(c) If (M_1, ϕ_1) and (M_2, ϕ_2) are two tensor products of E and F, then there is a bijective linear map u such that the following diagram is commutative.

$$\begin{array}{ccc} E \times F & \xrightarrow{\phi_2} & M_2 \\ \Big\downarrow{\scriptstyle \phi_1} & \nearrow_{u} & \\ M_1 & & \end{array}$$

Proof.

(a) Let $\mathcal{H}$ be the vector space of all functions from $E \times F$ into $\mathbb{K}$ which vanish outside a finite set ($\mathcal{H}$ is often called the free space of $E \times F$). For any $(x, y) \in E \times F$, let us define the function $e_{(x,y)} : E \times F \to \mathbb{K}$ as follows:

$$e_{(x,y)}(z, w) := \begin{cases} 1 & \text{if } (z, w) = (x, y) \\ 0 & \text{otherwise} \end{cases} .$$

Then $\mathcal{B}_{\mathcal{H}} := \{e_{(x,y)} : (x, y) \in E \times F\}$ forms a basis of $\mathcal{H}$ and so $\forall\, h \in \mathcal{H}$, $\exists!\ \lambda_{xy} \in \mathbb{K}$ s.t. $h = \sum_{x \in E} \sum_{y \in F} \lambda_{xy} e_{(x,y)}$ with $\lambda_{xy} = 0$ for all but finitely many x's in E and y's in Y. Let us consider now the following linear subspace of $\mathcal{H}$:

$$N := span \left\{ e_{\left(\sum\limits_{i=1}^{n} a_i x_i, \sum\limits_{j=1}^{m} b_j y_j \right)} - \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j e_{(x_i, y_j)} \,:\, n, m \in \mathbb{N}, a_i, b_j \in \mathbb{K}, (x_i, y_j) \in E \times F \right\}.$$

We then denote by M the quotient vector space $\mathcal{H}/N$, by π the quotient map from $\mathcal{H}$ onto M and by

$$\begin{array}{rccl} \phi: & E \times F & \to & M \\ & (x, y) & \to & \phi(x, y) := \pi\left(e_{(x,y)}\right). \end{array}$$

It is easy to see that the map ϕ is bilinear. Let us just show the linearity in the first variable as the proof in the second variable is just symmetric. Fixed $y \in F$, for any $a, b \in \mathbb{K}$ and any $x_1, x_2 \in E$, we get that:

$$\begin{aligned} \phi(ax_1 + bx_2, y) - a\phi(x_1, y) - b\phi(x_2, y) &= \pi\left(e_{(ax_1+bx_2,y)}\right) - a\pi\left(e_{(x_1,y)}\right) - b\pi\left(e_{x_2,y)}\right) \\ &= \pi\left(e_{(ax_1+bx_2,y)} - ae_{(x_1,y)} - be_{(x_2,y)}\right) \\ &= 0, \end{aligned}$$

where the last equality holds since $e_{(ax_1+bx_2,y)} - ae_{(x_1,y)} - be_{(x_2,y)} \in N$.

We aim to show that (M, ϕ) is a tensor product of E and F. It is clear from the definition of ϕ that

$$span(\phi(E \times F)) = span(\pi(\mathcal{B}_{\mathcal{H}})) = \pi(\mathcal{H}) = M,$$

i.e. (TP1) holds. It remains to prove that E and F are $\phi-$linearly disjoint. Let $r \in \mathbb{N}$, $\{x_1, \ldots, x_r\} \subseteq E$ and $\{y_1, \ldots, y_r\} \subseteq F$ be such that $\sum_{i=1}^r \phi(x_i, y_i) = 0$. Suppose that the y_i's are linearly independent. For any $\varphi \in E^*$, let us define the linear mapping $A_\varphi : \mathcal{H} \to F$ by setting $A_\varphi(e_{(x,y)}) := \varphi(x)y$. Then it is easy to check that A_φ vanishes on N, so it induces a map $\tilde{A}_\varphi : M \to F$ s.t. $\tilde{A}_\varphi(\pi(f)) = A(f)$, $\forall\, f \in \mathcal{H}$. Hence, since $\sum_{i=1}^r \phi(x_i, y_i) = 0$ can be rewritten as $\pi\left(\sum_{i=1}^r e_{(x_i,y_i)}\right) = 0$, we get that

$$0 = \tilde{A}_\varphi\left(\pi\left(\sum_{i=1}^r e_{(x_i,y_i)}\right)\right) = A_\varphi\left(\sum_{i=1}^r e_{(x_i,y_i)}\right) = \sum_{i=1}^r A_\varphi(e_{(x_i,y_i)}) = \sum_{i=1}^r \varphi(x_i)y_i.$$

