by Theorem 4.1.14 in TVS-I, the topology generated by $\mathcal{B}_\pi$ is a locally convex topology $E \otimes F$ and it makes continuous the canonical map $\otimes$, since for any $U_\alpha \in \mathcal{B}_E$ and $V_\beta \in \mathcal{B}_F$ we have that $\otimes^{-1}(conv_b(U_\alpha \otimes V_\beta)) \supseteq \otimes^{-1}(U_\alpha \otimes V_\beta) = U_\alpha \times V_\beta$ which is a neighbourhood of the origin in $E \times F$. Hence, the topology generated by $\mathcal{B}_\pi$ is coarser than the $\pi-$topology. Moreover, the $\pi-$topology is by definition locally convex and so it has a basis $\mathcal{B}$ of convex balanced neighbourhoods of the origin in $E \otimes F$. Then, as the canonical mapping $\otimes$ is continuous w.r.t. the $\pi-$topology, we have that for any $C \in \mathcal{B}$ there exist $U_\alpha \in \mathcal{B}_E$ and $V_\beta \in \mathcal{B}_F$ s.t. $U_\alpha \times V_\beta \subseteq \otimes^{-1}(C)$. Hence, $U_\alpha \otimes V_\beta \subseteq C$ and so $conv_b(U_\alpha \otimes V_\beta) \subseteq conv_b(C) = C$, which yields that the topology generated by $\mathcal{B}_\pi$ is finer than the $\pi-$topology.

The $\pi-$topology on $E \otimes F$ can be described by means of the seminorms defining the locally convex topologies on E and F. Indeed, we have the following characterization of the $\pi-$topology.

Proposition 4.2.2. *Let E and F be two locally convex t.v.s. and let $\mathcal{P}$ (resp. $\mathcal{Q}$) be a family of seminorms generating the topology on E (resp. on F). The $\pi-$topology on $E \otimes F$ is generated by the family of seminorms*

$$\{p \otimes q : p \in \mathcal{P}, q \in \mathcal{Q}\},$$

where for any $p \in \mathcal{P}, q \in \mathcal{Q}, \theta \in E \otimes F$ we define:

$$(p \otimes q)(\theta) := \inf\{\rho > 0 : \theta \in \rho \ conv_b(U_p \otimes V_q)\}$$

with $U_p := \{x \in E : p(x) \leq 1\}$ and $V_q := \{y \in F : q(y) \leq 1\}$.

Proof. (Exercise Sheet 7) □

The seminorm $p \otimes q$ on $E \otimes F$ defined in the previous proposition is called *tensor product of the seminorms p and q* (or *projective cross seminorm*) and it can be represented in a more practical way that shows even more directly its relation to the seminorms defining the topologies on E and F.

Theorem 4.2.3.
Let E and F be two locally convex t.v.s. and let $\mathcal{P}$ (resp. $\mathcal{Q}$) be a family of seminorms generating the topology on E (resp. on F). Then for any $p \in \mathcal{P}$ and any $q \in \mathcal{Q}$ we have that the following hold.

a) *For all $\theta \in E \otimes F$,*

$$(p \otimes q)(\theta) = \inf\left\{\sum_{k=1}^{r} p(x_k)q(y_k) : \theta = \sum_{k=1}^{r} x_k \otimes y_k, \ , x_k \in E, y_k \in F, r \in \mathbb{N}\right\}.$$

b) *For all $x \in E$ and $y \in F$, $(p \otimes q)(x \otimes y) = p(x)q(y)$.*

Proof.
a) As above, we set $U_p := \{x \in E : p(x) \leq 1\}$, $V_q := \{y \in F : q(y) \leq 1\}$ and $W_{pq} := conv_b(U_p \otimes V_q)$. Let $\theta \in E \otimes F$ and $\rho > 0$ such that $\theta \in \rho W_{pq}$.

