Then for all $x' \in E'$ we have:

$$|L(x' - p(x'))| \overset{(3.7)}{\leq} C \sup_{x \in F} |\langle x' - p(x'), x\rangle| \overset{(3.9)}{=} 0$$

which give that $L(x') = L(p(x')) \overset{(3.8)}{=} \langle x', x_L\rangle = v_{x_L}(x')$. Hence, we have proved that for every $L \in (E'_\sigma)'$ there exists $x_L \in E$ s.t. $\varphi(x_L) \equiv v_{x_L} \equiv L$, i.e. $\varphi : E \to (E'_\sigma)'$ is surjective. Then we are done because the injectivity of $\varphi : E \to (E'_\sigma)'$ follows by applying Corollary 3.2.9 to this special case. □

Remark 3.2.12. *The previous result suggests that it is indeed more convenient to restrict our attention to locally convex Hausdorff t.v.s. when dealing with weak duals. Moreover, as showed in Proposition 3.2.8, considering locally convex Hausdorff t.v.s has the advantage of avoiding the pathological situation in which the topological dual of a non-trivial t.v.s. is reduced to the only zero functional (for an example of a t.v.s. on which there are no continuous linear functional than the trivial one, see Exercise Sheet 6).*

3.3 The polar of a neighbourhood in a locally convex t.v.s.

Let us come back now to the study of the weak topology and prove one of the milestones of the t.v.s. theory: the *Banach-Alaoglu-Bourbaki theorem.* To prove this important result we need to look for a moment at the algebraic dual E^* of a t.v.s. E. In analogy to what we did in the previous section, we can define *the weak topology on the algebraic dual* E^* (which we will denote by $\sigma(E^*, E)$) as the coarsest topology such that for any $x \in E$ the linear functional w_x is continuous, where

$$\begin{array}{rccl} w_x : & E^* & \to & \mathbb{K} \\ & x^* & \mapsto & \langle x^*, x\rangle := x^*(x). \end{array} \tag{3.10}$$

(Note that $w_x \restriction E' = v_x$). Equivalently, the weak topology on the algebraic dual E^* is the locally convex topology on E^* generated by the family $\{q_F : F \subseteq E, |F| < \infty\}$ of seminorms $q_F(x^*) := \sup_{x \in F} |\langle x^*, x\rangle|$ on E^*. It is then easy to see that $\sigma(E', E) = \sigma(E^*, E) \restriction E'$.

An interesting property of the weak topology on the algebraic dual of a t.v.s. is the following one:

Proposition 3.3.1. *If E is a t.v.s. over $\mathbb{K}$, then its algebraic dual E^* endowed with the weak topology $\sigma(E^*, E)$ is topologically isomorphic to the product of $dim(E)$ copies of the field $\mathbb{K}$ endowed with the product topology.*

Proof.
Let $\{e_i\}_{i\in I}$ be an algebraic basis of E, i.e. $\forall\, x \in E$, $\exists\, \{x_i\}_{i\in I} \in \mathbb{K}^{dim(E)}$ s.t. $x = \sum_{i\in I} x_i e_i$. For any linear functions $L : E \to \mathbb{K}$ and any $x \in E$ we then have $L(x) = \sum_{i\in I} x_i L(e_i)$. Hence, L is completely determined by the sequence $\{L(e_i)\}_{i\in I} \in \mathbb{K}^{dim(E)}$. Conversely, every element $u := \{u_i\}_{i\in I} \in \mathbb{K}^{dim(E)}$ uniquely defines the linear functional L_u on E via $L_u(e_i) := u_i$ for all $i \in I$. This completes the proof that E^* is algebraically isomorphic to $\mathbb{K}^{dim(E)}$. Moreover, the collection $\{W_\varepsilon(e_{i_1}, \dots, e_{i_r}) : \varepsilon > 0,\, r \in \mathbb{N},\, i_1, \dots, i_r \in I\}$, where

$$W_\varepsilon(e_{i_1}, \dots, e_{i_r}) := \{x^* \in E^* : |\langle x^*, e_{i_j}\rangle| \leq \varepsilon, \text{ for } j = 1, \dots, r\},$$