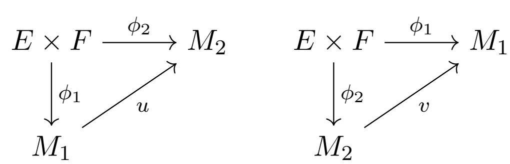
This together with the linear independence of the y_i's implies $\varphi(x_i) = 0$ for all $i \in \{1, \ldots, r\}$. Since the latter holds for all $\varphi \in E^*$, we have that $x_i = 0$ for all $i \in \{1, \ldots, r\}$. Exchanging the roles of the x_i's and the y_i's we get that (LD) holds, and so does (TP2) .

(b) Let (M, ϕ) be a tensor product of E and F, G a vector space and $b : E \times F \to G$ a bilinear map. Consider $\{x_\alpha\}_{\alpha \in A}$ and $\{y_\beta\}_{\beta \in B}$ bases of E and F, respectively. We know that $\{\phi(x_\alpha, y_\beta) : \alpha \in A, \beta \in B\}$ forms a basis of M, as *span*$(\phi(E \times F)) = M$ and, by Proposition 4.1.2, (LD') holds so the $\phi(x_\alpha, y_\beta)$'s for all $\alpha \in A$ and all $\beta \in B$ are linearly independent. The linear mapping $\tilde{b}$ will therefore be the unique linear map of M into G such that

$$\forall\, \alpha \in A, \forall\, \beta \in B,\ \tilde{b}(\phi(x_\alpha, y_\beta)) = b(x_\alpha, y_\beta).$$

Hence, the diagram in (b) commutes.

(c) Let (M_1, ϕ_1) and (M_2, ϕ_2) be two tensor products of E and F. Then using twice the universal property (b) we get that there exist unique linear maps $u : M_1 \to M_2$ and $v : M_2 \to M_1$ such that the following diagrams both commute:



Then combining $u \circ \phi_1 = \phi_2$ with $v \circ \phi_2 = \phi_1$, we get that u and v are one the inverse of the other. Hence, there is an algebraic isomorphism between M_1 and M_2. □

It is now natural to introduce the concept of tensor product of linear maps.

Proposition 4.1.5. *Let E, F, E_1, F_1 be four vector spaces over $\mathbb{K}$, and let $u : E \to E_1$ and $v : F \to F_1$ be linear mappings. There is a unique linear map of $E \otimes F$ into $E_1 \otimes F_1$, called the* tensor product of u and v *and denoted by $u \otimes v$, such that*

$$(u \otimes v)(x \otimes y) = u(x) \otimes v(y), \quad \forall\, x \in E, \forall\, y \in F.$$

Proof.
Let us define the mapping

$$\begin{array}{rccl} b: & E \times F & \to & E_1 \otimes F_1 \\ & (x, y) & \mapsto & b(x, y) := u(x) \otimes v(y), \end{array}$$

which is clearly bilinear because of the linearity of u and v and the bilinearity of the canonical map of the tensor product $E_1 \otimes F_1$. Then by the universal property there is a unique linear map $\tilde{b} : E \otimes F \to E_1 \otimes F_1$ s.t. the following diagram commutes:

$$\begin{array}{ccc} E \times F & \xrightarrow{\ b\ } & E_1 \otimes F_1 \\ \Big\downarrow \otimes & \nearrow_{\tilde{b}} & \\ E \otimes F & & \end{array}$$

i.e. $\tilde{b}(x \otimes y) = b(x, y), \forall\, (x, y) \in E \times F$. Hence, using the definition of b, we get that $\tilde{b} \equiv u \otimes v$. □

Examples 4.1.6.