Let us preliminarily observe that the condition "$\theta \in \rho W_{pq}$ for some $\rho > 0$" is equivalent to:

$$\begin{aligned} &\theta = \sum_{k=1}^{N} t_k x_k \otimes y_k \quad \text{with } N \in \mathbb{N},\ t_k \in \mathbb{K},\ x_k \in E \text{ and } \ y_k \in F \text{ s.t.} \\ &\sum_{k=1}^{N} |t_k| \leq \rho,\ p(x_k) \leq 1,\ q(y_k) \leq 1,\ \forall k \in \{1, \dots, N\}. \end{aligned} \tag{4.2}$$

If we set $\xi_k := t_k x_k$ and $\eta_k := y_k$, then we can rewrite the condition (4.2) as

$$\theta = \sum_{k=1}^{N} \xi_k \otimes \eta_k \text{ with } \sum_{k=1}^{N} p(\xi_k) q(\eta_k) \leq \rho.$$

Then $\inf \left\{ \sum_{k=1}^{N} p(\xi_k) q(\eta_k) : \theta = \sum_{k=1}^{N} \xi_k \otimes \eta_k, , \xi_k \in E, \eta_k \in F, N \in \mathbb{N} \right\} \leq \rho$. Since this is true for any $\rho > 0$ s.t. $\theta \in \rho W_{pq}$, we get:

$$\inf \left\{ \sum_{i=1}^{r} p(x_i) q(y_i) : \theta = \sum_{i=1}^{r} x_i \otimes y_i,\ x_i \in E, y_i \in F, r \in \mathbb{N} \right\} \leq (p \otimes q)(\theta).$$

Conversely, let us consider an arbitrary representation of θ, i.e.

$$\theta = \sum_{k=1}^{N} \xi_k \otimes \eta_k \text{ with } \xi_k \in E,\ \eta_k \in F,\ N \in \mathbb{N}.$$

Let $\rho > 0$ s.t. $\sum_{k=1}^{N} p(\xi_k) q(\eta_k) \leq \rho$ and $\varepsilon > 0$. Define

- $I_1 := \{k \in \{1, \dots, N\} : p(\xi_k) q(\eta_k) \neq 0\}$
- $I_2 := \{k \in \{1, \dots, N\} : p(\xi_k) \neq 0 \text{ and } q(\eta_k) = 0\}$
- $I_3 := \{k \in \{1, \dots, N\} : p(\xi_k) = 0 \text{ and } q(\eta_k) \neq 0\}$
- $I_4 := \{k \in \{1, \dots, N\} : p(\xi_k) = 0 \text{ and } q(\eta_k) = 0\}$

and set

- $\forall k \in I_1$, $x_k := \frac{\xi_k}{p(\xi_k)}$, $y_k := \frac{\eta_k}{q(\eta_k)}$, $t_k := p(\xi_k) q(\eta_k)$
- $\forall k \in I_2$, $x_k := \frac{\xi_k}{p(\xi_k)}$, $y_k := \frac{N}{\varepsilon} p(\xi_k) \eta_k$, $t_k := \frac{\varepsilon}{N}$
- $\forall k \in I_3$, $x_k := \frac{N}{\varepsilon} q(\eta_k) \xi_k$, $y_k := \frac{\eta_k}{q(\eta_k)}$, $t_k := \frac{\varepsilon}{N}$
- $\forall k \in I_4$, $x_k := \frac{N}{\varepsilon} \xi_k$, $y_k := \eta_k$, $t_k := \frac{\varepsilon}{N}$

Then $\forall k \in \{1, \dots, N\}$ we have that $p(x_k) \leq 1$ and $q(y_k) \leq 1$. Also we get:

$$\begin{aligned}
\sum_{k=1}^{N} t_k x_k \otimes y_k &= \sum_{k\in I_1} p(\xi_k)q(\eta_k)\frac{\xi_k}{p(\xi_k)} \otimes \frac{\eta_k}{q(\eta_k)} + \sum_{k\in I_2} \frac{\varepsilon}{N}\frac{\xi_k}{p(\xi_k)} \otimes \frac{N}{\varepsilon}p(\xi_k)\eta_k \\
&+ \sum_{k\in I_3} \frac{\varepsilon}{N}\frac{N}{\varepsilon}q(\eta_k)\xi_k \otimes \frac{\eta_k}{q(\eta_k)} + \sum_{k\in I_4} \frac{\varepsilon}{N}\frac{N}{\varepsilon}\xi_k \otimes \eta_k \\
&= \sum_{k=1}^{N} \xi_k \otimes \eta_k = \theta
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=1}^{N} |t_k| &= \sum_{k\in I_1} p(\xi_k)q(\eta_k) + \sum_{k\in(I_2\cup I_3\cup I_4)} \frac{\varepsilon}{N} \\
&= \sum_{k\in I_1} p(\xi_k)q(\eta_k) + |I_2 \cup I_3 \cup I_4|\frac{\varepsilon}{N} \\
&\leq \sum_{k=1}^{N} p(\xi_k)q(\eta_k) + \varepsilon \leq \rho + \varepsilon.
\end{aligned}$$

Hence, by (4.2) we get that $\theta \in (\rho + \varepsilon)W_{pq}$. As this holds for any $\varepsilon > 0$, we have $\theta \in \rho W_{pq}$. Therefore, we obtain that $(p \otimes q)(\theta) \leq \rho$ and in particular $(p \otimes q)(\theta) \leq \sum_{k=1}^{N} p(\xi_k)q(\eta_k)$. This yields that

$$(p \otimes q)(\theta) \leq \inf \left\{ \sum_{k=1}^{N} p(\xi_k)q(\eta_k) : \theta = \sum_{k=1}^{N} \xi_k \otimes \eta_k, \, , \xi_k \in E, \eta_k \in F, N \in \mathbb{N} \right\}.$$

b) Let $x \in E$ and $y \in F$. By using a), we immediately get that

$$(p \otimes q)(x \otimes y) \leq p(x)q(y).$$

Conversely, consider $M := span\{x\}$ and define $L : M \to \mathbb{K}$ as $L(\lambda x) := \lambda p(x)$ for all $\lambda \in \mathbb{K}$. Then clearly L is a linear functional on M and for any $m \in M$, i.e. $m = \lambda x$ for some $\lambda \in \mathbb{K}$, we have $|L(m)| = |\lambda|p(x) = p(\lambda x) = p(m)$. Therefore, Hahn-Banach theorem can be applied and provides that:

$$\exists\, x' \in E' \text{ s.t. } \langle x', x\rangle = p(x) \text{ and } |\langle x', x_1\rangle| \leq p(x_1), \forall\, x_1 \in E. \tag{4.3}$$

Repeating this reasoning for y we get that:

$$\exists\, y' \in F' \text{ s.t. } \langle y', y\rangle = q(y) \text{ and } |\langle y', y_1\rangle| \leq q(y_1), \forall\, y_1 \in F. \tag{4.4}$$

Let us consider now any representation of $x \otimes y$, namely $x \otimes y = \sum_{k=1}^{N} x_k \otimes y_k$ with $x_k \in E$, $y_k \in F$ and $N \in \mathbb{N}$. Then, combining Proposition 4.1.5 and the second part of both (4.3) and (4.4), we obtain:

$$
\begin{aligned}
\left|\langle x' \otimes y', x \otimes y\rangle\right| &\leq \sum_{k=1}^{N} |\langle x' \otimes y', x_k \otimes y_k\rangle| \\
&\overset{\text{Prop } 4.1.5}{=} \sum_{k=1}^{N} |\langle x', x_k\rangle| \cdot |\langle y', y_k\rangle| \\
&\overset{(4.3) \text{ and } (4.4)}{\leq} \sum_{k=1}^{N} p(x_k)q(x_k).
\end{aligned}
$$

Since this is true for any representation of $x \otimes y$, we deduce by a) that:

$$\left|\langle x' \otimes y', x \otimes y\rangle\right| \leq (p \otimes q)(x \otimes y).$$

The latter together with the first part of (4.3) and (4.4) gives:

$$p(x)q(y) = |p(x)| \cdot |q(y)| = |\langle x', x\rangle| \cdot |\langle y', y\rangle| = \left|\langle x' \otimes y', x \otimes y\rangle\right| \leq (p \otimes q)(x \otimes y).$$

