that $U_{n+1} \cap E_n = U_n$. Then, by induction, we get that for any $k \in \mathbb{N}$ there exists a convex neighbourhood U_{n+k} of the origin in (E_{n+k}, τ_{n+k}) such that $U_{n+k} \cap E_{n+k-1} = U_{n+k-1}$. Hence, for any $k \in \mathbb{N}$, we get $U_{n+k} \cap E_n = U_n$. If we consider now $U := \bigcup_{k=1}^{\infty} U_{n+k}$, then $U \cap E_n = U_n$ and U is a neighbourhood of the origin in (E, τ_{ind}). Indeed, for any $m \in \mathbb{N}$ we have $U \cap E_m = \bigcup_{k=1}^{\infty} U_{n+k} \cap E_m = \bigcup_{k=m-n}^{\infty} U_{n+k} \cap E_m$, which is a countable union of neighbourhoods of the origin in τ_m as for $k \geq m-n$ we get $n+k \geq m$ and so $\tau_{n+k} \restriction E_m = \tau_m$. We can then conclude that $\tau_n \subseteq \tau_{ind} \restriction E_n$, $\forall n \in \mathbb{N}$. □

Corollary 1.3.6. *Any LF-space is a locally convex Hausdorff. t.v.s..*

Proof. Let (E, τ_{ind}) be the LF-space with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$ and denote by $\mathcal{F}(o)$ [resp. $\mathcal{F}_n(o)$] the filter of neighbourhoods of the origin in (E, τ_{ind}) [resp. in (E_n, τ_n)]. Then:

$$\bigcap_{V \in \mathcal{F}(o)} V = \bigcap_{V \in \mathcal{F}(o)} V \cap \left(\bigcup_{n \in \mathbb{N}} E_n \right) = \bigcup_{n \in \mathbb{N}} \bigcap_{V \in \mathcal{F}(o)} (V \cap E_n) = \bigcup_{n \in \mathbb{N}} \bigcap_{U_n \in \mathcal{F}_n(o)} U_n = \{o\},$$

which implies that (E, τ_{ind}) is Hausdorff by Corollary 2.2.4 in TVS-I. □

As a particular case of Proposition 1.3.2 we easily get that:

Proposition 1.3.7.
Let (E, τ_{ind}) be an LF-space with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$ and (F, τ) an arbitrary locally convex t.v.s..

1. *A linear mapping u from E into F is continuous if and only if, for each $n \in \mathbb{N}$, the restriction $u \restriction E_n$ of u to E_n is continuous.*
2. *A linear form on E is continuous if and only if its restrictions to each E_n are continuous.*

Note that Propositions 1.3.5 and 1.3.7 and Corollary 1.3.6 hold for any countable strict inductive limit of an increasing sequence of locally convex Hausdorff t.v.s. (even when they are not Fréchet).

The next theorem is instead typical of LF-spaces as it heavily relies on the completeness of the t.v.s. of the defining sequence. Before introducing it, let us recall the concept of accumulation point of a filter on a topological space together with some basic useful properties.

Definition 1.3.8. *Let $\mathcal{F}$ be a filter on a topological space X. A point $x \in X$ is called an* accumulation point *of $\mathcal{F}$ if x belongs to the closure of every set which belongs to $\mathcal{F}$, i.e. $x \in \overline{M}$, $\forall M \in \mathcal{F}$.*

Proposition 1.3.9. *If a filter $\mathcal{F}$ of a topological space X converges to a point x, then x is an accumulation point of $\mathcal{F}$.*

Proof. If x were not an accumulation point of $\mathcal{F}$, then there would be a set $M \in \mathcal{F}$ such that $x \notin \overline{M}$. Hence, $X \setminus \overline{M}$ is open in X and contains x, so it is a neighbourhood of x. Then $X \setminus \overline{M} \in \mathcal{F}$ as $\mathcal{F} \to x$ by assumption. But $\mathcal{F}$ is a filter and so $M \cap \left(X \setminus \overline{M}\right) \in \mathcal{F}$ and so $M \cap \left(X \setminus \overline{M}\right) \neq \emptyset$, which is a contradiction. □

Proposition 1.3.10. *If a Cauchy filter $\mathcal{F}$ of a t.v.s. X has an accumulation point x, then $\mathcal{F}$ converges to x.*