1. *Let $n, m \in \mathbb{N}$, $E = \mathbb{K}^n$ and $F = \mathbb{K}^m$. Then $E \otimes F = \mathbb{K}^{n \times m}$ is a tensor product of E and F whose canonical bilinear map ϕ is given by:*

$$\begin{array}{rccl} \phi: & E \times F & \to & \mathbb{K}^{n \times m} \\ & \Big((x_i)_{i=1}^n, (y_j)_{j=1}^m \Big) & \mapsto & (x_i y_j)_{1 \leq i \leq n, 1 \leq j \leq m}. \end{array}$$

2. *Let X and Y be two sets. For any functions $f : X \to \mathbb{K}$ and $g : Y \to \mathbb{K}$, we define:*

$$\begin{array}{rccl} f \otimes g: & X \times Y & \to & \mathbb{K} \\ & (x, y) & \mapsto & f(x)g(y). \end{array}$$

Let E (resp. F) be the linear space of all functions from X (resp. Y) to $\mathbb{K}$ endowed with the usual addition and multiplication by scalars. We *denote by M the linear subspace of the space of all functions from $X \times Y$ to $\mathbb{K}$ spanned by the elements of the form $f \otimes g$ for all $f \in E$ and $g \in F$. Then M is actually a tensor product of E and F (see Exercise Sheet 7).*

Given X and Y open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, we can use the definitions in Example 2 above to construct the tensors $\mathcal{C}^k(X) \otimes \mathcal{C}^l(Y)$ for any $1 \leq k, l \leq \infty$. Then it is possible to show the following result (see e.g. [5, Theorem 39.2] for a proof).

Theorem 4.1.7. *Let X and Y open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Then $\mathcal{C}_c^\infty(X) \otimes \mathcal{C}_c^\infty(Y)$ is sequentially dense in $\mathcal{C}_c^\infty(X \times Y)$ endowed with the $\mathcal{C}^\infty-$topology.*

4.2 Topologies on the tensor product of locally convex t.v.s.

Given two locally convex t.v.s. E and F, there various ways to construct a topology on the tensor product $E \otimes F$ which makes the vector space $E \otimes F$ in a t.v.s.. Indeed, starting from the topologies on E and F, one can define a topology on $E \otimes F$ either relying directly on the seminorms on E and F, or using an embedding of $E \otimes F$ in some space related to E and F over which a natural topology already exists. The first method leads to the so-called $\pi-$topology. The second method may lead instead to a variety of topologies, the most important of which is the so-called $\varepsilon-$topology that is based on the isomorphism between $E \otimes F$ and $B(E'_\sigma, F'_\sigma)$ (see Proposition **??**).

4.2.1 $\pi-$topology

Let us define the first main topology on $E \otimes F$ which we will see can be directly characterized by mean of the seminorms generating the topologies on the starting locally convex t.v.s. E and F.

Definition 4.2.1 ($\pi-$topology)**.**
Given two locally convex t.v.s. E and F, we define the $\pi-$topology *(or* projective topology*) on $E \otimes F$ to be the finest locally convex topology on this vector space for which the canonical mapping $E \times F \to E \otimes F$ is continuous. The space $E \otimes F$ equipped with the $\pi-$topology will be denoted by $E \otimes_\pi F$.*

A basis of neighbourhoods of the origin in $E \otimes_\pi F$ is given by the family:

$$\mathcal{B}_\pi := \{conv_b(U_\alpha \otimes V_\beta) : U_\alpha \in \mathcal{B}_E, V_\beta \in \mathcal{B}_F\},$$