□

Proposition 4.2.4. *Let E and F be two locally convex t.v.s.. $E \otimes_\pi F$ is Hausdorff if and only if E and F are both Hausdorff.*

Proof. (Exercise Sheet 7) □

Corollary 4.2.5. *Let (E, p) and (F, q) be seminormed spaces. Then $p \otimes q$ is a norm on $E \otimes F$ if and only if p and q are both norms.*

Proof.
Under our assumptions, the $\pi-$topology on $E \otimes F$ is generated by the single seminorm $p \otimes q$. Then, recalling that a seminormed space is normed iff it is Hausdorff and using Proposition 4.2.4, we get: $(E \otimes F, p \otimes q)$ is normed $\Leftrightarrow$ $E \otimes_\pi F$ is Hausdorff $\Leftrightarrow$ E and F are both Hausdorff $\Leftrightarrow$ (E, p) and (F, q) are both normed. □

Definition 4.2.6. *Let (E, p) and (F, q) be normed spaces. The normed space $(E \otimes F, p \otimes q)$ is called the* projective tensor product *of E and F and $p \otimes q$ is said to be the corresponding* projective tensor norm.

In analogy with the algebraic case (see Theorem 4.1.4-b), we also have a universal property for the space $E \otimes_\pi F$.

Proposition 4.2.7.
Let E, F be locally convex spaces. The $\pi-$topology on $E \otimes_\pi F$ is the unique locally convex topology on $E \otimes F$ such that the following property holds:

(UP) *For every locally convex space G, the algebraic isomorphism between the space of bilinear mappings from $E \times F$ into G and the space of all linear mappings from $E \otimes F$ into G (given by Theorem 4.1.4-b) induces an algebraic isomorphism between $B(E, F; G)$ and $L(E \otimes F; G)$, where $B(E, F; G)$ denotes the space of all continuous bilinear mappings from $E \times F$ into G and $L(E \otimes F; G)$ the space of all continuous linear mappings from $E \otimes F$ into G.*

Proof. We first show that the $\pi-$topology fulfills (UP). Let (G, ω) be a locally convex space and $b \in B(E, F; G)$, then Theorem 4.1.4-b) ensures that there exists a unique $\tilde{b} : E \otimes F \to G$ linear s.t. $\tilde{b} \circ \phi = b$, where $\phi : E \times F \to E \otimes F$ is the canonical mapping. Let U basic neighbourhood of the origin in G, so w.l.o.g. we can assume U convex and balanced. Then the continuity of b implies that there exist V basic neighbourhood of the origin in E and W basic neighbourhood of the origin in E s.t. $\tilde{b}(\phi(V \times W)) = b(V \times W) \subseteq U$. Hence, $\phi(V \times W) \subseteq \tilde{b}^{-1}(U)$ and so $conv_b(\phi(V \times W)) \subseteq conv_b(\tilde{b}^{-1}(U)) = \tilde{b}^{-1}(U)$, which shows the continuity of $\tilde{b} : E \otimes_\pi F \to (G, \omega)$ as $conv_b(\phi(V \times W)) \in \mathcal{B}_\pi$.

Let τ be a locally convex topology on $E \otimes F$ such that the property (UP) holds. Then (UP) holds in particular for $G = (E \otimes F, \tau)$. Therefore, since in the algebraic isomorphism given by Theorem 4.1.4-b) in this case the canonical mapping $\phi : E \times F \to E \otimes F$ corresponds to the identity $id : E \otimes F \to E \otimes F$, we get that $\phi : E \times F \to E \otimes_\tau F$ has to be continuous.

$$\begin{tikzcd} E \times F \arrow[r, "\phi"] \arrow[d, "\phi"] & E \otimes_\tau F \\ E \otimes_\tau F \arrow[ur, "id"'] & \end{tikzcd}$$