Proof. Let us denote by $\mathcal{F}(o)$ the filter of neighbourhoods of the origin in X and consider $U \in \mathcal{F}(o)$. Since X is a t.v.s., there exists $V \in \mathcal{F}(o)$ such that $V + V \subseteq U$. Then there exists $M \in \mathcal{F}$ such that $M - M \subseteq V$ as $\mathcal{F}$ is a Cauchy filter in X. Being x an accumulation point of $\mathcal{F}$ guarantees that $x \in \overline{M}$ and so that $(x+V) \cap M \neq \emptyset$. Then $M - ((x+V) \cap M) \subseteq M - M \subseteq V$ and so $M \subseteq V + ((x+V) \cap M) \subseteq V + V + x \subseteq U + x$. Since $\mathcal{F}$ is a filter and $M \in \mathcal{F}$, the latter implies that $U + x \in \mathcal{F}$. This proves that $\mathcal{F}(x) \subseteq \mathcal{F}$, i.e. $\mathcal{F} \to x$. □

Theorem 1.3.11. *Any LF-space is complete.*

Proof.
Let (E, τ_{ind}) be an LF-space with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$. Let $\mathcal{F}$ be a Cauchy filter on (E, τ_{ind}). Denote by $\mathcal{F}_E(o)$ the filter of neighbourhoods of the origin in (E, τ_{ind}) and consider

$$\mathcal{G} := \{A \subseteq E : \ A \supseteq M + V \ \text{ for some } M \in \mathcal{F}, V \in \mathcal{F}_E(o)\}.$$

1) $\mathcal{G}$ is a filter on E.
Indeed, it is clear from its definition that $\mathcal{G}$ does not contain the empty set and that any subset of E containing a set in $\mathcal{G}$ has to belong to $\mathcal{G}$. Moreover, for any $A_1, A_2 \in \mathcal{G}$ there exist $M_1, M_2 \in \mathcal{F}$, $V_1, V_2 \in \mathcal{F}_E(o)$ s.t. $M_1 + V_1 \subseteq A_1$ and $M_2 + V_2 \subseteq A_2$; and therefore

$$A_1 \cap A_2 \supseteq (M_1 + V_1) \cap (M_2 + V_2) \supseteq (M_1 \cap M_2) + (V_1 \cap V_2).$$

The latter proves that $A_1 \cap A_2 \in \mathcal{G}$, since $\mathcal{F}$ and $\mathcal{F}_E(o)$ are both filters and so $M_1 \cap M_2 \in \mathcal{F}$ and $V_1 \cap V_2 \in \mathcal{F}_E(o)$.

2) $\mathcal{G} \subseteq \mathcal{F}$.
In fact, for any $A \in \mathcal{G}$ there exist $M \in \mathcal{F}$ and $V \in \mathcal{F}_E(o)$ s.t.

$$A \supseteq M + V \supset M + \{0\} = M$$

which implies that $A \in \mathcal{F}$ since $\mathcal{F}$ is a filter.

3) $\mathcal{G}$ is a Cauchy filter on E.
Let $U \in \mathcal{F}_E(o)$. Then there always exists $V \in \mathcal{F}_E(o)$ balanced such that $V + V - V \subseteq U$. As $\mathcal{F}$ is a Cauchy filter on (E, τ_{ind}), there exists $M \in \mathcal{F}$ such that $M - M \subseteq V$. Then

$$(M + V) - (M + V) \subseteq (M - M) + (V - V) \subseteq V + V - V \subseteq U$$

which proves that $\mathcal{G}$ is a Cauchy filter since $M + V \in \mathcal{G}$.

It is possible to show (and we do it later on) that

$$\exists\, p \in \mathbb{N} : \forall A \in \mathcal{G},\ A \cap E_p \neq \emptyset. \tag{1.12}$$

This property together with the fact that $\mathcal{G}$ is a filter ensures that the family

$$\mathcal{G}_p := \{A \cap E_p : A \in \mathcal{G}\}$$

is a filter on E_p. Moreover, since $\mathcal{G}$ is a Cauchy filter on (E, τ_{ind}) and since by Proposition 1.3.5 we have $\tau_{ind} \restriction E_p = \tau_p$, $\mathcal{G}_p$ is a Cauchy filter on (E_p, τ_p). Hence, the completeness of E_p guarantees that there exists $x \in E_p$ s.t. $\mathcal{G}_p \to x$ which implies in turn that x is an accumulation point for $\mathcal{G}_p$ by Proposition 1.3.9. In particular, this gives that for any $A \in \mathcal{G}$ we have $x \in \overline{A \cap E_p}^{\tau_p} = \overline{A \cap E_p}^{\tau_{ind}} \subseteq \overline{A}^{\tau_{ind}}$, i.e. x is an accumulation point for the Cauchy filter $\mathcal{G}$. Then, by Proposition 1.3.10, we get that $\mathcal{G} \to x$ and so $\mathcal{F}_E(x) \subseteq \mathcal{G} \subseteq \mathcal{F}$. Hence, we proved that $\mathcal{F} \to x \in E$. □

