1.3 Inductive topologies and LF-spaces

Let $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ be a family of locally convex t.v.s. over the field $\mathbb{K}$ of real or complex numbers (A is an arbitrary index set), E a vector space over the same field $\mathbb{K}$ and, for each $\alpha \in A$, let $g_\alpha : E_\alpha \to E$ be a linear mapping. The ***inductive topology*** τ_{ind} on E w.r.t. the family $\{(E_\alpha, \tau_\alpha, g_\alpha) : \alpha \in A\}$ is the topology generated by the following basis of neighbourhoods of the origin in E:

$$\begin{aligned}\mathcal{B}_{ind} : \; = \; & \{U \subset E \,\text{convex, balanced, absorbing} : \forall \alpha \in A,\, g_\alpha^{-1}(U) \text{ is} \\ & \text{a neighbourhood of the origin in}\,(E_\alpha, \tau_\alpha)\}.\end{aligned}$$

Then it easily follows that the space (E, τ_{ind}) is a l.c. t.v.s.. Indeed, $\mathcal{B}_{ind}$ is a family of absorbing and absolutely convex subsets of E such that

a) $\forall\, U, V \in \mathcal{B}_{ind}$, $U \cap V \in \mathcal{B}_{ind}$, since $g_\alpha^{-1}(U \cap V) = g_\alpha^{-1}(U) \cap g_\alpha^{-1}(V)$ is a neighbourhood of the origin in (E_α, τ_α) (as finite intersection of such neighbourhoods).

b) $\forall\, \rho > 0, \forall\, U \in \mathcal{B}_{ind}$, $\rho U \in \mathcal{B}_{ind}$, since $g_\alpha^{-1}(\rho U) = \rho g_\alpha^{-1}(U)$ which is a neighbourhood of the origin in (E_α, τ_α)(as a dilation of such a neighbourhood).

Then Theorem 4.1.14 in TVS-I ensures that τ_{ind} makes E into a l.c. t.v.s..

Note that τ_{ind} is the finest locally convex topology on E for which all the mappings g_α ($\alpha \in A$) are continuous. Suppose there exists a locally convex topology τ on E s.t. all the g_α's are continuous and $\tau_{ind} \subseteq \tau$. As (E, τ) is locally convex, there always exists a basis of neighbourhood of the origin consisting of convex, balanced, absorbing subsets of E. Then for any such a neighbourhood U of the origin in (E, τ) we have, by continuity, that $g_\alpha^{-1}(U)$ is a neighbourhood of the origin in (E_α, τ_α). Hence, $U \in \mathcal{B}_{ind}$ and so $\tau \equiv \tau_{ind}$.

It is also worth to underline that (E, τ_{ind}) is not necessarily a Hausdorff t.v.s., even when all the spaces (E_α, τ_α) are Hausdorff t.v.s..

Example 1.3.1. *Let (X, τ) be a l.c. Hausdorff t.v.s., M a non-closed subspace of X and $\varphi : X \to X/M$ the quotient map. Then the inductive limit topology on X/M w.r.t. (X, τ, ϕ) (here the index set A is just a singleton) coincides with the quotient topology on X/M, which is not Hausdorff since M is not closed (see Proposition 2.3.5 in TVS-I).*

Proposition 1.3.2. *Let E be a vector space over the field $\mathbb{K}$ endowed with the inductive topology τ_{ind} w.r.t. a family $\{(E_\alpha, \tau_\alpha, g_\alpha) : \alpha \in A\}$, where each (E_α, τ_α) is a locally convex t.v.s. over $\mathbb{K}$ and each $g_\alpha : E_\alpha \to E$ is a linear map. A linear map u from (E, τ_{ind}) to any locally convex t.v.s. (F, τ) is continuous if and only if for each $\alpha \in A$ the map $u \circ g_\alpha : E_\alpha \to F$ is continuous.*

Proof. Suppose u is continuous and fix $\alpha \in A$. Since g_α is also continuous, we have that $u \circ g_\alpha$ is continuous as composition of continuous mappings.

