1.4 Projective topologies and examples of projective limits

Let $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ be a family of locally convex t.v.s. over the field $\mathbb{K}$ of real or complex numbers (A is an arbitrary index set). Let E be a vector space over the same field $\mathbb{K}$ and, for each $\alpha \in A$, let $f_\alpha : E \to E_\alpha$ be a linear mapping. The ***projective topology*** τ_{proj} on E w.r.t. the family $\{(E_\alpha, \tau_\alpha, f_\alpha) : \alpha \in A\}$ is the locally convex topology generated by the following basis of neighbourhoods of the origin in E:

$$\mathcal{B}_{proj} := \left\{ \bigcap_{\alpha \in F} f_\alpha^{-1}(U_\alpha) : \ F \subseteq A \text{ finite, } U_\alpha \text{ basic nbhood of o in } (E_\alpha, \tau_\alpha), \forall \alpha \in F \right\}.$$

Hence, (E, τ_{proj}) is a locally convex t.v.s.. Indeed, since all (E_α, τ_α) are locally convex t.v.s., we can always choose the U_α's to be convex, balanced and absorbing and so, by the linearity of the f_α's, we get that the corresponding $\mathcal{B}_{proj}$ is a collection of convex, balanced and absorbing subsets of E such that:

a) $\forall\ U, V \in \mathcal{B}_{proj},\ U \cap V \in \mathcal{B}_{proj}$, because $U = \bigcap_{\alpha \in F} f_\alpha^{-1}(U_\alpha)$ and $V = \bigcap_{\alpha \in G} f_\alpha^{-1}(U_\alpha)$ for some $F, G \subseteq A$ finite and some U_α basic neighbourhoods of the origin in (E_α, τ_α) and so $U \cap V = \bigcap_{\alpha \in F \cup G} f_\alpha^{-1}(U_\alpha) \in \mathcal{B}_{proj}$.

b) $\forall\ \rho > 0, \forall\ U \in \mathcal{B}_{proj}, \rho U \in \mathcal{B}_{proj}$, since $U = \bigcap_{\alpha \in F} f_\alpha^{-1}(U_\alpha)$ for some $F \subseteq A$ finite and some U_α basic neighbourhoods of the origin in (E_α, τ_α) and so $\rho U = \bigcap_{\alpha \in F} f_\alpha^{-1}(\rho U_\alpha) \in \mathcal{B}_{proj}$.

Then Theorem 4.1.14 in TVS-I ensures that τ_{proj} makes E into a l.c. t.v.s..

Note that τ_{proj} is the coarsest topology on E for which all the mappings f_α ($\alpha \in A$) are continuous. Suppose there exists another topology τ on E such that all the f_α's are continuous and $\tau \subseteq \tau_{proj}$. Then for any neighbourhood U of the origin in τ_{proj} there exists $F \subseteq A$ finite and for each $\alpha \in F$ there exists U_α basic neighbourhood of the origin in (E_α, τ_α) such that $\bigcap_{\alpha \in F} f_\alpha^{-1}(U_\alpha) \subseteq U$. Since the τ-continuity of the f_α's ensures that each $f_\alpha^{-1}(U_\alpha)$ is a neighbourhood of the origin in τ, we have that U is itself a neighbourhood of the origin in τ. Hence, $\tau \equiv \tau_{proj}$.

Proposition 1.4.1. *Let E be a vector space over $\mathbb{K}$ endowed with the projective topology τ_{proj} w.r.t. the family $\{(E_\alpha, \tau_\alpha, f_\alpha) : \alpha \in A\}$, where each (E_α, τ_α) is a locally convex t.v.s. over $\mathbb{K}$ and each f_α a linear mapping from E to E_α. Then τ_{proj} is Hausdorff if and only if for each $0 \neq x \in E$, there exists an $\alpha \in A$ and a neighbourhood U_α of the origin in (E_α, τ_α) such that $f_\alpha(x) \notin U_\alpha$.*

Proof. Suppose that (E, τ_{proj}) is Hausdorff and let $0 \neq x \in E$. By Proposition 2.2.3 in TVS-I, there exists a neighbourhood U of the origin in E not containing x. Then, by definition of τ_{proj} there exists a finite subset $F \subseteq A$

and, for any $\alpha \in F$, there exists U_α neighbourhood of the origin in (E_α, τ_α) s.t. $\bigcap_{\alpha\in F} f_\alpha^{-1}(U_\alpha) \subseteq U$. Hence, as $x \notin U$, there exists $\alpha \in F$ s.t. $x \notin f_\alpha^{-1}(U_\alpha)$ i.e. $f_\alpha(x) \notin U_\alpha$. Conversely, suppose that there exists an $\alpha \in A$ and a neighbourhood of the origin in (E_α, τ_α) such that $f_\alpha(x) \notin U_\alpha$. Then $x \notin f_\alpha^{-1}(U_\alpha)$, which implies by Proposition 2.2.3 in TVS-I that τ_{proj} is a Hausdorff topology, as $f_\alpha^{-1}(U_\alpha)$ is a neighbourhood of the origin in (E, τ_{proj}) not containing x. □