Proof. of (1.12)
Suppose that (1.12) is false, i.e. $\forall n \in \mathbb{N}$, $\exists\, A_n \in \mathcal{G}$ s.t. $A_n \cap E_n = \emptyset$. By the definition of $\mathcal{G}$, this implies that

$$\forall n \in \mathbb{N},\ \exists\, M_n \in \mathcal{F},\ V_n \in \mathcal{F}_E(o), \text{ s.t. } (M_n + V_n) \cap E_n = \emptyset. \tag{1.13}$$

Since E is a locally convex t.v.s., we may assume that each V_n is balanced, convex, and such that $V_{n+1} \subseteq V_n$. For each $n \in \mathbb{N}$, define

$$W_n := conv\left(V_n \cup \bigcup_{k=1}^{n-1} (V_k \cap E_k)\right).$$

Moreover, if for some $n \in \mathbb{N}$ there exists $h \in (W_n + M_n) \cap E_n$ then $h \in E_n$ and $h \in (W_n + M_n)$. Therefore, we can write $h = x + w$ with $x \in M_n$ and $w \in W_n \subseteq conv\,(V_n \cup (V_1 \cap E_{n-1}))$. As V_n and $V_1 \cap E_{n-1}$ are both convex, we get that $h = x + ty + (1-t)z$ with $x \in M_n$, $y \in V_n$, $z \in V_1 \cap E_{n-1}$ and $t \in [0,1]$. Then $x + ty = h - (1-t)z \in E_n$, but we also have $x + ty \in M_n + V_n$ (since V_n is balanced). Hence, $x + ty \in (M_n + V_n) \cap E_n$ which contradicts (1.13), proving that

$$(W_n + M_n) \cap E_n = \emptyset, \forall n \in \mathbb{N}.$$

Now let us define

$$W := conv\left(\bigcup_{k=1}^{\infty}(V_k \cap E_k)\right).$$

As W is convex and as $W \cap E_k$ contains $V_k \cap E_k$ for all $k \in \mathbb{N}$, W is a neighbourhood of the origin in (E, τ_{ind}). Moreover, as $(V_n)_{n\in\mathbb{N}}$ is decreasing, we have that for all $n \in \mathbb{N}$

$$W = conv\left(\bigcup_{k=1}^{n-1}(V_k \cap E_k) \cup \bigcup_{k=n}^{\infty}(V_k \cap E_k)\right) \subseteq conv\left(\bigcup_{k=1}^{n-1}(V_k \cap E_k) \cup V_n\right) = W_n.$$

Since $\mathcal{F}$ is a Cauchy filter on (E, τ_{ind}), there exists $B \in \mathcal{F}$ such that $B - B \subseteq W$ and so $B - B \subseteq W_n, \forall n \in \mathbb{N}$. We also have that $B \cap M_n \neq \emptyset, \forall n \in \mathbb{N}$, as both B and M_n belong to $\mathcal{F}$. Hence, for all $n \in \mathbb{N}$ we get

$$B - (B \cap M_n) \subseteq B - B \subseteq W_n,$$

which implies

$$B \subseteq W_n + (B \cap M_n) \subseteq W_n + M_n$$

and so

$$B \cap E_n \subseteq (W_n + M_n) \cap E_n \overset{(1.13)}{=} \emptyset.$$

Therefore, we have got that $B \cap E_n = \emptyset$ for all $n \in \mathbb{N}$ and so that $B = \emptyset$, which is impossible as $B \in \mathcal{F}$. Hence, (1.12) must hold true. □

Example I: The space of polynomials

Let $n \in \mathbb{N}$ and $\mathbf{x} := (x_1, \ldots, x_n)$. Denote by $\mathbb{R}[\mathbf{x}]$ the space of polynomials in the n variables $x_1, \ldots, x_n$ with real coefficients. A canonical algebraic basis for $\mathbb{R}[\mathbf{x}]$ is given by all the monomials

$$\mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \forall \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n.$$

For any $d \in \mathbb{N}_0$, let $\mathbb{R}_d[\mathbf{x}]$ be the linear subpace of $\mathbb{R}[\mathbf{x}]$ spanned by all monomials $\mathbf{x}^\alpha$ with $|\alpha| := \sum_{i=1}^n \alpha_i \leq d$, i.e.