This implies that τ is coarser than the $\pi-$topology. On the other hand, (UP) also holds for $G = (E \otimes F, \pi)$. Hence,

$$\begin{tikzcd} E \times F \arrow[r, "\phi"] \arrow[d, "\phi"] & E \otimes_\pi F \\ E \otimes_\tau F \arrow[ur, "id"'] & \end{tikzcd}$$

since by definition of $\pi-$topology $\phi : E \times F \to E \otimes_\pi F$ is continuous, the $id : E \otimes_\tau F \to E \otimes_\pi F$ has to be also continuous. This means that the $\pi-$topology is coarser than τ, which completes the proof. $\square$

Corollary 4.2.8. *$(E \otimes_\pi F)' \cong B(E, F)$, where $B(E, F) := B(E, F; \mathbb{K})$.*

Proof. By taking $G = \mathbb{K}$ in Proposition 4.2.7, we get the conclusion. $\square$

4.2.2 $\varepsilon-$topology

The definition of $\varepsilon-$topology strongly relies strongly relies on the algebraic isomorphism between $E \otimes F$ and the space $B(E'_\sigma, F'_\sigma)$ of continuous bilinear forms on the product $E'_\sigma \times F'_\sigma$ of the weak duals of E and F (see Section 3.2 for the definition of weak topology). More precisely, the following hold.

Proposition 4.2.9. *Let E and F be non-trivial locally convex t.v.s. over $\mathbb{K}$ with non-trivial topological duals. The space $B(E'_\sigma, F'_\sigma)$ is a tensor product of E and F.*

Proof.
Let us consider the bilinear mapping:

$$\begin{array}{rccclcl} \phi: & E \times F & \to & B(E'_\sigma, F'_\sigma) & & & \\ & (x, y) & \mapsto & \phi(x, y): & E'_\sigma \times F'_\sigma & \to & \mathbb{K} \\ & & & & (x', y') & \mapsto & \langle x', x\rangle\langle y', y\rangle. \end{array} \tag{4.5}$$

We first show that E and F are ϕ-linearly disjoint. Let $r, s \in \mathbb{N}$, $x_1, \ldots, x_r$ be linearly independent in E and $y_1, \ldots, y_s$ be linearly independent in F. In their correspondence, select[1] $x'_1, \ldots, x'_r \in E'$ and $y'_1, \ldots, y'_s \in F'$ such that

$$\langle x'_m, x_j\rangle = \delta_{mj},\ \forall\, m, j \in \{1, \ldots, r\} \quad \text{and} \quad \langle y'_n, y_k\rangle = \delta_{nk}\, \forall\, n, k \in \{1, \ldots, s\}.$$

Then we have that:

$$\phi(x_j, y_k)(x'_m, y'_n) = \langle x'_m, x_j\rangle\langle y'_n, y_k\rangle = \begin{cases} 1 & \text{if } m = j \text{ and } n = k \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

This implies that the set $\{\phi(x_j, y_k) : j = 1, \ldots, r,\ k = 1, \ldots, s\}$ consists of linearly independent elements. Indeed, if there exists $\lambda_{jk} \in \mathbb{K}$ s.t.

$$\sum_{j=1}^{r}\sum_{k=1}^{s} \lambda_{jk}\phi(x_j, y_k) = 0$$

then for all $m \in \{1, \ldots, r\}$ and all $n \in \{1, \ldots, r\}$ we have that:

$$\sum_{j=1}^{r}\sum_{k=1}^{s} \lambda_{jk}\phi(x_j, y_k)(x'_m, y'_n) = 0$$

[1] This can be done using Lemma 3.2.10 together with the assumption that E' and F' are not trivial.

and so by using (4.6) that all $\lambda_{mn} = 0$.