Conversely, suppose that for each $\alpha \in A$ the mapping $u \circ g_\alpha$ is continuous. As (F, τ) is locally convex, there always exists a basis of neighbourhoods of the origin consisting of convex, balanced, absorbing subsets of F. Let W be such a neighbourhood. Then, by the linearity of u, we get that $u^{-1}(W)$ is a convex, balanced and absorbing subset of E. Moreover, the continuity of all $u \circ g_\alpha$ guarantees that each $(u \circ g_\alpha)^{-1}(W)$ is a neighbourhood of the origin in (E_α, τ_α), i.e. $g_\alpha^{-1}(u^{-1}(W))$ is a neighbourhood of the origin in (E_α, τ_α). Then $u^{-1}(W)$, being also convex, balanced and absorbing, must be in $\mathcal{B}_{ind}$ and so it is a neighbourhood of the origin in (E, τ_{ind}). Hence, u is continuous. □

Let us consider now the case when we have a total order $\leq$ on the index set A and $\{E_\alpha : \alpha \in A\}$ is a family of linear subspaces of a vector space E over $\mathbb{K}$ which is directed under inclusion, i.e. $E_\alpha \subseteq E_\beta$ whenever $\alpha \leq \beta$, and s.t. $E = \bigcup_{\alpha \in A} E_\alpha$. For each $\alpha \in A$, let i_α be the canonical embedding of E_α in E and τ_α a topology on E_α s.t. (E_α, τ_α) is a locally convex Hausdorff t.v.s. and, whenever $\alpha \leq \beta$, the topology induced by τ_β on E_α is coarser than τ_α. The space E equipped with the inductive topology τ_{ind} w.r.t. the family $\{(E_\alpha, \tau_\alpha, i_\alpha) : \alpha \in A\}$ is said to be the ***inductive limit*** of the family of linear subspaces $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$.

An inductive limit of a family of linear subspaces $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ is said to be a ***strict inductive limit*** if, whenever $\alpha \leq \beta$, the topology induced by τ_β on E_α coincides with τ_α.

There are even more general constructions of inductive limits of a family of locally convex t.v.s. but in the following we will focus on a more concrete family of inductive limits which are more common in applications. Namely, we are going to consider the so-called ***LF-spaces***, i.e. countable strict inductive limits of increasing sequences of Fréchet spaces. For convenience, let us explicitly write down the definition of an LF-space.

Definition 1.3.3. *Let $\{E_n : n \in \mathbb{N}\}$ be an increasing sequence of linear subspaces of a vector space E over $\mathbb{K}$, i.e. $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$, such that $E = \bigcup_{n \in \mathbb{N}} E_n$. For each $n \in \mathbb{N}$ let i_n be the canonical embedding of E_n in E and (E_n, τ_n) be a Fréchet space such that the topology induced by τ_{n+1} on E_n coincides with τ_n (i.e. the natural embedding of E_n into E_{n+1} is a topological embedding). The space E equipped with the inductive topology τ_{ind} w.r.t. the family $\{(E_n, \tau_n, i_n) : n \in \mathbb{N}\}$ is said to be the* LF-space with defining sequence *$\{(E_n, \tau_n) : n \in \mathbb{N}\}$.*

A basis of neighbourhoods of the origin in the LF-space (E, τ_{ind}) with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$ is given by:

$\{U \subset E$ convex, balanced, abs. : $\forall n \in \mathbb{N}$, $U \cap E_n$ is a nbhood of o in $(E_n, \tau_n)\}$.

Note that from the construction of the LF-space (E, τ_{ind}) with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$ we know that each E_n is topologically embedded in the subsequent ones, but a priori we do not know if E_n is topologically embedded in E, i.e. if the topology induced by τ_{ind} on E_n is identical to the topology τ_n initially given on E_n. This is indeed true and it will be a consequence of the following lemma.

Lemma 1.3.4. *Let X be a locally convex t.v.s., X_0 a linear subspace of X equipped with the subspace topology, and U a convex neighbourhood of the origin in X_0. Then there exists a convex neighbourhood V of the origin in X such that $V \cap X_0 = U$.*

Proof.
As X_0 carries the subspace topology induced by X, there exists a neighbourhood W of the origin in X such that $U = W \cap X_0$. Since X is a locally convex t.v.s., there exists a convex neighbourhood W_0 of the origin in X such that $W_0 \subseteq W$. Let V be the convex hull of $U \cup W_0$. Then by construction we have that V is a convex neighbourhood of the origin in X and that $U \subseteq V$ which implies $U = U \cap X_0 \subseteq V \cap X_0$. We claim that actually $V \cap X_0 = U$. Indeed, let $x \in V \cap X_0$; as $x \in V$ and as U and W_0 are both convex, we may write $x = ty + (1-t)z$ with $y \in U, z \in W_0$ and $t \in [0,1]$. If $t = 1$, then $x = y \in U$ and we are done. If $0 \leq t < 1$, then $z = (1-t)^{-1}(x - ty)$ belongs to X_0 and so $z \in W_0 \cap X_0 \subseteq W \cap X_0 = U$. This implies, by the convexity of U, that $x \in U$. Hence, $V \cap X_0 \subseteq U$. □