$$\mathbb{R}_d[\mathbf{x}] := \{f \in \mathbb{R}[\mathbf{x}] |\deg f \leq d\}.$$

Since there are exactly $\binom{n+d}{d}$ monomials $\mathbf{x}^\alpha$ with $|\alpha| \leq d$, we have that

$$dim(\mathbb{R}_d[\mathbf{x}]) = \frac{(d+n)!}{d!n!},$$

and so that $\mathbb{R}_d[\mathbf{x}]$ is a finite dimensional vector space. Hence, by Tychonoff Theorem (see Corollary 3.1.4 in TVS-I) there is a unique topology τ_e^d that makes $\mathbb{R}_d[\mathbf{x}]$ into a Hausdorff t.v.s. which is also complete and so Fréchet (as it topologically isomorphic to $\mathbb{R}^{dim(\mathbb{R}_d[\underline{x}])}$ equipped with the euclidean topology).

As $\mathbb{R}[\mathbf{x}] := \bigcup_{d=0}^\infty \mathbb{R}_d[\mathbf{x}]$, we can then endow it with the inductive topology τ_{ind} w.r.t. the family of F-spaces $\left\{(\mathbb{R}_d[\mathbf{x}], \tau_e^d) : d \in \mathbb{N}_0\right\}$; thus $(\mathbb{R}[\mathbf{x}], \tau_{ind})$ is a LF-space and the following properties hold (proof in Exercise Sheet 3):

a) τ_{ind} is the finest locally convex topology on $\mathbb{R}[\mathbf{x}]$,

b) every linear map f from $(\mathbb{R}[\mathbf{x}], \tau_{ind})$ into any t.v.s. is continuous.

Example II: The space of test functions

Let $\Omega \subseteq \mathbb{R}^d$ be open in the euclidean topology. For any integer $0 \leq s \leq \infty$, we have defined in Section 1.2 the set $\mathcal{C}^s(\Omega)$ of all real valued *$s-$times continuously differentiable functions* on Ω, which is a real vector space w.r.t. pointwise addition and scalar multiplication. We have equipped this space with the *$\mathcal{C}^s$-topology* (i.e. the topology of uniform convergence on compact sets of the functions and their derivatives up to order s) and showed that this turns $\mathcal{C}^s(\Omega)$ into a Fréchet space.

Let K be a compact subset of Ω, which means that it is bounded and closed in $\mathbb{R}^d$ and that its closure is contained in Ω. For any integer $0 \leq s \leq \infty$, consider the subset $\mathcal{C}_c^k(K)$ of $\mathcal{C}^s(\Omega)$ consisting of all the functions $f \in \mathcal{C}^s(\Omega)$ whose support lies in K, i.e.

$$\mathcal{C}_c^s(K) := \{f \in \mathcal{C}^s(\Omega) : supp(f) \subseteq K\},$$

where $supp(f)$ denotes the support of the function f on Ω, that is the closure in Ω of the subset $\{x \in \Omega : f(x) \neq 0\}$.

For any integer $0 \leq s \leq \infty$, $\mathcal{C}_c^s(K)$ is always a closed linear subspace of $\mathcal{C}^s(\Omega)$. Indeed, for any $f, g \in \mathcal{C}_c^s(K)$ and any $\lambda \in \mathbb{R}$, we clearly have $f + g \in \mathcal{C}^s(\Omega)$ and $\lambda f \in \mathcal{C}^s(\Omega)$ but also $supp(f + g) \subseteq supp(f) \cup supp(g) \subseteq K$ and $supp(\lambda f) = supp(f) \subseteq K$, which gives $f + g, \lambda f \in \mathcal{C}_c^s(K)$. To show that $\mathcal{C}_c^s(K)$ is closed in $\mathcal{C}^s(\Omega)$, it suffices to prove that it is sequentially closed as $\mathcal{C}^s(\Omega)$ is a F-space. Consider a sequence $(f_j)_{j\in\mathbb{N}}$ of functions in $\mathcal{C}^s_c(K)$ converging to f in the $\mathcal{C}^s-$topology. Then clearly $f \in \mathcal{C}^s(\Omega)$ and since all the f_j vanish in the open set $\Omega \setminus K$, obviously their limit f must also vanish in $\Omega \setminus K$. Thus, regarded as a subspace of $\mathcal{C}^s(\Omega)$, $\mathcal{C}^s_c(K)$ is also complete (see Proposition 2.5.8 in TVS-I) and so it is itself an F-space.