Proposition 1.4.2. *Let E be a vector space over $\mathbb{K}$ endowed with the projective topology τ_{proj} w.r.t. the family $\{(E_\alpha, \tau_\alpha, f_\alpha) : \alpha \in A\}$, where each (E_α, τ_α) is a locally convex t.v.s. over $\mathbb{K}$ and each f_α a linear mapping from E to E_α. Let (F, τ) be an arbitrary t.v.s. and u a linear mapping from F into E. The mapping $u : F \to E$ is continuous if and only if, for each $\alpha \in A$, $f_\alpha \circ u : F \to E_\alpha$ is continuous.*

Proof. (Exercise Sheet 3) □

Example I: The product of locally convex t.v.s

Let $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ be a family of locally convex t.v.s. The product topology τ_{prod} on $E = \prod_{\alpha\in A} E_\alpha$ (see Definition 1.1.20 in TVS-I) is the coarsest topology for which all the canonical projections $p_\alpha : E \to E_\alpha$ (defined by $p_\alpha(x) := x_\alpha$ for any $x = (x_\beta)_{\beta\in A} \in E$) are continuous. Hence, τ_{prod} coincides with the projective topology on E w.r.t. $\{(E_\alpha, \tau_\alpha, p_\alpha) : \alpha \in A\}$.

Let us consider now the case when we have a directed partially ordered index set $(A, \leq)$, a family $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ of locally convex t.v.s. over $\mathbb{K}$ and for any $\alpha \leq \beta$ a continuous linear mapping $g_{\alpha\beta} : E_\beta \to E_\alpha$. Let E be the subspace of $\prod_{\alpha\in A} E_\alpha$ whose elements $x = (x_\alpha)_{\alpha\in A}$ satisfy the relation $x_\alpha = g_{\alpha\beta}(x_\beta)$ whenever $\alpha \leq \beta$. For any $\alpha \in A$, let f_α be the canonical projection $p_\alpha : \prod_{\beta\in A} E_\beta \to E_\alpha$ restricted to E. The space E endowed with the projective topology w.r.t. the family $\{(E_\alpha, \tau_\alpha, f_\alpha) : \alpha \in A\}$ is said to be the ***projective limit*** of the family $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ w.r.t. the mappings $\{g_{\alpha\beta} : \alpha, \beta \in A, \alpha \leq \beta\}$ and $\{f_\alpha : \alpha \in A\}$. If each $f_\alpha(E)$ is dense in E_α then the projective limit is said to be ***reduced***.

Remark 1.4.3. *Given a family $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ of locally convex t.v.s. over $\mathbb{K}$ which is directed by topological embeddings (i.e. for any $\alpha, \beta \in A$ there exists $\gamma \in A$ s.t. $E_\gamma \subseteq E_\alpha$ and $E_\gamma \subseteq E_\beta$ with continuous embeddings) and such that the set $E := \bigcap_{\alpha\in A} E_\alpha$ is dense in each E_α, we denote by i_α the embedding of E into E_α. The directedness of the family induces a partial order on A making A directed, i.e. $\alpha \leq \beta$ if and only if $E_\beta \subseteq E_\alpha$. For any $\alpha \leq \beta$, let us denote by $i_{\alpha\beta}$ the continuous embedding of E_β in E_α. Then the set E endowed with the projective topology τ_{proj} w.r.t. the family $\{(E_\alpha, \tau_\alpha, i_\alpha) : \alpha \in A\}$*

is the reduced projective limit of $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ *w.r.t. the mappings* $\{i_{\alpha\beta} : \alpha, \beta \in A, \alpha \leq \beta\}$ *and* $\{i_\alpha : \alpha \in A\}$*. For convenience, in such cases,* (E, τ_{proj}) *is called just reduced projective limit of* $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ *(omitting the maps as they are all natural embeddings).*

Example II: The space of test functions

Let $\Omega \subseteq \mathbb{R}^d$ be open in the euclidean topology. The space of test functions $\mathcal{C}_c^\infty(\Omega)$, i.e. the space of all the functions belonging to $\mathcal{C}^\infty(\Omega)$ which have a compact support, can be constructed as a reduced projective limit of the kind introduced in Remark 1.4.3. Consider the index set

$$T := \{t := (t_1, t_2) : t_1 \in \mathbb{N}_0,\ t_2 \in \mathcal{C}^\infty(\Omega) \text{ with } t_2(x) \geq 1,\ \forall\, x \in \Omega\}$$

and for each $t \in T$, let us introduce the following norm on $\mathcal{C}_c^\infty(\Omega)$:

$$\|\varphi\|_t := \sup_{x \in \Omega} \left(t_2(x) \sum_{|\alpha| \leq t_1} |(D^\alpha \varphi)(x)| \right).$$

