding to the family Σ of all bounded subsets of E and it is usually denoted by $b(E', E)$. As a filter in E' converges to the origin in the strong topology if and only if it converges to the origin uniformly on every bounded subset of E (see Proposition 3.2.2), the strong topology on E' is sometimes also referred as *the topology of bounded convergence*. When E' carries the strong topology, it is usually called the *strong dual of* E and denoted by E'_b.

Let us look now at some general properties of polar topologies and how they relate to the above examples.

Proposition 3.2.2. *A filter $\mathcal{F}'$ on E' converges to an element $x' \in E'$ in the Σ-topology on E' if and only if $\mathcal{F}'$ converges uniformly to x' on each subset A belonging to Σ, i.e. the following holds:*

$$\forall \varepsilon > 0, \forall A \in \Sigma, \; \exists \; M' \in \mathcal{F}' \; s.t. \sup_{x \in A} |\langle x', x\rangle - \langle y', x\rangle| \leq \varepsilon, \, \forall \, y' \in M'. \tag{3.3}$$

This proposition explains why the $\Sigma-$topology on E' is often referred as *topology of the uniform converge over the sets of* Σ.

Proof.

Suppose that (3.3) holds and let U be a neighbourhood of the origin in the $\Sigma-$topology on E'. Then there exists $\varepsilon > 0$ and $A \in \Sigma$ s.t. $W_\varepsilon(A) \subseteq U$ and so

$$x' + W_\varepsilon(A) \subseteq x' + U. \tag{3.4}$$

On the other hand, since we have that

$$\begin{aligned} x' + W_\varepsilon(A) &= \left\{ x' + y' \in E' : \sup_{x \in A} |\langle y', x \rangle| \leq \varepsilon \right\} \\ &= \left\{ z' \in E' : \sup_{x \in A} |\langle z' - x', x \rangle| \leq \varepsilon \right\}, \end{aligned} \tag{3.5}$$

the condition (3.3) together with (3.4) gives that

$$\exists\ M' \in \mathcal{F}' \text{ s.t. } M' \subseteq x' + W_\varepsilon(A) \subseteq x' + U.$$

The latter implies that $x' + U \in \mathcal{F}'$ since $\mathcal{F}'$ is a filter and so the family of all neighbourhoods of x' in the $\Sigma-$topology on E' is contained in $\mathcal{F}'$, i.e. $\mathcal{F}' \to x'$.

Conversely, if $\mathcal{F}' \to x'$, then for any neighbourhood V of x' in the $\Sigma-$topology on E' we have $V \in \mathcal{F}'$. In particular, for all $A \in \Sigma$ and for all $\varepsilon > 0$ we have $x' + W_\varepsilon(A) \in \mathcal{F}'$. Then by taking $M' := x' + W_\varepsilon(A)$ and using (3.5), we easily get (3.3). □

Remark 3.2.3. *Using the previous result, one can easily show that sequence $\{x'_n\}_{n\in\mathbb{N}}$ of elements in E' converges to the origin in the weak topology if and only if at each point $x \in E$ the sequence of their values $\{\langle x'_n, x\rangle\}_{n\in\mathbb{N}}$ converges to zero in $\mathbb{K}$ (see Exercise Sheet 6). In other words, the weak topology on E' is nothing else but the topology of pointwise convergence in E, when we look at continuous linear functionals on E simply as functions on E.*

In general we can compare two polar topologies by using the following criterion: *If Σ_1 and Σ_2 are two families of bounded subsets of a t.v.s. E such that (P1) and (P2) hold and $\Sigma_1 \supseteq \Sigma_2$, then the Σ_1-topology is finer than the Σ_2-topology.* In particular, this gives the following comparison relations between the three polar topologies on E' introduced above:

$$\sigma(E', E) \subseteq c(E', E) \subseteq b(E', E).$$

Proposition 3.2.4. *Let Σ be a family of bounded subsets of a t.v.s. E s.t. (P1) and (P2) hold. If the union of all subsets in Σ is dense in E, then E'_Σ is Hausdorff.*

Proof. Assume that the union of all subsets in Σ is dense in E. As the $\Sigma-$topology is locally convex, to show that E'_Σ is Hausdorff is enough to check that the family of seminorms in (3.1) is separating (see Proposition 4.3.3 in TVS-I). Suppose that $p_A(x') = 0$ for all $A \in \Sigma$, then

$$\sup_{x \in A} |\langle x', x \rangle| = 0, \ \forall A \in \Sigma,$$

which gives
$$\langle x', x\rangle = 0, \, \forall x \in \bigcup_{A\in\Sigma} A.$$

As the continuous functional x' is zero on a dense subset of E, it has to be identically zero on the whole E. Hence, the family $\{p_A : A \in \Sigma\}$ is a separating family of seminorms which generates the $\Sigma-$topology on E'. □

Corollary 3.2.5. *The topology of compact convergence, the weak and the strong topologies on E' are all Hausdorff.*

Let us consider now for any $x \in E$ the linear functional v_x on E' which associates to each element of the dual E' its *"value at the point x"*, i.e.

$$\begin{array}{rccc} v_x : & E' & \to & \mathbb{K} \\ & x' & \mapsto & \langle x', x\rangle. \end{array}$$

Clearly, each $v_x \in (E')^*$ but when can we say that $v_x \in (E'_\Sigma)'$? Can we find conditions on Σ which guarantee the continuity of v_x w.r.t. the $\Sigma-$topology?