is a basis of neighbourhoods of the origin in $(E^*, \sigma(E^*, E))$. Via the isomorphism described above, we have that for any $\varepsilon > 0$, $r \in \mathbb{N}$, and $i_1, \dots, i_r \in I$:

$$\begin{aligned} W_\varepsilon(e_{i_1}, \dots, e_{i_r}) &\approx \left\{\{u_i\}_{i\in I} \in \mathbb{K}^{dim(E)} : |u_{i_j}| \leq \varepsilon, \text{ for } j = 1, \dots, r\right\} \\ &= \prod_{j=1}^{r} \bar{B}_\varepsilon(0) \times \prod_{I\setminus\{i_1,\dots,i_r\}} \mathbb{K} \end{aligned}$$

and so $W_\varepsilon(e_{i_1}, \dots, e_{i_r})$ is a neighbourhood of the product topology τ_{prod} on $\mathbb{K}^{dim(E)}$ (recall that we always consider the euclidean topology on $\mathbb{K}$). Therefore, $(E^*, \sigma(E^*, E))$ is topological isomorphic to $\left(\mathbb{K}^{dim(E)}, \tau_{prod}\right)$. □

Let us now focus our attention on the polar of a neighbourhood U of the origin in a non-trivial locally convex Hausdorff t.v.s. E. We are considering here only non-trivial locally convex Hausdorff t.v.s. in order to be sure to have non-trivial continuous linear functionals (see Remark 3.2.12) and so to make a meaningful analysis on the topological dual.

First of all let us observe that:

$$\{x^* \in E^* : \sup_{x\in U} |\langle x^*, x\rangle| \leq 1\} \equiv U^\circ := \{x' \in E' : \sup_{x\in U} |\langle x', x\rangle| \leq 1\}. \quad (3.11)$$

Indeed, since $E' \subseteq E^*$, we clearly have $U^\circ \subseteq \{x^* \in E^* : \sup_{x\in U} |\langle x^*, x\rangle| \leq 1\}$. Moreover, any linear functional $x^* \in E^*$ s.t. $\sup_{x\in U} |\langle x^*, x\rangle| \leq 1$ is continuous on E and it is therefore an element of E'.

It is then quite straightforward to show that:

Proposition 3.3.2. *The polar of a neighbourhood U of the origin in E is closed w.r.t. $\sigma(E^*, E)$.*

Proof. By (3.11) and (3.10), it is clear that $U^\circ = \bigcap_{x\in U} w_x^{-1}([-1,1])$. On the other hand, by definition of $\sigma(E^*, E)$ we have that w_x is continuous on $(E^*, \sigma(E^*, E))$ for all $x \in E$ and so each $w_x^{-1}([-1,1])$ is closed in $(E^*, \sigma(E^*, E))$. Hence, U° is closed in $(E^*, \sigma(E^*, E))$ as the intersection of closed subsets of $(E^*, \sigma(E^*, E))$. □

We are ready now to prove the famous Banach-Alaoglu-Bourbaki Theorem

Theorem 3.3.3 (Banach-Alaoglu-Bourbaki Theorem)**.**
The polar of a neighbourhood U of the origin in a locally convex Hausdorff t.v.s. $E \neq \{o\}$ is compact in E'_σ.

Proof.
Since U is a neighbourhood of the origin in E, U is absorbing in E, i.e. $\forall\, x \in E$, $\exists\, M_x > 0$ s.t. $M_x x \in U$. Hence, for all $x \in E$ and all $x' \in U^\circ$ we have $|\langle x', M_x x\rangle| \leq 1$, which is equivalent to:

$$\forall\, x \in E,\, \forall\, x' \in U^\circ,\ |\langle x', x\rangle| \leq \frac{1}{M_x}. \tag{3.12}$$

Moreover, for any $x \in E$, the subset

$$D_x := \left\{\alpha \in \mathbb{K} :\ |\alpha| \leq \frac{1}{M_x}\right\}$$

is compact in $\mathbb{K}$ w.r.t. to the euclidean topology.

Consider an algebraic basis $\mathcal{B}$ of E, then by Tychnoff's theorem[2] the subset $P := \prod_{x\in\mathcal{B}} D_x$ is compact in $\left(\mathbb{K}^{dim(E)}, \tau_{prod}\right)$.

