Let us introduce now three general properties of all metrizable t.v.s. (not necessarily l.c.), which are well-known in the theory of metric spaces.

Proposition 1.1.6. *A metrizable t.v.s. X is complete if and only if X is sequentially complete.*

Proof. (Exercise Sheet 1) □

(For the definitions of completeness and sequentially completeness of a t.v.s., see Definition 2.5.5 and Definition 2.5.6 in TVS-I. See also Proposition 2.5.7 and Example 2.5.11 n TVS-I for more details on the relation between these two notions for general t.v.s..)

Proposition 1.1.7. *Let X be a metrizable t.v.s. and Y be any t.v.s. (not necessarily metrizable). A mapping $f : X \to Y$ (not necessarily linear) is continuous if and only if it is sequentially continuous.*

Proof. (Exercise Sheet 1) □

Recall that a mapping f from a topological space X into a topological space Y is said to be *sequentially continuous* if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ convergent to a point $x \in X$ the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $f(x)$ in Y.

The proof that continuity of $f : X \to Y$ always implies its sequentially continuity is pretty straightforward and holds under the general assumption that X and Y are topological spaces (see Proposition 1.1.39 in TVS-I). The converse does not hold in general as the following example shows.

Example 1.1.8.
Let us consider the set $\mathcal{C}([0,1])$ of all real-valued continuous functions on $[0,1]$. This is a vector space w.r.t. the pointwise addition and multiplication by real scalars. We endow $\mathcal{C}([0,1])$ with two topologies which both make it into a t.v.s.. The first topology σ is the one give by the metric:

$$d(f,g) := \int_0^1 \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}, \quad \forall f,g \in \mathcal{C}([0,1]).$$

The second topology τ is instead the topology generated by the family $(p_x)_{x\in[0,1]}$ of seminorms on $\mathcal{C}([0,1])$, where

$$p_x(f) := |f(x)|, \forall f \in \mathcal{C}([0,1]).$$

We will show that the identity map $I : (\mathcal{C}([0,1]),\tau) \to (\mathcal{C}([0,1]),\sigma)$ is sequentially continuous but not continuous.

• *I is sequentially continuous*
Let $(f_n)_{n\in\mathbb{N}}$ *be a sequence of elements in* $\mathcal{C}([0,1])$ *which is* $\tau-$*convergent to* $f\in\mathcal{C}([0,1])$ *as* $n\to\infty$, *i.e.* $|f_n(x)-f(x)|\to 0$, $\forall x\in[0,1]$ *as* $n\to\infty$. *Set*

$$g_n(x):=\frac{|f_n(x)-f(x)|}{1+|f_n(x)-f(x)|},\quad \forall x\in[0,1],\,\forall n\in\mathbb{N}.$$

Then $|g_n(x)|\leq 1$, $\forall x\in[0,1]$, $\forall n\in\mathbb{N}$ *and* $g_n(x)\to 0$, $\forall x\in[0,1]$ *as* $n\to\infty$. *Hence, by the Lebesgue dominated convergence theorem, we get* $\int_0^1 g_n(x)dx\to 0$ *as* $n\to\infty$, *that is,* $d(f_n,f)\to 0$ *as* $n\to\infty$, *i.e. the sequence* $(I(f_n))_{n\in\mathbb{N}}$ *is* $\sigma-$*convergent to* f *as* $n\to\infty$.
• *I is not continuous*
Suppose that I *is continuous at* $o\in\mathcal{C}([0,1])$ *and fix* $\varepsilon\in(0,1)$. *Then there exists a neighbourhood* N *of the origin in* $(\mathcal{C}([0,1]),\tau)$ *s.t.* $N\subset I^{-1}(B^d_\varepsilon(o))$, *where* $B^d_\varepsilon(o):=\{f\in\mathcal{C}([0,1]):d(f,0)\leq\varepsilon\}$. *This means that there exist* $n\in\mathbb{N}$, $x_1,\ldots,x_n\in[0,1]$ *and* $\delta>0$ *s.t.*