Proposition 1.3.5.
Let (E, τ_{ind}) be an LF-space with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$. Then

$$\tau_{ind} \restriction E_n \equiv \tau_n, \ \forall n \in \mathbb{N}.$$

Proof.
($\subseteq$) Let $V \in \mathcal{B}_{ind}$. Then, by definition, for each $n \in \mathbb{N}$ we have that $V \cap E_n$ is a neighbourhood of the origin in (E_n, τ_n). Hence, $\tau_{ind} \restriction E_n \subseteq \tau_n$, $\forall n \in \mathbb{N}$.

($\supseteq$) Given $n \in \mathbb{N}$, let U_n be a convex, balanced, absorbing neighbourhood of the origin in (E_n, τ_n). Since E_n is a linear subspace of E_{n+1}, we can apply Lemma 1.3.4 (for $X = E_{n+1}$, $X_0 = E_n$ and $U = U_n$) which ensures the existence of a convex neighbourhood U_{n+1} of the origin in (E_{n+1}, τ_{n+1}) such that $U_{n+1} \cap E_n = U_n$. Then, by induction, we get that for any $k \in \mathbb{N}$ there exists a convex neighbourhood U_{n+k} of the origin in (E_{n+k}, τ_{n+k}) such that $U_{n+k} \cap E_{n+k-1} = U_{n+k-1}$. Hence, for any $k \in \mathbb{N}$, we get $U_{n+k} \cap E_n = U_n$. If we consider now $U := \bigcup_{k=1}^{\infty} U_{n+k}$, then $U \cap E_n = U_n$ and U is a neighbourhood of the origin in (E, τ_{ind}). Indeed, for any $m \in \mathbb{N}$ we have $U \cap E_m = \bigcup_{k=1}^{\infty} U_{n+k} \cap E_m = \bigcup_{k=m-n}^{\infty} U_{n+k} \cap E_m$, which is a countable union of neighbourhoods of the origin in τ_m as for $k \geq m-n$ we get $n+k \geq m$ and so $\tau_{n+k} \restriction E_m = \tau_m$. We can then conclude that $\tau_n \subseteq \tau_{ind} \restriction E_n$, $\forall n \in \mathbb{N}$. □

Corollary 1.3.6. *Any LF-space is a locally convex Hausdorff. t.v.s..*

Proof. Let (E, τ_{ind}) be the LF-space with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$ and denote by $\mathcal{F}(o)$ [resp. $\mathcal{F}_n(o)$] the filter of neighbourhoods of the origin in (E, τ_{ind}) [resp. in (E_n, τ_n)]. Then:

$$\bigcap_{V \in \mathcal{F}(o)} V = \bigcap_{V \in \mathcal{F}(o)} V \cap \left(\bigcup_{n \in \mathbb{N}} E_n \right) = \bigcup_{n \in \mathbb{N}} \bigcap_{V \in \mathcal{F}(o)} (V \cap E_n) = \bigcup_{n \in \mathbb{N}} \bigcap_{U_n \in \mathcal{F}_n(o)} U_n = \{o\},$$

which implies that (E, τ_{ind}) is Hausdorff by Corollary 2.2.4 in TVS-I. □

As a particular case of Proposition 1.3.2 we easily get that:

Proposition 1.3.7.
Let (E, τ_{ind}) be an LF-space with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$ and (F, τ) an arbitrary locally convex t.v.s..

1. *A linear mapping u from E into F is continuous if and only if, for each $n \in \mathbb{N}$, the restriction $u \restriction E_n$ of u to E_n is continuous.*
2. *A linear form on E is continuous if and only if its restrictions to each E_n are continuous.*

Note that Propositions 1.3.5 and 1.3.7 and Corollary 1.3.6 hold for any countable strict inductive limit of an increasing sequence of locally convex Hausdorff t.v.s. (even when they are not Fréchet).