Using the isomorphism introduced in Proposition 3.3.1 and (3.11), we get that

$$U^\circ \approx \{(\langle x^*, x\rangle)_{x\in\mathcal{B}} : x^* \in U^\circ\}$$

and so by (3.12) we have that $U^\circ \subset P$. Since Corollary 3.3.2 and Proposition 3.3.1 ensure that U° is closed in $\left(\mathbb{K}^{dim(E)}, \tau_{prod}\right)$, we get that U° is a closed subset of P. Hence, by Proposition 2.1.4–1, U° is compact $\left(\mathbb{K}^{dim(E)}, \tau_{prod}\right)$ and so in $(E^*, \sigma(E^*, E))$. As $U^\circ = E' \cap U^\circ$ we easily see that U° is compact in $(E', \sigma(E', E))$. □

[2]**Tychnoff's theorem**: The product of an arbitrary family of compact spaces endowed with the product topology is also compact.

We briefly introduce now a nice consequence of Banach-Alaoglu-Bourbaki theorem. Let us start by introducing a norm on the topological dual space E' of a seminormed space (E, ρ):

$$\rho'(x') := \sup_{x \in E: \rho(x) \leq 1} |\langle x', x \rangle|.$$

ρ' is usually called the *operator norm* on E'.

Corollary 3.3.4. *Let (E, ρ) be a non-trivial normed space. The closed unit ball in E' w.r.t. the operator norm ρ' is compact in E'_σ.*

Proof. First of all, let us note that a normed space it is indeed a locally convex Hausdorff t.v.s.. Then, by applying Banach-Alaoglu-Borubaki theorem to the closed unit ball $\bar{B}_1(o)$ in (E, ρ), we get that $\left(\bar{B}_1(o)\right)^\circ$ is compact in E'_σ. The conclusion then easily follow by the observation that $\left(\bar{B}_1(o)\right)^\circ$ actually coincides with the closed unit ball in (E', ρ'):

$$\begin{aligned}
\left(\bar{B}_1(o)\right)^\circ &= \{x' \in E' : \sup_{x \in \bar{B}_1(o)} |\langle x', x \rangle| \leq 1\} \\
&= \{x' \in E' : \sup_{x \in E', \rho(x) \leq 1} |\langle x', x \rangle| \leq 1\} \\
&= \{x' \in E' : \rho'(x') \leq 1\}.
\end{aligned}$$

□

Chapter 4

Tensor products of t.v.s.

4.1 Tensor product of vector spaces

As usual, we consider only vector spaces over the field $\mathbb{K}$ of real numbers or of complex numbers.

Definition 4.1.1.
*Let E, F, M be three vector spaces over $\mathbb{K}$ and $\phi : E \times F \to M$ be a bilinear map. E and F are said to be ϕ-*linearly disjoint *if:*

(LD) *For any $r \in \mathbb{N}$, any $\{x_1, \dots, x_r\}$ finite subset of E and any $\{y_1, \dots, y_r\}$ finite subset of F s.t. $\sum_{i=1}^{r} \phi(x_i, y_j) = 0$, we have that both the following conditions hold:*

- *if $x_1, \dots, x_r$ are linearly independent in E, then $y_1 = \cdots = y_r = 0$*
- *if $y_1, \dots, y_r$ are linearly independent in F, then $x_1 = \cdots = x_r = 0$*

Recall that, given three vector spaces over $\mathbb{K}$, a map $\phi : E \times F \to M$ is said to be *bilinear* if:

$$\forall\, x_0 \in E, \quad \phi_{x_0} : \begin{array}{ccc} F & \to & M \\ y & \to & \phi(x_0, y) \end{array} \qquad \text{is linear}$$

and

$$\forall\, y_0 \in F, \quad \phi_{y_0} : \begin{array}{ccc} E & \to & M \\ x & \to & \phi(x, y_0) \end{array} \qquad \text{is linear.}$$

Let us give a useful characterization of $\phi-$linear disjointness.