Fixed an arbitrary $x \in E$, v_x is continuous on E'_Σ if and only if for any $\varepsilon > 0$, $v_x^{-1}(\bar{B}_\varepsilon(0))$ is a neighbourhood of the origin in E' w.r.t. the $\Sigma-$topology ($\bar{B}_\varepsilon(0)$ denotes the closed ball of radius ε and center 0 in $\mathbb{K}$). This means that

$$\forall\, \varepsilon > 0, \exists\, A \in \Sigma : \; A^\circ \subseteq v_x^{-1}(\bar{B}_\varepsilon(0)) = \{x' \in E' : |\langle x', x\rangle| \leq \varepsilon\}$$

i.e.

$$\forall\, \varepsilon > 0, \exists\, A \in \Sigma : \left|\langle x', \frac{1}{\varepsilon} x\rangle\right| \leq 1, \forall\, x' \in A^\circ. \tag{3.6}$$

Then it is easy to see that the following holds:

Proposition 3.2.6. *Let Σ be a family of bounded subsets of a t.v.s. E s.t. (P1) and (P2) hold. If Σ covers E then for every $x \in E$ the value at x is a continuous linear functional on E'_Σ, i.e. $v_x \in (E'_\Sigma)'$.*

Proof. If $E \subseteq \bigcup_{A\in\Sigma} A$ then for any $x \in E$ and any $\varepsilon > 0$ we have $\frac{1}{\varepsilon} \in A$ for some $A \in \Sigma$ and so $|\langle x', \frac{1}{\varepsilon} x\rangle| \leq 1$ for all $x' \in A^\circ$. This means that (3.6) holds, which is equivalent to v_x being continuous w.r.t. the $\Sigma-$topology on E'. □

The previous proposition is useful to get the following characterization of the weak topology on E', which is often taken as a definition for this topology.

Proposition 3.2.7. *Let E be a t.v.s.. The weak topology on E' is the coarsest topology on E' such that, for all $x \in E$, v_x is continuous.*

Proof.
Since the weak topology $\sigma(E', E)$ is by definition the $\Sigma-$topology on E' corresponding to the family Σ of all finite subsets of E which clearly covers E, Proposition 3.2.6 ensures that all v_x are continuous on E'_σ.[1] Moreover, if there would exist a topology τ on E' strictly coarser that $\sigma(E', E)$ and such that all v_x were continuous, then in particular $\forall\, \varepsilon > 0$, $\forall\, r \in \mathbb{N}$, $\forall\, x_1, \ldots, x_r \in E$, each $v_{x_i}^{-1}(\bar{B}_\varepsilon(0))$ would be a neighbourhood of the origin in (E', τ) for $i = 1, \ldots, r$. Hence, each $W_\varepsilon(x_1, \ldots, x_r)$ would be a neighbourhood of the origin in (E', τ), since $W_\varepsilon(x_1, \ldots, x_r) = \bigcap_{i=1}^{r} v_{x_i}^{-1}(\bar{B}_\varepsilon(0))$ (cf. (3.2)). Therefore, any element of a basis of neighborhoods of the origin in E'_σ is also a neighbourhood of the origin in (E', τ). This implies that the two topologies τ and $\sigma(E', E)$ must necessarily coincide. □

Proposition 3.2.6 means that, if Σ covers E then the image of E under the canonical map

$$\begin{array}{rccc} \varphi: & E & \to & (E'_\Sigma)^* \\ & x & \mapsto & v_x. \end{array}$$

is contained in the topological dual of E'_Σ, i.e. $\varphi(E) \subseteq (E'_\Sigma)'$. In general, the canonical map $\varphi : E \to (E'_\Sigma)'$ is neither injective or surjective. However, when we restrict our attention to locally convex Hausdorff t.v.s., the following consequence of Hahn-Banach theorem guarantees the injectivity of the canonical map.