$$\bigcap_{i=1}^{n}\delta U_{p_{x_i}}\subset B^d_\varepsilon(o),\tag{1.4}$$

where $U_{p_{x_i}}:=\{f\in\mathcal{C}([0,1]):|f(x_i)|\leq 1\}$.
Take now $f_k(x):=k(x-x_1)\cdots(x-x_n)$, $\forall k\in\mathbb{N},\forall x\in[0,1]$. *Then* $f_k\in\mathcal{C}([0,1])$ *for all* $k\in\mathbb{N}$ *and* $f_k(x_i)=0<\delta$ *for all* $i=1,\ldots,n$. *Hence,*

$$f_k\in\bigcap_{i=1}^{n}\{f\in\mathcal{C}([0,1]):|f(x_i)|\leq\delta\}=\bigcap_{i=1}^{n}\delta U_{p_{x_i}}\overset{(1.4)}{\subset}B^d_\varepsilon(o),\ \forall k\in\mathbb{N}\tag{1.5}$$

Set

$$h_k(x):=\frac{|f_k(x)|}{1+|f_k(x)|},\quad\forall x\in[0,1],\,\forall k\in\mathbb{N}.$$

Then $|h_k(x)|\leq 1$, $\forall x\in[0,1]$, $\forall k\in\mathbb{N}$ *and* $h_k(x)\to 1\,\forall x\in[0,1]\setminus\{x_1,\ldots,x_n\}$ *as* $k\to\infty$. *Hence, by the Lebesgue dominated convergence theorem, we get* $\int_0^1 h_k(x)dx\to\int_0^1 1dx=1$ *as* $k\to\infty$, *that is,* $d(f_k,f)\to 1$ *as* $k\to\infty$. *This together with (1.5) gives tha* $\varepsilon\geq 1$ *which contradicts our assumption* $\varepsilon\in(0,1)$.

By Proposition 1.1.7, we then conclude that $(\mathcal{C}([0,1]),\tau)$ *is not metrizable.*

Proposition 1.1.9. *A complete metrizable t.v.s. X is a Baire space, i.e. X fulfills any of the following properties:*
(B) the union of any countable family of closed sets, none of which has interior points, has no interior points.
(B') the intersection of any countable family of everywhere dense open sets is an everywhere dense set.

Note that the equivalence of (B) and (B') is easily given by taking the complements. Indeed, the complement of a closed set C without interior points is clearly open and we get: $X \setminus (\overline{X \setminus C}) = \mathring{C} = \emptyset$ which is equivalent to $\overline{X \setminus C} = X$, i.e. $X \setminus C$ is everywhere dense.

Example 1.1.10. *An example of Baire space is $\mathbb{R}$ with the euclidean topology. Instead $\mathbb{Q}$ with the subset topology given by the euclidean topology on $\mathbb{R}$ is not a Baire space. Indeed, for any $q \in \mathbb{Q}$ the subset $\{q\}$ is closed and has empty interior in $\mathbb{Q}$, but $\cup_{q\in\mathbb{Q}}\{q\} = \mathbb{Q}$ which has interior points in $\mathbb{Q}$ (actually its interior is the whole $\mathbb{Q}$).*

Before proving Proposition 1.1.9, let us observe that the converse of the proposition does not hold because there exist Baire spaces which are not metrizable. Moreover, the assumptions of Proposition 1.1.9 cannot be weakened, because there exist complete non-metrizable t.v.s and metrizable non-complete t.v.s which are not Baire spaces.

Proof. of Proposition 1.1.9
We are going to prove that Property (B') holds in any complete metrizable t.v.s.. Let $\{\Omega_k\}_{k\in\mathbb{N}}$ be a sequence of dense open subsets of X and let us denote by A their intersection. We need to show that A intersects every non-empty open subset of X (this implies that A is dense, since every neighbourhood of every point in X contains some open set and hence some point of A).