For each $t \in T$, let $\mathscr{D}_t(\Omega)$ be the completion of $\mathcal{C}_c^\infty(\Omega)$ w.r.t. $\|\cdot\|_t$ and denote by τ_t the topology induced by the norm $\|\cdot\|_t$. Then the family $\{(\mathscr{D}_t(\Omega), \tau_t, i_t) : t \in T\}$ is directed by topological embeddings, since for any $t := (t_1, t_2), s := (s_1, s_2) \in T$ we always have that $r := (t_1 + s_1, t_2 + s_2) \in T$ is such that $\mathscr{D}_r(\Omega) \subseteq \mathscr{D}_t(\Omega)$ and $\mathscr{D}_r(\Omega) \subseteq \mathscr{D}_s(\Omega)$. Moreover, we have that as sets

$$\mathcal{C}_c^\infty(\Omega) = \bigcap_{t \in T} \mathscr{D}_t(\Omega).$$

Hence, the space of test functions $\mathcal{C}_c^\infty(\Omega)$ endowed with the projective topology τ_{proj} w.r.t. the family $\{(\mathscr{D}_t(\Omega), \tau_t, i_t) : t \in T\}$, where (for each $t \in T$) i_t denotes the natural embedding of $\mathcal{C}_c^\infty(\Omega)$ into $\mathscr{D}_t(\Omega)$ is the reduced projective limit of the family $\{(\mathscr{D}_t(\Omega), \tau_t) : t \in T\}$.

Using Sobolev embeddings theorems, it can be showed that the space of test functions $\mathcal{C}_c^\infty(\Omega)$ can be actually written as projective limit of a family of weighted Sobolev spaces which are Hilbert spaces (see [1, Chapter I, Section 3.10]).

1.5 Open mapping theorem

In this section we are going to come back for a moment to the general theory of metrizable t.v.s. to give one of the most celebrated theorems in this framework, the so-called open mapping theorem. Let us first try to motivate the question on which such a theorem is based on.

Let X and Y be two t.v.s. over $\mathbb{K}$ and $f : X \to Y$ a linear map. Then there exists a unique linear map $\bar{f} : X/\mathrm{Ker}(f) \to \mathrm{Im}(f)$ making the following diagram commutative, i.e.

$$\forall\, x \in X,\ f(x) = \bar{f}(\phi(x)).$$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\phi"] & \mathrm{Im}(f) \arrow[r, "i"] & Y \\
X/\mathrm{Ker}(f) \arrow[ur, "\bar{f}"] & &
\end{tikzcd}$$

where i is the natural injection of $\mathrm{Im}(f)$ into Y, i.e. the mapping which to each element y of $\mathrm{Im}(f)$ assigns that same element y regarded as an element of Y; ϕ is the canonical map of X onto its quotient $X/\mathrm{Ker}(f)$ (since we are between t.v.s. ϕ is continuous and open).

Note that

- $\bar{f}$ is well-defined.
  Indeed, if $\phi(x) = \phi(y)$, i.e. $x - y \in \mathrm{Ker}(f)$, then $f(x - y) = 0$ that is $f(x) = f(y)$ and so $\bar{f}(\phi(x)) = \bar{f}(\phi(y))$.
- $\bar{f}$ is linear.
  This is an immediate consequence of the linearity of f and of the linear structure of $X/\mathrm{Ker}(f)$.
- $\bar{f}$ is a one-to-one map of $X/\mathrm{Ker}(f)$ onto $\mathrm{Im}(f)$.
  The onto property is evident from the definition of $\mathrm{Im}(f)$ and of $\bar{f}$. As for the one-to-one property, note that $\bar{f}(\phi(x)) = \bar{f}(\phi(y))$ means by definition that $f(x) = f(y)$, i.e. $f(x - y) = 0$. This is equivalent, by linearity of f, to say that $x-y \in \mathrm{Ker}(f)$, which means that $\phi(x) = \phi(y)$.