Lemma 3.2.8. *If E is a locally convex Hausdorff t.v.s with $E \neq \{o\}$, then for every $o \neq x_0 \in E$ there exists $x' \in E'$ s.t. $\langle x', x_0 \rangle \neq 0$, i.e. $E' \neq \{o\}$.*

Proof. (see Interactive Sheet 3) □

Corollary 3.2.9. *Let E be a non-trivial locally convex Hausdorff t.v.s and Σ a family of bounded subsets of E s.t. (P1) and (P2) hold and Σ covers E. Then the canonical map $\varphi : E \to (E'_\Sigma)'$ is injective.*

Proof. Let $o \neq x_0 \in E$. By Proposition 3.2.8, we know that there exists $x' \in E'$ s.t. $v_x(x') \neq 0$ which proves that v_x is not identically zero on E' and so that $Ker(\varphi) = \{o\}$. Hence, φ is injective. □

[1] Fixed $x \in E$, one could also show the continuity of v_x w.r.t. $\sigma(E', E)$ by simply noticing that $|v_x(x')| = p_{\{x\}}(x')$ for any $x' \in E'$ and using Corollary 4.6.2. in TVS-I about continuity of functionals on locally convex t.v.s.

In the particular case of the weak topology on E' the canonical map $\varphi : E \to (E'_\sigma)'$ is also surjective, and so E can be regarded as the dual of its weak dual E'_σ. To show this result we will need to use the following consequence of Hahn-Banach theorem:

Lemma 3.2.10. *Let Y be a closed linear subspace of a locally convex t.v.s. X. If $Y \neq X$, then there exists $f \in X'$ s.t. f is not identically zero on X but identically vanishes on Y.*

Proof. (see Exercise Sheet 6)

Proposition 3.2.11. *Let E be a locally convex Hausdorff t.v.s. with $E \neq \{o\}$. Then the canonical map $\varphi : E \to (E'_\sigma)'$ is an isomorphism.*

Proof. Let $L \in (E'_\sigma)'$. By the definition of $\sigma(E', E)$ and Proposition 4.6.1 in TVS-I, we have that there exist $F \subset E$ with $|F| < \infty$ and $C > 0$ s.t.

$$|L(x')| \leq C p_F(x') = C \sup_{x \in F} |\langle x', x \rangle|. \tag{3.7}$$

Take $M := span(F)$ and $d := dim(M)$. Consider an algebraic basis $\mathcal{B} := \{e_1, \ldots, e_d\}$ of M and for each $j \in \{1, \ldots, d\}$ apply Lemma 3.2.10 to $Y := span\{\mathcal{B} \setminus \{e_j\}\}$ and $X := M$. Then for each $j \in \{1, \ldots, d\}$ there exists $f_j : M \to \mathbb{K}$ linear and continuous such that $\langle f_j, e_k \rangle = 0$ if $k \neq j$ and $\langle f_j, e_j \rangle \neq 0$. W.l.o.g. we can assume $\langle f_j, e_j \rangle = 1$. By applying the Hahn-Banach theorem (see Theorem 5.1.1 in TVS-I), we get that for each $j \in \{1, \ldots, d\}$ there exists $e'_j : E \to \mathbb{K}$ linear and continuous such that $e'_j \restriction_M = f_j$, in particular $\langle e'_j, e_k \rangle = 0$ for $k \neq j$ and $\langle e'_j, e_j \rangle = 1$.

Let $M' := span\{e'_1, \ldots, e'_d\} \subset E'$, $x_L := \sum_{j=1}^d L(e'_j) e_j \in M$ and for any $x' \in E'$ define $p(x') := \sum_{j=1}^d \langle x', e_j \rangle e'_j \in M'$. Then for any $x' \in E'$ we get that:

$$\langle x', x_L \rangle = \sum_{j=1}^d L(e'_j) \langle x', e_j \rangle = L(p(x')) \tag{3.8}$$

and also

$$\langle x' - p(x'), e_k \rangle = \langle x', e_k \rangle - \sum_{j=1}^d \langle x', e_j \rangle \langle e'_j, e_k \rangle = \langle x', e_k \rangle - \langle x', e_k \rangle \langle e_k, e_k \rangle = 0$$

which gives

$$\langle x' - p(x'), m \rangle = 0, \forall m \in M. \tag{3.9}$$

Then for all $x' \in E'$ we have:

$$|L(x' - p(x'))| \overset{(3.7)}{\leq} C \sup_{x \in F} |\langle x' - p(x'), x \rangle| \overset{(3.9)}{=} 0$$

which give that $L(x') = L(p(x')) \overset{(3.8)}{=} \langle x', x_L \rangle = v_{x_L}(x')$. Hence, we have proved that for every $L \in (E'_\sigma)'$ there exists $x_L \in E$ s.t. $\varphi(x_L) \equiv v_{x_L} \equiv L$, i.e. $\varphi : E \to (E'_\sigma)'$ is surjective. Then we are done because the injectivity of $\varphi : E \to (E'_\sigma)'$ follows by applying Corollary 3.2.9 to this special case. □

Remark 3.2.12. *The previous result suggests that it is indeed more convenient to restrict our attention to locally convex Hausdorff t.v.s. when dealing with weak duals. Moreover, as showed in Proposition 3.2.8, considering locally convex Hausdorff t.v.s has the advantage of avoiding the pathological situation in which the topological dual of a non-trivial t.v.s. is reduced to the only zero functional (for an example of a t.v.s. on which there are no continuous linear functional than the trivial one, see Exercise Sheet 6).*