Let O be an arbitrary non-empty open subset of X. Since X is a metrizable t.v.s., there exists a countable basis $\{U_k\}_{k\in\mathbb{N}}$ of neighbourhoods of the origin which we may take all closed and s.t. $U_{k+1} \subseteq U_k$ for all $k \in \mathbb{N}$. As Ω_1 is open and dense we have that $O \cap \Omega_1$ is open and non-empty. Therefore, there exists $x_1 \in O \cap \Omega_1$ and $k_1 \in \mathbb{N}$ s.t. $x_1 + U_{k_1} \subseteq O \cap \Omega_1$. Let us denote by G_1 the interior of $x_1 + U_{k_1}$, which is non-empty since it contains x_1 (Indeed, $x_1 + U_{k_1}$ is a neighbourhood of x_1 and so there exists an open set V such that $x_1 \in V \subset x_1 + U_{k_1}$, i.e. x_1 belongs to the interior of $x_1 + U_{k_1}$).

As Ω_2 is dense and G_1 is a non-empty open set, we have that $G_1 \cap \Omega_2$ is open and non-empty. Hence, there exists $x_2 \in G_1 \cap \Omega_2$ and $k_2 \in \mathbb{N}$ s.t. $x_2 + U_{k_2} \subseteq G_1 \cap \Omega_2$. Let us choose $k_2 > k_1$ and call G_2 the interior of $x_2 + U_{k_2}$, which is non-empty since it contains x_2. Proceeding in this way, we get a sequence of non-empty open sets $\mathcal{G} := \{G_l\}_{l\in\mathbb{N}}$ with the following properties for any $l \in \mathbb{N}$:

1. $\overline{G_l} \subseteq \Omega_l \cap O$
2. $G_{l+1} \subseteq G_l$
3. $G_l \subseteq x_l + U_{k_l}$.

Note that the family $\mathcal{G}$ does not contain the empty set and Property 2 implies that for any $G_j, G_k \in \mathcal{G}$ the intersection $G_j \cap G_k = G_{\max\{j,k\}} \in \mathcal{G}$. Hence, $\mathcal{G}$ is a basis of a filter $\mathcal{F}$ in X[1]. Moreover, Property 3 implies that

$$\forall l \in \mathbb{N}, G_l - G_l \subseteq U_{k_l} - U_{k_l} \tag{1.6}$$

which guarantees that $\mathcal{F}$ is a Cauchy filter in X. Indeed, for any neighbourhood U of the origin in X there exists a balanced neighbourhood of the origin such that $V - V \subseteq U$ and so there exists $k \in \mathbb{N}$ such that $U_k \subseteq V$. Hence, there exists $l \in \mathbb{N}$ s.t. $k_l \geq l$ and so $U_{k_l} \subseteq U_k$. Then by (1.6) we have that $G_l - G_l \subseteq U_{k_l} - U_{k_l} \subseteq V - V \subseteq U$. Since $G_l \in \mathcal{G}$ and so in $\mathcal{F}$, we have got that $\mathcal{F}$ is a Cauchy filter.

As X is complete, the Cauchy filter $\mathcal{F}$ has a limit point $x \in X$, i.e. the filter of neighbourhoods of x is contained in the filter $\mathcal{F}$. This implies that $x \in \overline{G_l}$ for all $l \in \mathbb{N}$ (If there would exists $l \in \mathbb{N}$ s.t. $x \notin \overline{G_l}$ then there would exists a neighbourhood N of x s.t. $N \cap G_l = \emptyset$. As $G_l \in \mathcal{F}$ and any neighbourhood of x belongs to $\mathcal{F}$, we get $\emptyset \in \mathcal{F}$ which contradicts the definition of filter.) Hence:

$$x \in \bigcap_{l \in \mathbb{N}} \overline{G_l} \subseteq O \cap \bigcap_{l \in \mathbb{N}} \Omega_l = O \cap A.$$

□

[1] Recall that a basis of a filter on X is a family $\mathcal{G}$ of non-empty subsets of X such that $\forall G_1, G_2 \in \mathcal{G}, \exists G_3 \in \mathcal{G}$ s.t. $G_3 \subset G_1 \cap G_2$.