We have therefore showed that (LD') holds and so, by Proposition 4.1.2, E and F are ϕ-linearly disjoint. Let us briefly sketch the main steps of the proof that $span(\phi(E \times F)) = B(E'_\sigma, F'_\sigma)$.

a) Take any $\varphi \in B(E'_\sigma, F'_\sigma)$. By the continuity of φ, it follows that there exist finite subsets $A \subset E$ and $B \subset F$ s.t. $|\varphi(x', y')| \leq 1$, $\forall\, x' \in A^\circ$, $\forall\, y' \in B^\circ$.

b) Set $E_A := span(A)$ and $F_B := span(B)$. Since E_A and E_B are finite dimensional, their orthogonals $(E_A)^\circ$ and $(F_B)^\circ$ have finite codimension and so

$$E' \times F' = (M' \oplus (E_A)^\circ) \times (N' \oplus (F_B)^\circ) = (M' \times N') \oplus ((E_A)^\circ \times F') \oplus (E' \times (F_B)^\circ),$$

where M' and N' finite dimensional subspaces of E' and F', respectively.

c) Using a) and b) one can prove that φ vanishes on the direct sum $((E_A)^\circ \times F') \oplus (E' \times (F_B)^\circ)$ and so that φ is completely determined by its restriction to a finite dimensional subspace $M' \times N'$ of $E' \times F'$.

d) Let $r := dim(E_A)$ and $s := dim(F_B)$. Then there exist $x_1, \ldots, x_r \in E_A$ and $y_1, \ldots, y_s \in F_B$ s.t. the restriction of φ to $M' \times N'$ is given by

$$(x', y') \mapsto \sum_{i=1}^{r} \sum_{j=1}^{s} \langle x', x_i \rangle \langle y', y_j \rangle.$$

Hence, by c), we can conclude that $\phi \in span(\phi(E \times F))$.

□

The $\varepsilon-$topology on $E \otimes F$ will be then naturally defined by the so-called *topology of bi-equicontinuous convergence* on the space $B(E'_\sigma, F'_\sigma)$. As the name suggests this topology is intimately related to the notion *equicontinuous sets of linear mappings* between t.v.s..

Definition 4.2.10. *Let X and Y be two t.v.s.. A set S of linear mappings of X into Y is said to be* equicontinuous *if for any neighbourhood V of the origin in Y there exists a neighbourhood U of the origin in X such that*

$$\forall\, f \in S,\, x \in U \;\Rightarrow\; f(x) \in V$$

i.e.

$$\forall\, f \in S,\, f(U) \subseteq V \quad (or\ U \subseteq f^{-1}(V)).$$

The equicontinuity condition can be also rewritten as follows: S is equicontinuous if for any neighbourhood V of the origin in Y there exists a neighbourhood U of the origin in X such that $\bigcup_{f \in S} f(U) \subseteq V$ or, equivalently, if for any neighbourhood V of the origin in Y the set $\bigcap_{f \in S} f^{-1}(V)$ is a neighbourhood of the origin in X.

Note that if S is equicontinuous then each mapping $f \in S$ is continuous but clearly the converse does not hold.

A first property of equicontinuous sets which is clear from the definition is that any subset of an equicontinuous set is itself equicontinuous. We are going to introduce now few more properties of equicontinuous sets of linear functionals on a t.v.s. which will be useful in the following.

Proposition 4.2.11. *A set of continuous linear functionals on a t.v.s. X is equicontinuous if and only if it is contained in the polar of some neighbourhood of the origin in X.*

Proof.
For any $\rho > 0$, let us denote by $D_\rho := \{k \in \mathbb{K} : |k| \leq \rho\}$. Let H be an equicontinuous set of linear forms on X. Then there exists a neighbourhood U of the origin in X s.t. $\bigcup_{f \in H} f(U) \subseteq D_1$, i.e. $\forall f \in H$, $|\langle f, x\rangle| \leq 1, \forall\, x \in U$, which means exactly that $H \subseteq U^\circ$.