where $\mathcal{B}_E$ (resp. $\mathcal{B}_F$) is a basis of neighbourhoods of the origin in E (resp. in F), $U_\alpha \otimes V_\beta := \{x \otimes y \in E \otimes F : x \in U_\alpha, y \in V_\beta\}$ and $conv_b(U_\alpha \otimes V_\beta)$ denotes the smallest convex balanced subset of $E \otimes F$ containing $U_\alpha \otimes V_\beta$. Indeed, by Theorem 4.1.14 in TVS-I, the topology generated by $\mathcal{B}_\pi$ is a locally convex topology $E \otimes F$ and it makes continuous the canonical map $\otimes$, since for any $U_\alpha \in \mathcal{B}_E$ and $V_\beta \in \mathcal{B}_F$ we have that $\otimes^{-1}(conv_b(U_\alpha \otimes V_\beta)) \supseteq \otimes^{-1}(U_\alpha \otimes V_\beta) = U_\alpha \times V_\beta$ which is a neighbourhood of the origin in $E \times F$. Hence, the topology generated by $\mathcal{B}_\pi$ is coarser than the $\pi-$topology. Moreover, the $\pi-$topology is by definition locally convex and so it has a basis $\mathcal{B}$ of convex balanced neighbourhoods of the origin in $E \otimes F$. Then, as the canonical mapping $\otimes$ is continuous w.r.t. the $\pi-$topology, we have that for any $C \in \mathcal{B}$ there exist $U_\alpha \in \mathcal{B}_E$ and $V_\beta \in \mathcal{B}_F$ s.t. $U_\alpha \times V_\beta \subseteq \otimes^{-1}(C)$. Hence, $U_\alpha \otimes V_\beta \subseteq C$ and so $conv_b(U_\alpha \otimes V_\beta) \subseteq conv_b(C) = C$, which yields that the topology generated by $\mathcal{B}_\pi$ is finer than the $\pi-$topology.

The $\pi-$topology on $E \otimes F$ can be described by means of the seminorms defining the locally convex topologies on E and F. Indeed, we have the following characterization of the $\pi-$topology.

Proposition 4.2.2. *Let E and F be two locally convex t.v.s. and let $\mathcal{P}$ (resp. $\mathcal{Q}$) be a family of seminorms generating the topology on E (resp. on F). The $\pi-$topology on $E \otimes F$ is generated by the family of seminorms*

$$\{p \otimes q : \, p \in \mathcal{P}, q \in \mathcal{Q}\},$$

where for any $p \in \mathcal{P}, q \in \mathcal{Q}, \theta \in E \otimes F$ we define:

$$(p \otimes q)(\theta) := \inf\{\rho > 0 : \, \theta \in \rho conv_b(U_p \otimes V_q)\}$$

with $U_p := \{x \in E : p(x) \leq 1\}$ and $V_q := \{y \in F : q(y) \leq 1\}$.

Proof. (Exercise Sheet 7) □

The seminorm $p \otimes q$ on $E \otimes F$ defined in the previous proposition is called *tensor product of the seminorms p and q* (or *projective cross seminorm*) and it can be represented in a more practical way that shows even more directly its relation to the seminorms defining the topologies on E and F.

Theorem 4.2.3.

a) *For any $\theta \in E \otimes F$, we have:*

$$(p \otimes q)(\theta) := \inf\left\{\sum_{k=1}^{r} p(x_k)q(y_k) : \theta = \sum_{k=1}^{r} x_k \otimes y_k, \, , x_k \in E, y_k \in F, r \in \mathbb{N}\right\}.$$

b) *For all $x \in E$ and $y \in F$, $(p \otimes q)(x \otimes y) = p(x)q(y)$.*

Proof. (see Lect 14) □

Proposition 4.2.4. *Let E and F be two locally convex t.v.s.. $E \otimes_\pi F$ is Hausdorff if and only if E and F are both Hausdorff.*

Proof. (Exercise Sheet 7) □

Corollary 4.2.5. *Let (E, p) and (F, q) be seminormed spaces. Then $p \otimes q$ is a norm on $E \otimes F$ if and only if p and q are both norms.*

Proof.
Under our assumptions, the $\pi-$topology on $E \otimes F$ is generated by the single seminorm $p \otimes q$. Then, recalling that a seminormed space is normed iff it is Hausdorff and using Proposition 4.2.4, we get: $(E \otimes F, p \otimes q)$ is normed $\Leftrightarrow$ $E \otimes_\pi F$ is Hausdorff $\Leftrightarrow$ E and F are both Hausdorff $\Leftrightarrow$ (E, p) and (F, q) are both normed. □

Definition 4.2.6. *Let (E, p) and (F, q) be normed spaces. The normed space $(E \otimes F, p \otimes q)$ is called the* projective tensor product *of E and F and $p \otimes q$ is said to be the corresponding* projective tensor norm.