Proposition 1.5.1. *Let $f : X \to Y$ a linear map between two t.v.s. X and Y. The map f is continuous if and only if the map $\bar{f}$ is continuous.*

Proof. Suppose f continuous and let U be an open subset in $\mathrm{Im}(f)$ (endowed with the subspace topology induced by the topology on Y). Then $f^{-1}(U)$ is open in X. By definition of $\bar{f}$, we have $\bar{f}^{-1}(U) = \phi(f^{-1}(U))$. Since the quotient map $\phi : X \to X/\mathrm{Ker}(f)$ is open, $\phi(f^{-1}(U))$ is open in $X/\mathrm{Ker}(f)$. Hence, $\bar{f}^{-1}(U)$ is open in $X/\mathrm{Ker}(f)$ and so the map $\bar{f}$ is continuous. Viceversa, suppose that $\bar{f}$ is continuous. Since $f = \bar{f} \circ \phi$ and ϕ is continuous, f is also continuous as composition of continuous maps. □

In general, the inverse of $\bar{f}$, which is well defined on $\mathrm{Im}(f)$ since $\bar{f}$ is injective, is not continuous, i.e. $\bar{f}$ is not necessarily open. However, combining the previous proposition with the definition of $\bar{f}$, it is easy to see that

Proposition 1.5.2. *Let $f : X \to Y$ a linear map between two t.v.s. X and Y. The map f is a topological homomorphism (i.e. linear, continuous and open) if and only if $\bar{f}$ is a topological isomorphism (i.e. bijective topological homomorphism).*

Now if Y is additionally Hausdorff and $\mathrm{Im}(f)$ finite dimensional, then whenever f is continuous we have that $\bar{f}$ is not only continuous but also open (see Theorem 3.1.1-c in TVS-I and recall that in this case $\mathrm{Ker}(f)$ is closed and so $X/\mathrm{Ker}(f)$ is a Hausdorff t.v.s..). Hence, any linear continuous map from a t.v.s. into a Hausdorff t.v.s. whose image is finite dimensional is also open. It is then natural to ask for which classes of t.v.s. any linear continuous map is also open. Of course, we are really interested in loosening the restriction of the finite dimensionality of $\mathrm{Im}(f)$ but we do expect that in doing so we shall give up some of the generality on the domain X of f. The open mapping theorem exactly provides an answer to this question.

Theorem 1.5.3. *Let X and Y be two metrizable and complete t.v.s.. Every continuous linear surjective map $f : X \to Y$ is open.*

The proof consists of two rather distinct parts. In the first one, we make use only of the fact that the mapping under consideration is onto and that Y is metrizable and complete (and so Baire). In the second part, we take advantage of the fact that both X and Y can be turned into metric spaces, and that Y is also complete.

Proof. Since Y is metrizable and complete, it is a Baire t.v.s. by Proposition 1.1.9. This together with the fact that $f : X \to Y$ is linear, continuous, onto map (and so $\mathrm{Im}(f)$ has non-empty interior) implies that the assumptions of Lemma 1.5.4 below are satisfied and so we get that $\overline{f(V)}$ is a neighbourhood of the origin in Y whenever V is a neighbourhood of the origin in X. This provides in particular that, for any $r > 0$ there exists $\rho > 0$ such that $B_\rho(o) \subseteq \overline{f(B_r(o))}$ since X and Y are both metrizable t.v.s.. Since the metrics employed can be always chosen to be translation invariant (see Proposition 1.1.3), we easily obtain that the assumption (1.14) in Lemma 1.5.5 below holds.

Let U be a neighbourhood of the origin in X. Then there exists $s > 0$ s.t. $B_s(o) \subseteq U$ and so $f(B_s(o)) \subseteq f(U)$. By applying (1.14) for $r = \frac{s}{2}$, we obtain that $\exists\, \rho := \rho_{\frac{s}{2}} > 0$ s.t. $B_\rho(o) \subseteq \overline{f(B_{\frac{s}{2}}(o))}$ and so, by Lemma 1.5.5, we have $B_\rho(o) \subseteq \overline{f(B_s(o))} \subseteq f(U)$ since $s > \frac{s}{2}$. Hence, $f(U)$ is a neighbourhood of the origin in Y. □

Lemma 1.5.4. *Let X be a t.v.s., Y a Baire t.v.s. and $f\colon X \to Y$ a continuous linear map. If $f(X)$ has non-empty interior, then $\overline{f(V)}$ is a neighbourhood of the origin in Y whenever V is a neighbourhood of the origin in X.*

Proof. (see Exercise Sheet 1) □

Lemma 1.5.5. *Let X be a metrizable and complete t.v.s. and Y a metrizable (not necessarily complete) t.v.s.. If $f\colon X \to Y$ is a continuous linear map such that*

$$\forall r > 0, \exists\ \rho_r > 0 \ \ s.t.\ B_{\rho_r}(f(x)) \subseteq \overline{f(B_r(x))}, \forall x \in X, \tag{1.14}$$

then for any $a > r$ we have that $B_{\rho_r}(f(x)) \subseteq f(B_a(x))$ for all $x \in X$.