Conversely, let U be an arbitrary neighbourhood of the origin in X and let us consider the polar $U^\circ := \{f \in X' : \sup_{x \in U} |\langle f, x\rangle| \leq 1\}$. Then for any $\rho > 0$

$$\forall\, f \in U^\circ,\, |\langle f, y\rangle| \leq \rho,\, \forall\, y \in \rho U,$$

which is equivalent to

$$\bigcup_{f \in U^\circ} f(\rho U) \subseteq D_\rho.$$

This means that U° is equicontinuous and so any subset H of U° is also equicontinuous, which yields the conclusion. □

Proposition 4.2.12. *Let X be a non-trivial locally convex Hausdorff t.v.s.*[2]*. Any equicontinuous subset of X' is bounded in X'_σ.*

Proof. Let H be an equicontinuous subset of X'. Then, by Proposition 4.2.11, we get that there exists a neighbourhood U of the origin in X such that $H \subseteq U^\circ$. By Banach-Alaoglu theorem (see Theorem 3.3.3), we know that $U^\circ$is compact in X'_σ and so bounded by Proposition 2.2.4. Hence, by Proposition 2.2.2-4, H is also bounded in X'_σ. □

It is also possible to show, but we are not going to prove this here, that the following holds.

Proposition 4.2.13. *Let X be a non-trivial locally convex Hausdorff t.v.s.. The union of all equicontinuous subsets of X' is dense in X'_σ.*

[2] Recall that non-trivial locally convex Hausdorff t.v.s. have non-trivial topological dual by Proposition 3.2.8

Now let us come back to the space $B(X,Y;Z)$ of continuous bilinear mappings from $X\times Y$ to Z, where X,Y and Z are non-trivial locally convex t.v.s.. The following is a very natural way of introducing a topology on $B(X,Y;Z)$ and is a kind of generalization of the method we have used to define polar topologies in Chapter 3.

Consider a family Σ (resp. Γ) of bounded subsets of X (resp. Y) satisfying the following properties:
(P1) If $A_1, A_2 \in \Sigma$, then $\exists\, A_3 \in \Sigma$ s.t. $A_1 \cup A_2 \subseteq A_3$.
(P2) If $A_1 \in \Sigma$ and $\lambda \in \mathbb{K}$, then $\exists\, A_2 \in \Sigma$ s.t. $\lambda A_1 \subseteq A_2$.
(resp. satisfying (P1) and (P2) replacing Σ by Γ). The Σ-Γ-topology on $B(X,Y;Z)$, or topology of uniform convergence on subsets of the form $A\times B$ with $A\in\Sigma$ and $B\in\Gamma$, is defined by taking as a basis of neighbourhoods of the origin in $B(X,Y;Z)$ the following family:

$$\mathcal{U} := \{\mathcal{U}(A,B;W) :\, A\in\Sigma, B\in\Gamma, W\in\mathcal{B}_Z(o)\},$$

where

$$\mathcal{U}(A,B;W) := \{\varphi\in B(X,Y;Z) : \varphi(A,B)\subseteq W\}$$

and $\mathcal{B}_Z(o)$ is a basis of neighbourhoods of the origin in Z. It is not difficult to verify that (c.f. [5, Chapter 32]):

a) each $\mathcal{U}(A,B;W)$ is an absorbing, convex, balanced subset of $B(X,Y;Z)$;
b) the Σ-Γ-topology makes $B(X,Y;Z)$ into a locally convex t.v.s. (by Theorem 4.1.14 of TVS-I);
c) If Z is Hausdorff, the union of all subsets in Σ is dense in X and the union of all subsets in Γ is dense in Y, then the Σ-Γ-topology on $B(X,Y;Z)$ is Hausdorff.

In particular, given two non-trivial locally convex Hausdorff t.v.s. E and F, we call *topology of bi-equicontinuous convergence* on $B(E'_\sigma,F'_\sigma)$ the Σ-Γ-topology when Σ is the family of all equicontinuous subsets of E' and Γ is the family of all equicontinuous subsets of F'. Note that we can make this choice of Σ and Γ, because by Proposition 4.2.12 all equicontinuous subsets of E' (resp. F') are bounded in E'_σ (resp. F'_σ) and satisfy the properties $(P1)$ and $(P2)$. A basis for the topology of bi-equicontinuous convergence $B(E'_\sigma,F'_\sigma)$ is then given by:

$$\mathcal{U} := \{\mathcal{U}(A,B;\varepsilon) :\, A\in\Sigma, B\in\Gamma, \varepsilon>0\}$$

where

$$\begin{aligned}\mathcal{U}(A,B;\varepsilon) &:= \{\varphi\in B(E'_\sigma,F'_\sigma) : \varphi(A,B)\subseteq D_\varepsilon\}\\ &= \{\varphi\in B(E'_\sigma,F'_\sigma) : |\varphi(x',y')|\leq\varepsilon, \forall x'\in A, \forall y'\in B\}\end{aligned}$$

and $D_\varepsilon := \{k \in \mathbb{K} : |k| \leq \varepsilon\}$. By using a) and b), we get that $B(E'_\sigma, F'_\sigma)$ endowed with the topology of bi-equicontinuous convergence is a locally convex t.v.s.. Also, by using Proposition 4.2.13 together with c), we can prove that the topology of bi-equicontinuous convergence on $B(E'_\sigma, F'_\sigma)$ is Hausdorff (as E and F are both assumed to be Hausdorff).

We can then use the isomorphism between $E \otimes F$ and $B(E'_\sigma, F'_\sigma)$ provided by Proposition 4.2.9 to carry the topology of bi-equicontinuous convergence on $B(E'_\sigma, F'_\sigma)$ over $E \otimes F$.

Definition 4.2.14 ($\varepsilon-$topology)**.**
Given two non-trivial locally convex Hausdorff t.v.s. E and F, we define the $\varepsilon-$topology *on $E \otimes F$ to be the topology carried over from $B(E'_\sigma, F'_\sigma)$ endowed with the topology of bi-equicontinuous convergence, i.e. topology of uniform convergence on the products of an equicontinuous subset of E' and an equicontinuous subset of F'. The space $E \otimes F$ equipped with the $\varepsilon-$topology will be denoted by $E \otimes_\varepsilon F$.*

It is clear then $E \otimes_\varepsilon F$ is a locally convex Hausdorff t.v.s.. Moreover, we have that:

Proposition 4.2.15. *Given two non-trivial locally convex Hausdorff t.v.s. E and F, the canonical mapping from $E \times F$ into $E \otimes_\varepsilon F$ is continuous. Hence, the $\pi-$topology is finer than the $\varepsilon-$topology on $E \otimes F$.*

Proof.
By definition of $\pi-$topology and $\varepsilon-$topology, it is enough to show that the canonical mapping ϕ from $E \times F$ into $B(E'_\sigma, F'_\sigma)$ defined in (4.5) is continuous w.r.t. the topology of bi-equicontinuous convergence on $B(E'_\sigma, F'_\sigma)$. Let $\varepsilon > 0$, A any equicontinuous subset of E' and B any equicontinuous subset of F', then by Proposition 4.2.11 we get that there exist a neighbourhood N_A of the origin in E and a neighbourhood N_B of the origin in F s.t. $A \subseteq (N_A)^\circ$ and $B \subseteq (N_B)^\circ$. Hence, we obtain that

$$\begin{aligned}
\phi^{-1}(\mathcal{U}(A, B; \varepsilon)) &= \{(x, y) \in E \times F : \phi(x, y) \in \mathcal{U}(A, B; \varepsilon)\} \\
&= \left\{(x, y) \in E \times F : |\phi(x, y)(x', y')| \leq \varepsilon, \forall\, x' \in A, \forall\, y' \in B\right\} \\
&= \left\{(x, y) \in E \times F : |\langle x', x\rangle\langle y', y\rangle| \leq \varepsilon, \forall\, x' \in A, \forall\, y' \in B\right\} \\
&\supseteq \left\{(x, y) \in E \times F : |\langle x', x\rangle\langle y', y\rangle| \leq \varepsilon, \forall\, x' \in (N_A)^\circ, \forall\, y' \in (N_B)^\circ\right\} \\
&\supseteq \varepsilon N_A \times N_B,
\end{aligned}$$

which proves the continuity of ϕ as $\varepsilon N_A \times N_B$ is a neighbourhood of the origin in $E \times F$. $\square$