Let us now denote by $\mathcal{C}^s_c(\Omega)$ the union of the subspaces $\mathcal{C}^s_c(K)$ as K varies in all possible ways over the family of compact subsets of Ω, i.e. $\mathcal{C}^s_c(\Omega)$ is linear subspace of $\mathcal{C}^s(\Omega)$ consisting of all the functions belonging to $\mathcal{C}^s(\Omega)$ which have a compact support (this is what is actually encoded in the subscript c). In particular, $\mathcal{C}^\infty_c(\Omega)$ (smooth functions with compact support in Ω) is called *space of test functions* and plays an essential role in the theory of distributions.

We will not endow $\mathcal{C}^s_c(\Omega)$ with the subspace topology induced by $\mathcal{C}^s(\Omega)$, but we will consider a finer one, which will turn $\mathcal{C}^s_c(\Omega)$ into an LF-space. Let us consider a sequence $(K_j)_{j\in\mathbb{N}}$ of compact subsets of Ω s.t. $K_j \subseteq K_{j+1}, \forall j \in \mathbb{N}$ and $\bigcup_{j=1}^\infty K_j = \Omega$. (Sometimes is even more advantageous to choose the K_j's to be relatively compact i.e. the closures of open subsets of Ω such that $K_j \subseteq \mathring{K}_{j+1}, \forall j \in \mathbb{N}$ and $\bigcup_{j=1}^\infty K_j = \Omega$.)

Then $\mathcal{C}^s_c(\Omega) = \bigcup_{j=1}^\infty \mathcal{C}^s_c(K_j)$, as an arbitrary compact subset K of Ω is contained in K_j for some sufficiently large j. Because of our way of defining the F-spaces $\mathcal{C}^s_c(K_j)$, we have that $\mathcal{C}^s_c(K_j) \subseteq \mathcal{C}^s_c(K_{j+1})$ and $\mathcal{C}^s_c(K_{j+1})$ induces on the subset $\mathcal{C}^s_c(K_j)$ the same topology as the one originally given on it, i.e. the subspace topology induced on $\mathcal{C}^s_c(K_j)$ by $\mathcal{C}^s(\Omega)$. Thus we can equip $\mathcal{C}^s_c(\Omega)$ with the inductive topology τ_{ind} w.r.t. the sequence of F-spaces $\{\mathcal{C}^s_c(K_j),\ j \in \mathbb{N}\}$, which makes $\mathcal{C}^s_c(\Omega)$ an LF-space. It is easy to check that τ_{ind} does not depend on the choice of the sequence of compact sets K_j's provided they fill Ω.

Note that $(\mathcal{C}^s_c(\Omega), \tau_{ind})$ is not metrizable since it is not Baire (proof in Exercise Sheet 3).

Proposition 1.3.12. *For any integer $0 \leq s \leq \infty$, consider $\mathcal{C}^s_c(\Omega)$ endowed with the LF-topology τ_{ind} described above. Then we have the following continuous injections:*

$$\mathcal{C}^\infty_c(\Omega) \to \mathcal{C}^s_c(\Omega) \to \mathcal{C}^{s-1}_c(\Omega), \quad \forall\, 0 < s < \infty.$$

Proof. Let us just prove the first inclusion $i : \mathcal{C}^\infty_c(\Omega) \to \mathcal{C}^s_c(\Omega)$ as the others follows in the same way. As $\mathcal{C}^\infty_c(\Omega) = \bigcup_{j=1}^\infty \mathcal{C}^\infty_c(K_j)$ is the inductive limit of the sequence of F-spaces $(\mathcal{C}^\infty_c(K_j))_{j\in\mathbb{N}}$, where $(K_j)_{j\in\mathbb{N}}$ is a sequence of compact subsets of Ω such that $K_j \subseteq K_{j+1}, \forall j \in \mathbb{N}$ and $\bigcup_{j=1}^\infty K_j = \Omega$, by Proposition 1.3.7 we know that i is continuous if and only if, for any $j \in \mathbb{N}$, $e_j := i \restriction \mathcal{C}^\infty_c(K_j)$ is continuous. But from the definition we gave of the topology on each $\mathcal{C}^s_c(K_j)$ and $\mathcal{C}^\infty_c(K_j)$, it is clear that both the inclusions $i_j : \mathcal{C}^\infty_c(K_j) \to \mathcal{C}^s_c(K_j)$ and $s_j : \mathcal{C}^s_c(K_j) \to \mathcal{C}^s_c(\Omega)$ are continuous. Hence, for each $j \in \mathbb{N}$, $e_j = s_j \circ i_j$ is indeed continuous. □