Proposition 4.1.2. *Let E, F, M be three vector spaces, and $\phi : E \times F \to M$ be a bilinear map. Then E and F are $\phi-$linearly disjoint if and only if:*

(LD') *For any $r, s \in \mathbb{N}$, $x_1, \dots, x_r$ linearly independent in E and $y_1, \dots, y_s$ linearly independent in F, the set $\{\phi(x_i, y_j) : i = 1, \dots, r,\ j = 1, \dots, s\}$ consists of linearly independent vectors in M.*

Proof.

(⇒) Let $x_1, \dots, x_r$ be linearly independent in E and $y_1, \dots, y_s$ be linearly independent in F. Suppose that $\sum_{i=1}^r \sum_{j=1}^s \lambda_{ij}\phi(x_i, y_j) = 0$ for some $\lambda_{ij} \in \mathbb{K}$. Then, using the bilinearity of ϕ and setting $z_i := \sum_{j=1}^s \lambda_{ij} y_j$, we easily get $\sum_{i=1}^r \phi(x_i, z_i) = 0$. As the x_i's are linearly independent in E, we derive from (LD) that all z_i's have to be zero. This means that for each $i \in \{1, \dots, r\}$ we have $\sum_{j=1}^s \lambda_{ij} y_j = 0$, which implies by the linearly independence of the y_j's that $\lambda_{ij} = 0$ for all $i \in \{1, \dots, r\}$ and all $j \in \{1, \dots, s\}$.

(⇐) Let $r \in \mathbb{N}$, $\{x_1, \dots, x_r\} \subseteq E$ and $\{y_1, \dots, y_r\} \subseteq F$ be such that $\sum_{i=1}^r \phi(x_i, y_i) = 0$. Suppose that the x_is are linearly independent and let $\{z_1, \dots, z_s\}$ be a basis of $span\{y_1, \dots, y_r\}$. Then for each $i \in \{1, \dots, r\}$ there exist $\lambda_{ij} \in \mathbb{K}$ s.t. $y_i = \sum_{j=1}^s \lambda_{ij} z_j$ and so by the bilinearity of ϕ we get:

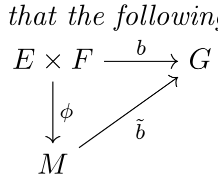
$$0 = \sum_{i=1}^r \phi(x_i, y_j) = \sum_{i=1}^r \sum_{j=1}^s \lambda_{ij}\phi(x_i, z_j). \tag{4.1}$$

By applying (LD') to the x_i's and $z_j's$, we get that all $\phi(x_i, z_j)$'s are linearly independent. Therefore, (4.1) gives that $\lambda_{ij} = 0$ for all $i \in \{1, \dots, r\}$ and all $j \in \{1, \dots, s\}$ and so $y_i = 0$ for all $i \in \{1, \dots, r\}$. Exchanging the roles of the x_i's and the y_i's we get that (LD) holds. □

Definition 4.1.3. *A tensor product of two vector spaces E and F over $\mathbb{K}$ is a pair (M, ϕ) consisting of a vector space M over $\mathbb{K}$ and of a bilinear map $\phi : E \times F \to M$ (canonical map) s.t. the following conditions are satisfied:*
(TP1) *The image of $E \times F$ spans the whole space M.*
(TP2) *E and F are $\phi-$linearly disjoint.*

We now show that the tensor product of any two vector spaces always exists, satisfies the "universal property" and it is unique up to isomorphisms. For this reason, the tensor product of E and F is usually denoted by $E \otimes F$ and the canonical map by $(x, y) \mapsto x \otimes y$.

Theorem 4.1.4. *Let E, F be two vector spaces over $\mathbb{K}$.*
(a) There exists a tensor product of E and F.
(b) Let (M, ϕ) be a tensor product of E and F. Let G be any vector space over $\mathbb{K}$, and b any bilinear mapping of $E \times F$ into G. There exists a unique linear map $\tilde{b} : M \to G$ such that the following diagram is commutative.



(c) If (M_1, ϕ_1) and (M_2, ϕ_2) are two tensor products of E and F, then there is a bijective linear map u such that the following diagram is commutative.

$$\begin{array}{ccc} E \times F & \xrightarrow{\phi_2} & M_2 \\ \big\downarrow{\scriptstyle \phi_1} & \nearrow{\scriptstyle u} & \\ M_1 & & \end{array}$$